Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

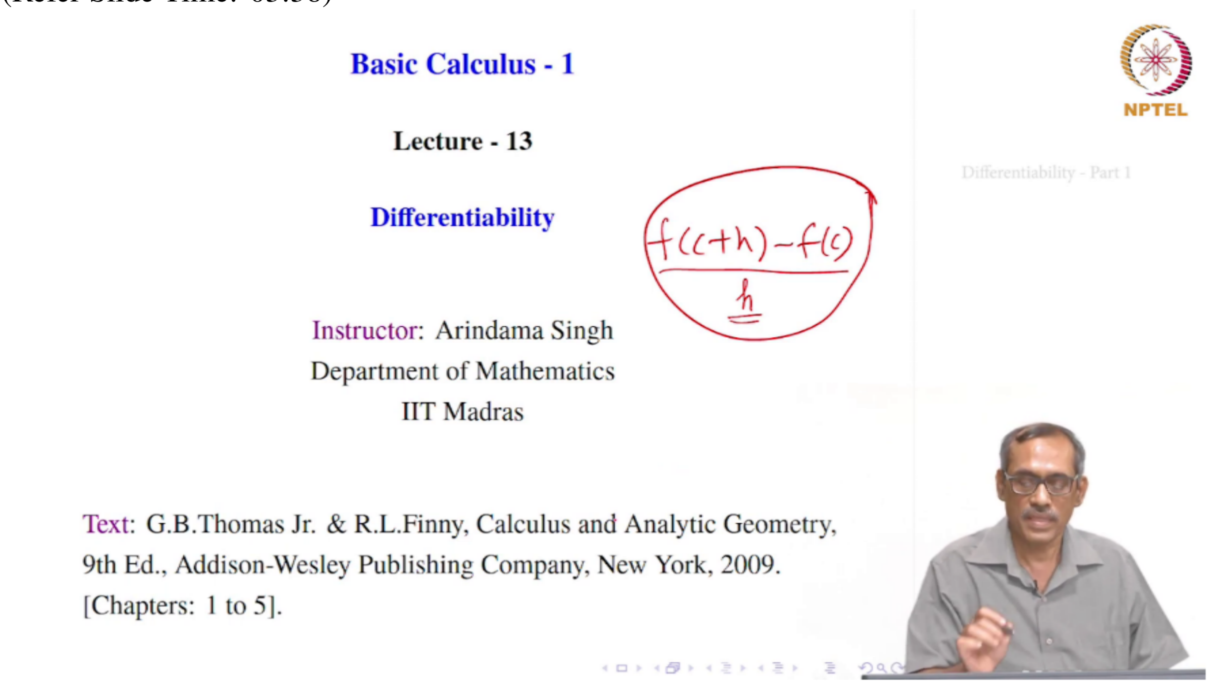
Lecture 13 - Part 1

Differentiability - Part 1

Well, this is lecture 13 of Basic Calculus 1. In the last two or three classes, we had covered the notion of continuity. Today we will be talking about another important concept, which is called differentiability of functions.

The idea is really simple. There is a cyclist, who covers one kilometer of distance in six minutes. You want to find out what is the speed of the cyclist. That is easy. The distance covered is 1 kilometer and the time taken is 6 minutes. So, it is 1 kilometer divided by 6 minutes; that will be the speed, that is the average speed. If you want to convert to different units, say, kilometer per hour. So, you write that 6 minutes as 1 divided by 10 hour; then it becomes 10 kilometer per hour. But see, what is happening here. We are computing the average speed. It might happen that on the road where the cyclist is riding is having some uphills, some downhills so that his speed is affected; sometimes it is faster, sometimes it is slower and so on. Finally, what we get is the average speed.

(Refer Slide Time: 03:38)



However, this notion of differentiability is something like instantaneous speed. So, what is the speed of the cyclists at a given point. Obviously, we cannot do that here. What we are doing is average speed only. But the average speed uses speed over some duration, say 6 minutes was here. Suppose it is not over 6 minutes, but we want to compute the instantaneous speed at a point. What do we do? We take that duration to be as small as possible, that is close to 0. Let us say h is the time taken. And, how do you get the distance? Where from you have started that is of course 0.

But you might have covered some other distance before coming to the final place. We measure the total distance.

Let us say that the time point is c. Write the distance as a function of time, say, $f(x)$. After the duration of h, the distance is $f(c+h)$. Now the distance covered from time c to time $c+h$ is $f(c+h) - f(c)$. The duration is of course h. This $[f(c+h) - f(c)]/h$ is the average speed in the duration h, starting from the instant of time c, and going up to $c+h$. Now when you speak about the instantaneous speed at the point c, at the time c, you take this duration of time close to 0. That means we will take the limit of this ratio where $f(x)$ is the distance as a function of time. This would give us the instantaneous speed. This is the notion of differentiability.

So, let us give the definition. Suppose we have a function f defined from a subset D of $\mathbb{R}$ to $\mathbb{R}$. And we are not going not for the units now, we are ignoring the units. On the x-axis you have time, and on the y-axis you have the distance covered. If the distance is in kilometers and time is in hours, then you would get the ratio as kilometer per hour. We are making it unit less. So, on the x-axis a number shows the time measured, and one on the y-axis shows the distance covered. The function f connects to the distance with time. Now, f is a function from D to $\mathbb{R}$, where $D \subseteq \mathbb{R}$.

Let c be an interior point of D. We need the time instant c to be an interior point because we will take a neighborhood of c. Recall that c is an interior point means, there is a neighborhood of c, which is contained inside D. That is, D contains some open interval of the type $(c-\delta, c+\delta)$. So, c is an interior point of D. In fact, we will not consider abstract subsets D of $\mathbb{R}$, we will consider just intervals. So, think of c as an interior point of the interval D, on which f is defined.

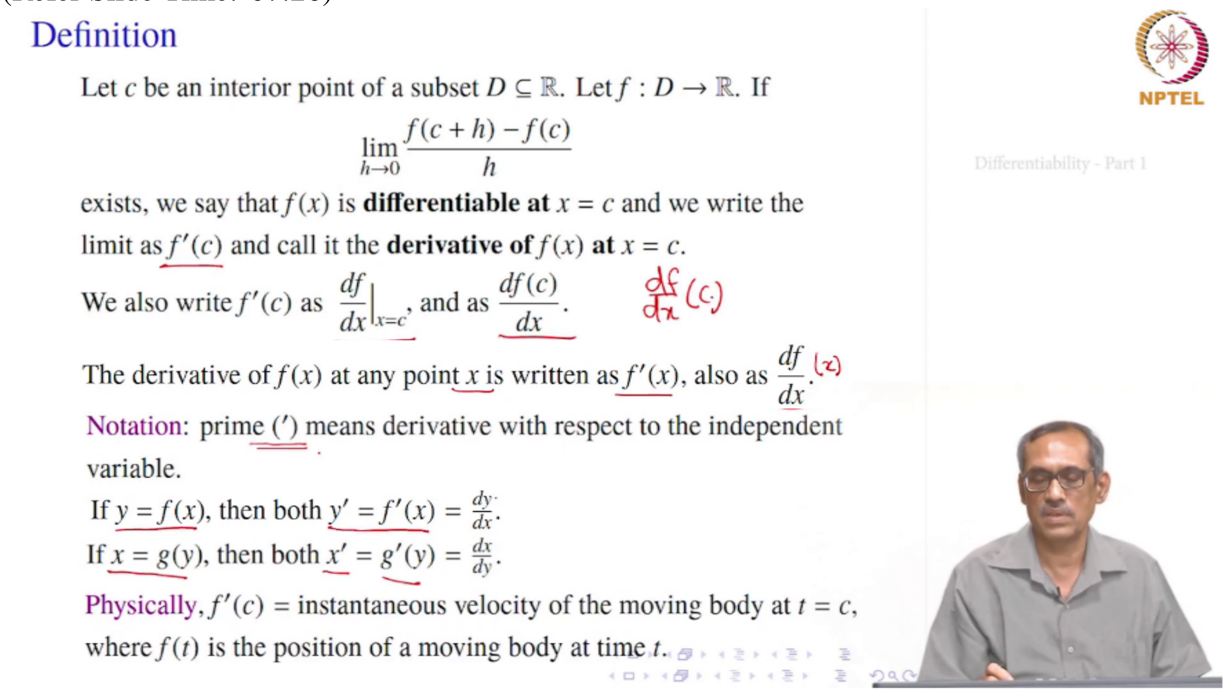
Now what we do? We take the ratio, which is $f[(c+h) - f(c)]/h$, and then take its limit as h goes to 0. Since there is a neighborhood around c, which is contained inside D, this limiting process makes sense. If this limit exists, that means it is a real number, then we would say that the function $f(x)$ is differentiable at $x = c$. It is quite possible that the limit does not exist. We will see some examples to that effect.

First of all we are assuming that c is an interior point, and then you take the ratio, take its limit as $h \to 0$. Now, this ratio is a new function of h. Since $f(c)$ is a number, the ratio $f(c+h) - f(c)$ divided by h is a function of h. We try to compute this value as limit h goes to 0. If that value is a real number, then we say that $f(x)$ is differentiable at $x = c$. Then we write this limit as $f'(c)$, and call it the derivative of $f(x)$ at $x = c$.

So, it is implicitly defined. That is, once the derivative exists, we say it is differentiable. And once it is differentiable, then the derivative exits. Both the things are interconnected. That is why we put it this way: if the limit exists, we call f to be differentiable; and whatever is the value of that limit, that is called $f'(c)$, the derivative of $f(x)$ at $x = c$.

There are different notations for writing the derivative. Sometimes we write like this: df/dx at $x = c$. We take that c to the top and write $df(c)/dx$; it is probably better to write this way: $(df/dx)(c)$. But think of this as an abbreviation of this expression: df/dx at c.

(Refer Slide Time: 07:26)



You see that the derivative of $f(x)$ at any point x is defined at one point c. Now, vary this c over all the interior points of D. At all those points x, the derivative of $f(x)$ will be written as $f'(x)$. As usual we forget that c here, assuming that it is as if at any x. We will write this as df/dx or as $f'(x)$; this is just a notation. We should be a bit careful about this prime notation. Here, prime means it is df/dx; where x is the independent variable of f, that is why we write dx here, and that is what prime means.

That means, suppose y is a function of x, then we would write $y'(x)$ and also as dy/dx. If we consider x as a function of y, say, $x = g(y)$, then $x'(y)$ is written as $g'(y)$, which is same as dx/dy. It is with respect to the independent variable we are taking. So, prime means derivative with respect to its independent variable. This notation will be very handy. We will see how.

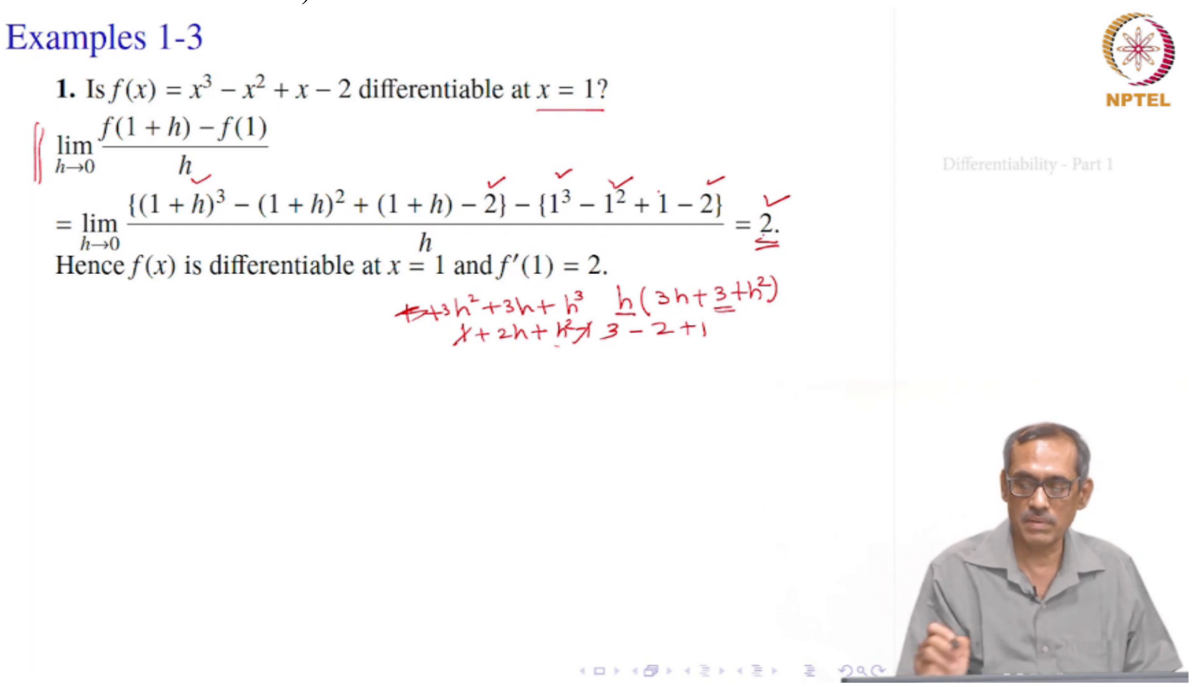
Let us consider an example and apply this notion. Suppose $f(x) = x^3 - x^2 + x - 2$. The question is whether this function is differentiable at $x = 1$? All that we have to do is, find the ratio and find the limit. Now that x is 1, $[f(1+h) - f(1)]/h$ is the ratio and we want to find its limit as h goes to 0. If this limit exists, then we would say "yes, it is differentiable at 1, and that value of the limit is the derivative".

So, let us work out: $f(1+h) = (1+h)^2 - (1+h)^2 + (1+h) - 2$, $f(1) = 1^3 - 1^2 + 1 - 2$. Then, $f(1+h) - f(1) = [(1+h)^3 - 1^3] - [(1+h)^2 - 1^2] + [1 + h - 1]$. We expand and see that there is always an h in each of the bracketed quantities. Now, we can cancel this h and simplify. It gives $h[(1+h)^2 + (1+h) + 1^2] - h[1+h+1] + h$ divided by h, or $(1+h)^2 + (1+h) + 1 - (2+h) + 1$. As h goes to 0, we get the limit as 2.

Or, directly expanding the powers, we have $(1+h)^3 = 1 + 3h^2 + 3h + h^3$ and from there you have taken away 1, so this has gone, you have only this factor. And that gives $h(3h + 3 + h^2)$. This h gets canceled, you get only 3 in the limit as h goes to 0. Similarly, from $(1+h)^2 - 1$, you get

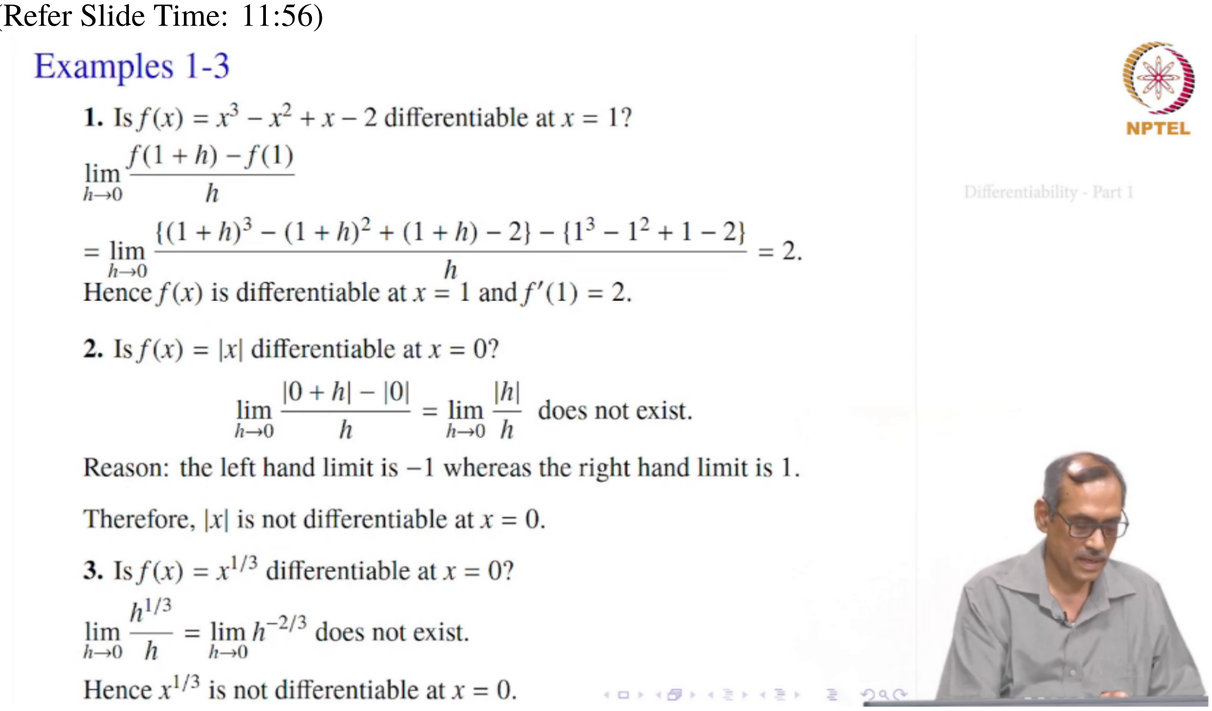
$1 + 2h + h^2 - 1$. So, 1 cancels, h cancels, and you get $2 + h$, which in the limit gives 2. Next, $1 + h - 1$ gives h, and that h cancels to give you 1. So, the answer is $3 - 2 + 1 = 2$.

(Refer Slide Time: 08:50)



Since the limit is equal to 2, we would say that "yes, $f(x)$ is differentiable at $x = 1$ with the derivative equal to 2, or that $f'(1) = 2$".

(Refer Slide Time: 11:56)

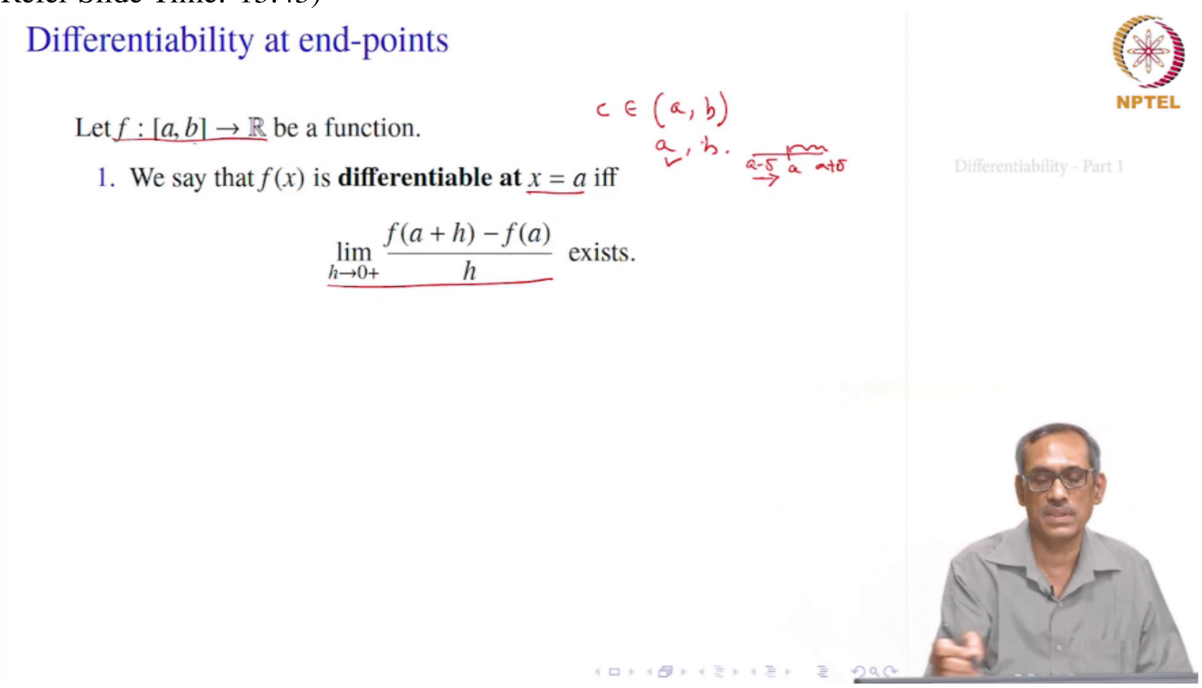


Let us consider the second example. Here, the function is $|x|$. The question is, whether it is differentiable at $x = 0$? Again we would proceed to evaluate the limit of $[|0 + h| - |0|]/h$ as h goes to 0. We want to find out this limit. This simplifies to $|h|/h$. When you take limit as h goes to 0, you may think of the left side limit and the right side limit. That will be easier.

So, when $h < 0$ but close to 0, $|h|$ becomes $-h$. Then, the ratio $|h|/h = -1$; so the left side limit is -1. If you take the right side limit, then $h > 0$ and close to 0; it gives $|h| = h$ so that $|h|/h = 1$. The limit is 1. Since these two limits are different, the limit does not exist. Therefore, this is not differentiable at $x = 0$. That is what we get.

Let us take the third problem. Here, $f(x)$ is given as $x^{1/3}$. The question is, whether this is differentiable at $x = 0$? Again, we would take the ratio. The ratio is $[f(0+h) - f(0)]/h = h^{1/3}/h = h^{-2/3}$. Again, this limit does not exist. This is in the form $1/h$; it will blow up. So, the limit does not exist. Therefore, this function is not differentiable at $x = 0$.

(Refer Slide Time: 13:43)



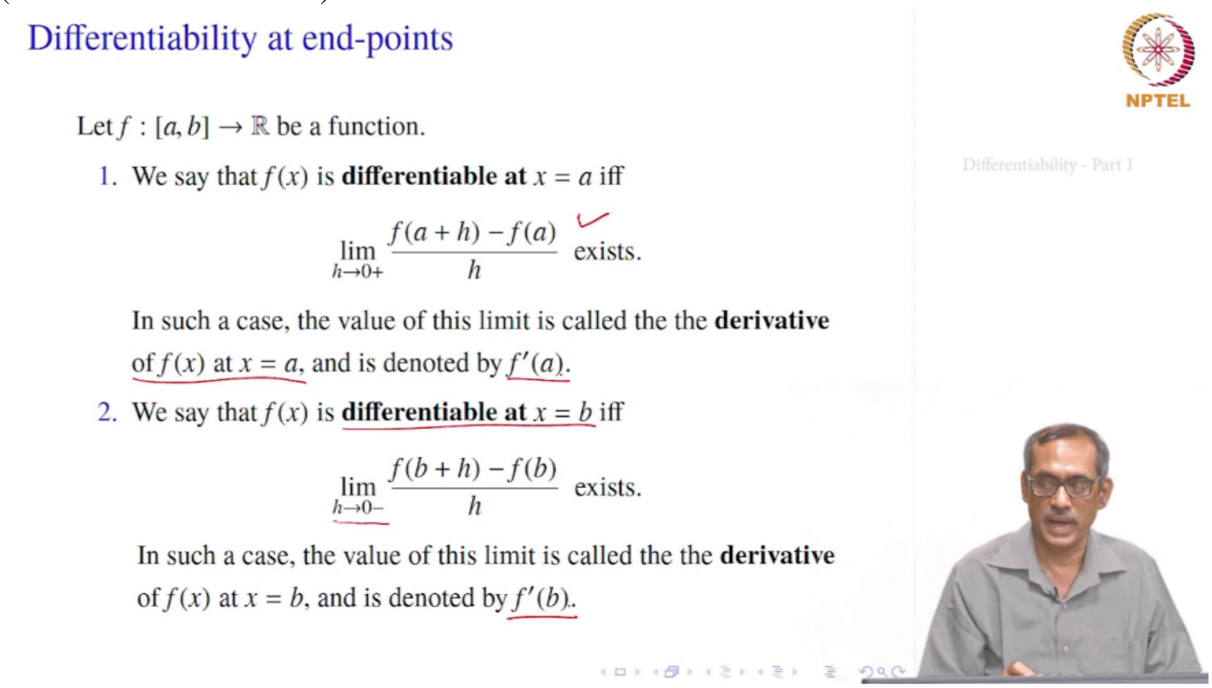
Consider this function f which is defined on an interval $[a, b]$. In our definition of differentiability, we have c, and at c when we are computing $f'(c)$, or when we are finding out whether f is differentiableat at c or not, this c has to be an interior point of $[a, b]$. That means, in this case, c must be a point inside the open interval (a, b). There are other points in the domain of f, which are a and b. What happens at these endpoints?

At the endpoint a, if you take any neighborhood, say $(a - \delta, a + \delta)$, this is not contained inside the domain of definition of the function, which is $[a, b]$. Only a right neighborhood is contained inside this, provided we make this δ small. But whatever δ you choose, the left neighborhood is never contained inside the domain. Therefore, the left side limit does not make any sense. Only the right side limit makes sense. So, what do we do?

At the end point a, we define the differentiability notion itself first. We say that $f(x)$ is differentiable at $x = a$, which is the left endpoint, if only the right side limit exists, because that is the only relevant thing. That is what this definition says. If f is defined on a closed interval $[a, b]$, then at the left endpoint a it is differentiable if the right side limit of the ratio $[f(a + h) - f(a)]/h$

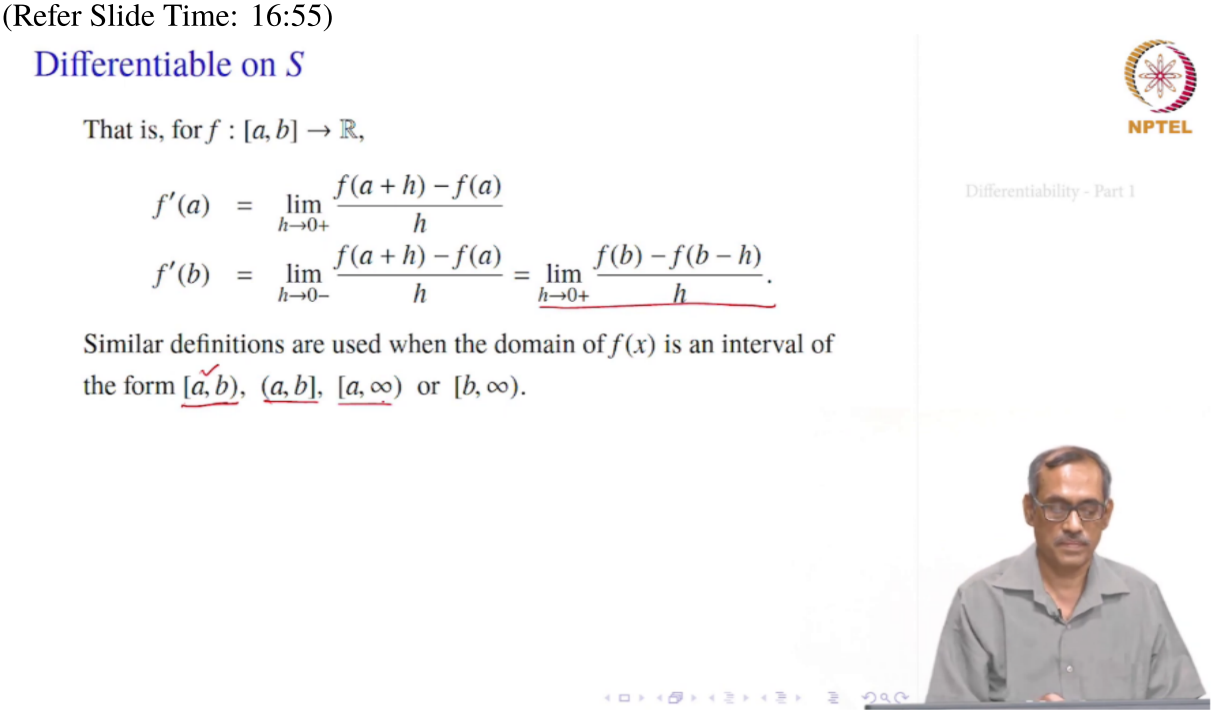
exists. Here, h is assumed to be positive.

(Refer Slide Time: 15:43)



Similarly, at b, the left side neighborhood makes sense. But the right side neighborhood is not contained inside the domain. So, we will say that the function is differentiable at $x = b$ if the left side limit exists. Here h is negative. What is the meaning of h is negative here? If you write $h = -k$, where k is positive, then the ratio gives $[f(b-k) - f(b)](-k)$, which is same as $[f(b) - f(b-k)]/k$. Here, k goes to 0+. Also, you can write the limit this way.

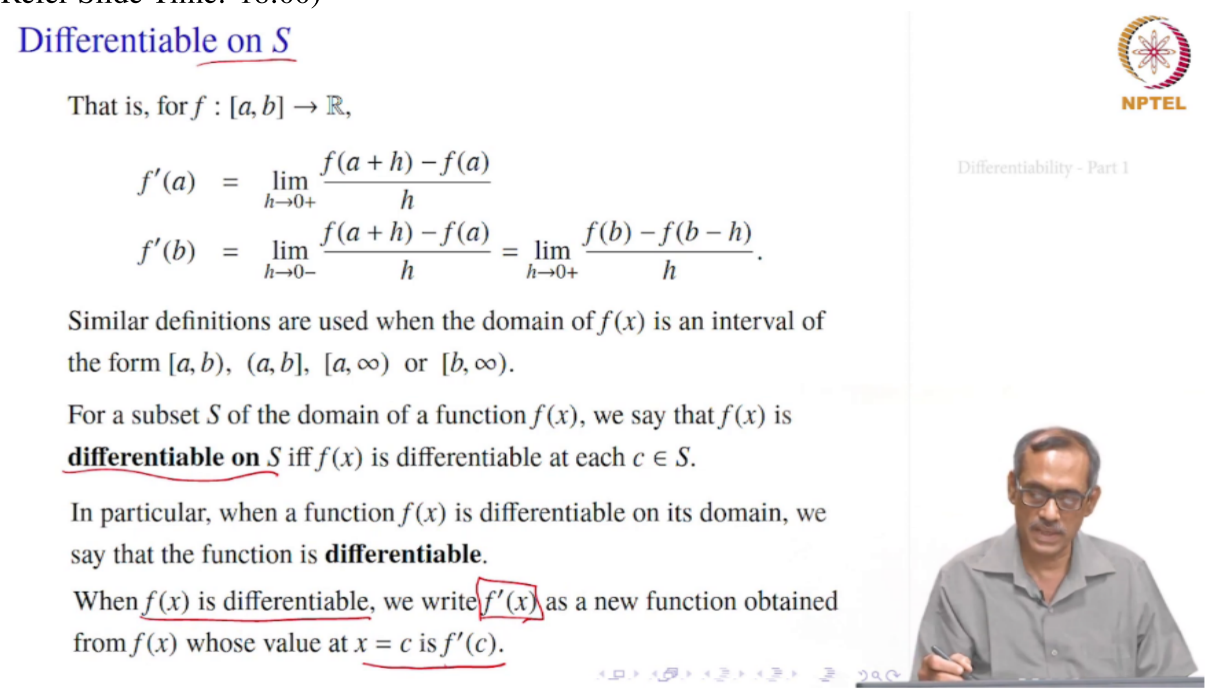
(Refer Slide Time: 16:55)



So whatever value we get from this limit, they will be taken, by definition, as the derivative of f at the endpoints. When $x = a$, we say that this right side limit is equal to $f'(a)$. And when $x = b$, we will say that this left side limit is equal to $f'(b)$. That is how we give definitions of the derivative at the endpoints. We may not have to worry about this now. Of course, when time comes, it will be helpful to use the notion of differentiability at the endpoints.

Suppose I have a function $f(x)$ defined on $[a, b]$ to $\mathbb{R}$. At a, we can write this as the right side limit, and at b, it is the left side limit, which is same as limit k goes to 0+ of $[f(b-k) - f(b)]k$, or we write the same thing using h. This way also it becomes helpful, because you are used to thinking in terms of h to be positive. But we should always remember that h can be both positive and negative, when we take the limit as h goes to 0. At the end points only, this 0− and this 0+ come up. The same way we can define at the endpoints when f is defined on intervals of this type. Here, at the left side endpoint a, the limit should be taken as $h \to 0+$.

(Refer Slide Time: 18:00)



Here, it is the left side limit at b; here again, the right side limit at a and so on. So that is how our definitions will carry over.

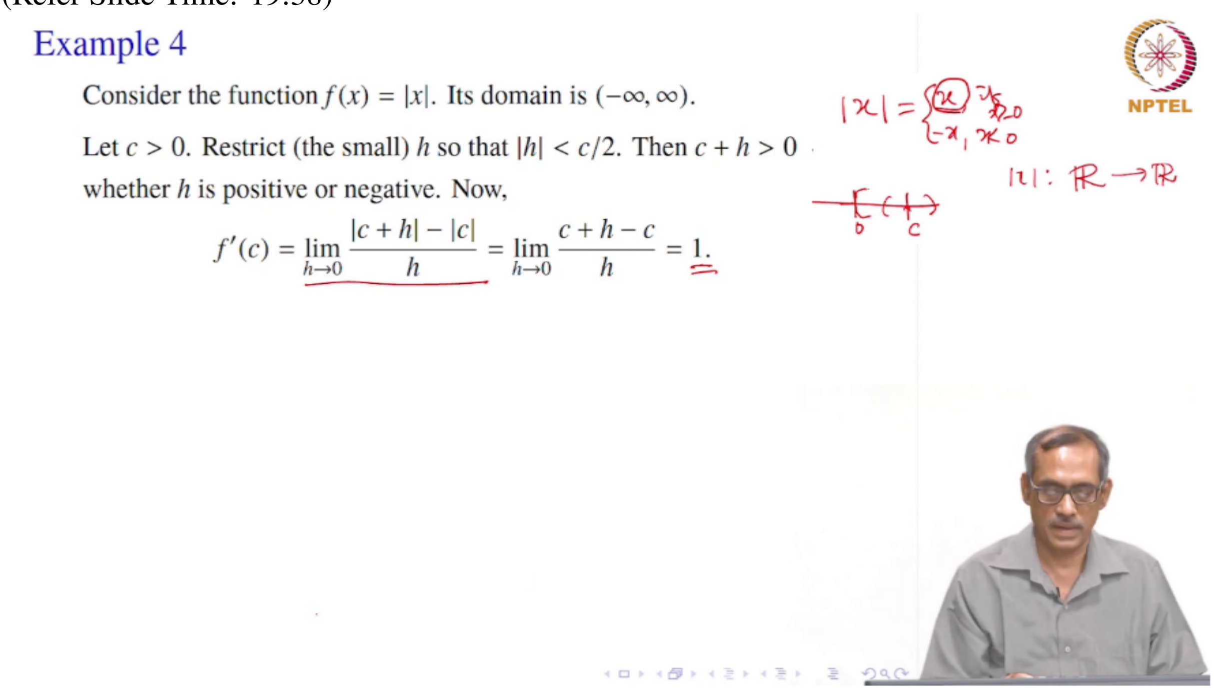
Now, if the function is defined on a set, we would say that the function is differentiable on that set, if and only if it is differentiable at each point of the set. Just like your idea of continuity, we define at a point, then generalize over all points. That is the way we are fixing the terminology.

So, it is differentiable means differentiable at every point. When a function is differentiable at every point of its domain, we would say that the function is differentiable, forgetting this S. If S is a subset of the domain, then 'differentiable on S' is also okay. But once you say only differentiable that means it is differentiable on its domain.

The thing is, if $f(x)$ is differentiable at every point x inside its domain, then $f'(x)$ at $x = c$ is $f'(c)$. At x equal to another point d, its value is $f'(d)$, and so on. So, $f'(x)$ becomes a new function, obtained from $f(x)$. This is called the derivative of $f(x)$ as a function, not only at a particular point. It is something like the derivative of $f(x)$ evaluated at $x = c$ gives $f'(c)$. Both the things are there in the notation, though this is not our definition. We have defined $f'(c)$ independently and we give that as the value of $f'(x)$ at $x = c$ now. Both the notations coincide.

Let us take another example. We consider the function $f(x) = |x|$. For this we had already considered whether it is differentiable at $x = 0$ or not. The domain of this function is the whole of $\mathbb{R}$; it is defined from $\mathbb{R}$ to $\mathbb{R}$. Of course, its range is only the non-negative real numbers. Negatives are mapped to positives. Once you ask whether this is differentiable or not, we have to be concerned about all points, all real numbers. At each real number we have to see whether it is differentiable or not. And our experience says at 0 there is some problem. But if if c is less than 0 or greater than 0, there can be different values, altogether different types of values for the derivative at c.

(Refer Slide Time: 19:38)



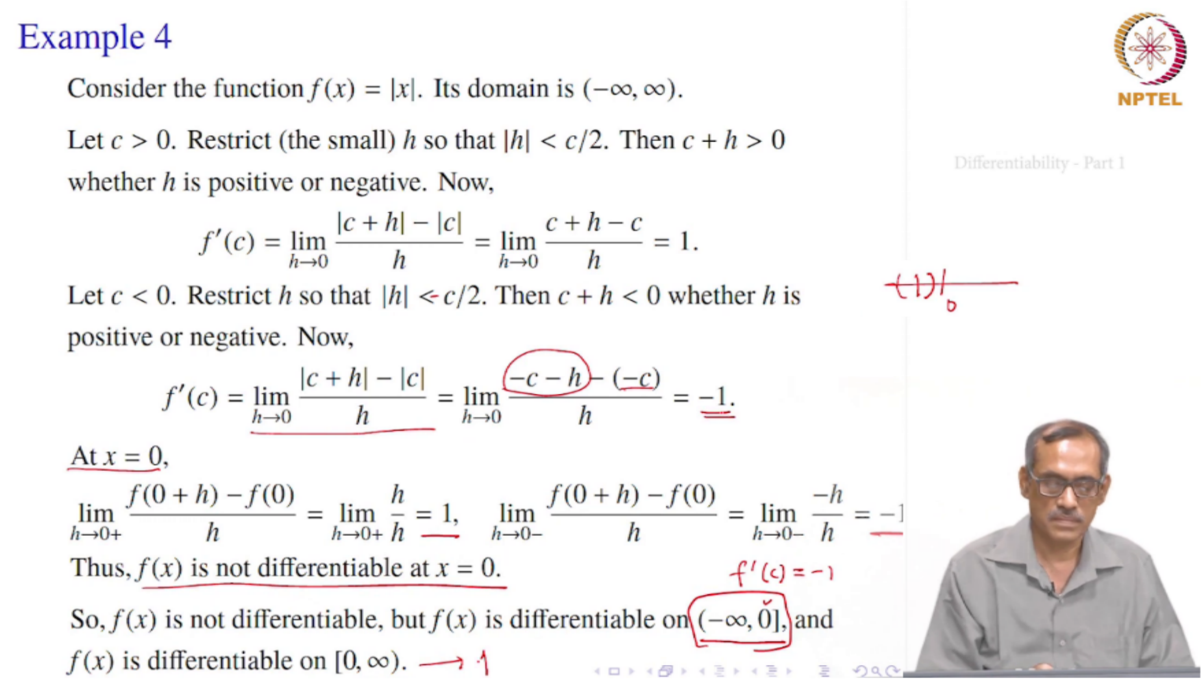
Let us consider first $c > 0$. We want to find whether the function is differentiable at c or not. As $c > 0$, it is not equal to 0; say 0 is here, and c is here somewhere. Now what happens, c is an interior point on the right side of 0. We have defined $|x|$ conditionally, that is, $|x| = x$ if $x|geq0$, and $|x| = -x$ if $x < 0$. When $x > 0$, we are considering the right side. Even this is possible, but we know the result at $x = 0$. On the other side, right interval, it is an interior point means there is a neighborhood which is inside the set; even for 0 that is true.

Let us see what happens at this point c, where c is positive. Since we will be taking the limit, we can really make our h to be smaller. Let us restrict our h to be smaller than $c/2$. So that this $c + h$ will not go beyond 0.

So, one we can really use the first one, because $c > 0$. Let us restrict that h, anyway it goes to 0. So, $c + h$ is now positive; it is lying to the right of 0; whether h is positive or negative, it does not matter. Then this limit of the ratio $[|c + h| - |c|]/h$ will be $(c + h - c)/h$, which is equal to 1. So, $f'(c) = 1$. Because h is small, we can limit it to a convenient neighborhood.

Now, when $c < 0$, we can again see what is happening. It is 0 and $c < 0$. We can limit our h to be smaller than say, $|c|/2 = -c/2$. It will be somewhere here. Then $c + h$ will remain here anyway, will be always negative. As $c + h < 0$, whether h is positive or negative, we compute the ratio. Now, $|c + h| = -(c + h) = -c - h$. Then, $|c + h| - |c| = -c - h - (-c)$. This c gets canceled and $-h/h$ gives you -1. Therefore, when $c < 0$, $f'(c)$ is negative.

(Refer Slide Time: 23:01)



And already we know at $x = 0$, its left side limit is -1, its right side limit is $+1$. So, $f(x)$ is not differentiable at $x = 0$. But taking care of our definition of right endpoint or left endpoint, suppose you consider f to be defined on only a subset, $(-\infty, 0]$. That is, this is our S now. On this subset we want to find whether it is differentiable or not.

In this interval $(-\infty, 0]$, 0 is the right endpoint. So, the limit of the ratio will be taken when h remains negative. At 0 it is the left side limit that becomes -1. We know for $c < 0$, its limit is also -1. Therefore, $f'(c) = -1$ for any point $c \in (-\infty, 0]$. Similarly, when you go to the right side, the right side limit is relevant. That gives you the right side limit of the ratio as 1 at 0. So, whenever $c \in [0, \infty)$, $f'(c) = 1$ and not -1.

So, we have different subsets; each includes 0; here $f'(c)$ is -1 but here, it is $+1$. That goes along with our definition of one-sided derivatives, for, the derivatives at the endpoints are really one sided now. It happens because the domains are like that.