Basic Calculus - 1

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Lecture 12 - Part 2

Results on Continuity - Part 2

This is the intermediate value theorem. But what it says: the conclusion of the intermediate value theorem may not hold true if you do not have continuous function. However, continuity is not really required for the conclusion in IVT to hold. We will see both the things, both these points. If these points do not hold, then this property would have been equivalent to continuity. Let us see the first point. You have an example. We can give many examples. Let us see one.

Define the function g; $[0, 2] \to \mathbb{R}$ by $g(x) = x$ for $x \in [0, 1]$, and $g(x) = x + 2$ otherwise, that is, when $x \in (1, 2]$. How does the graph look like here? It is x from 0 to 1; it is something like this. And it is $x + 2$ for $1 < x \leq 2$, so at $x = 1$, it is 3. It is something like this here. It will be again a straight line. It goes up to from 3 to 4, something like this. Then it is not continuous, there is a break.

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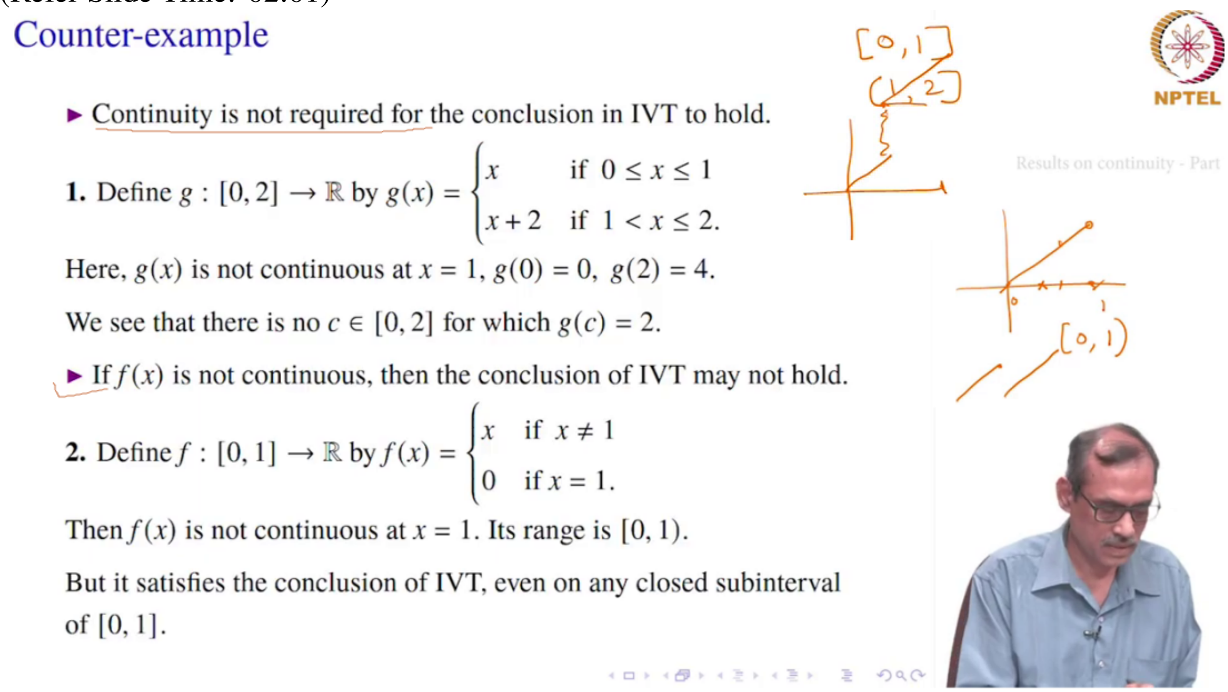

Now, we find that it is not continuous at $x = 1$. There is a break point; it is 1, and $g(0) = 0$ and $g(2) = 4$. We see that 2 lies between 0 and 4, but there is no point c in its domain of definition such that $g(c) = 2$ because all that we get is g achieves values from 0 to 1 as x varies 0 to 1, and again from 3 to 4 as x varies from 1 to 2. From 1 to 3 nothing is mapped.

So, this is not equal to 2 for any c. It is not continuous; at least at $x = 1$ it is not continuous, and the intermediate value theorem does not hold. That is what it says. Our earlier example also in the picture has shown that.

Next, we are going to see that continuity is not required for the conclusion in IVT to hold. That means there can be some point where it is not continuous but intermediate value theorem holds.

For example, define $f : [0,1] \to \mathbb{R}$ by $f(x) = x$ if $x \neq 1$ and $f(x) = 0$ if $x = 1$. How does it look? It is defined from 0 to 1. It is x up to this, but this part is excluded, 1 is excluded. And at 1, it is really 0, it is this point. So, it is not continuous at 1. It assumes values from 0 to 1 that is its range is $[0,1)$. And you give any value in this range $[0,1)$, there is a point corresponding to each such value so that $f(c)$ is equal to that. That is what it means. So, the conclusion of intermediate value theorem holds. However, the function is not continuous there. That is what it says.

The first example shows this one: $f(x)$ is not continuous, then the conclusion of IVT may not hold. And the second example says that continuity is not required for the conclusion in IVT to hold. Of course, you can give some functions not continuous even at an interior point. You can give a function like this: it starts from here and there is another from here; at this point include only one of them. There also the conclusion of intermediate value theorem will hold but the function is not continuous at that point.

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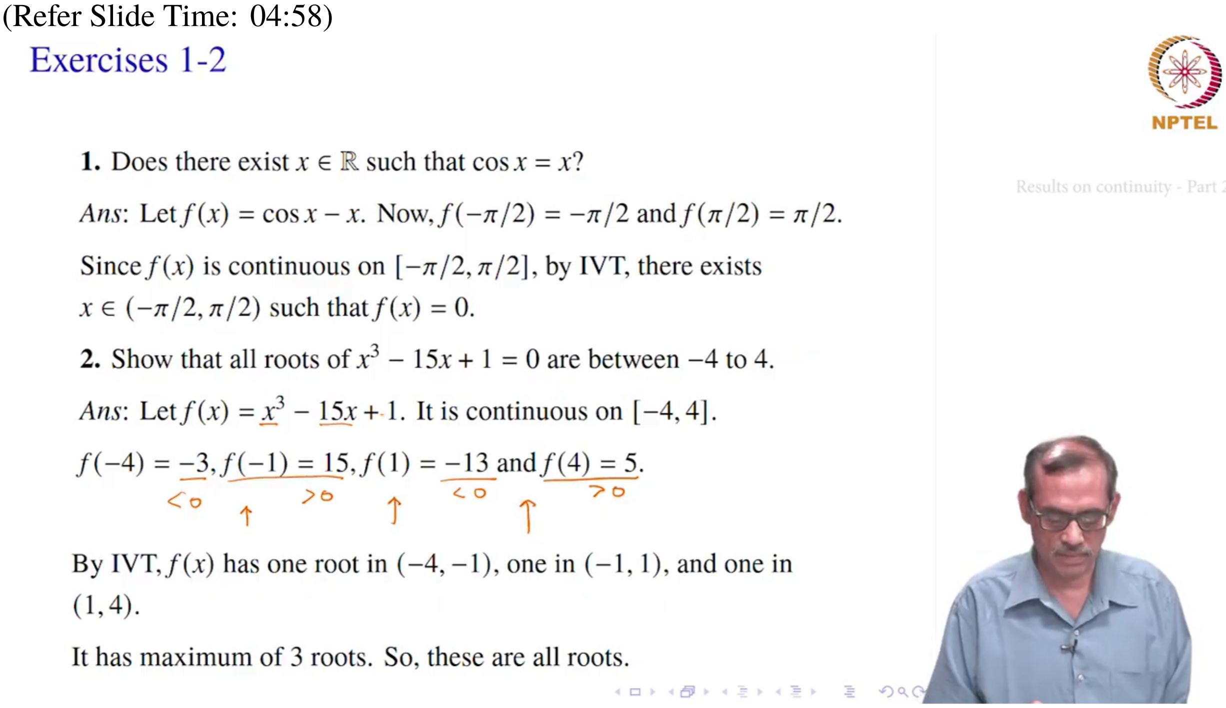

we take some problems. The question here is: does there exist one point x, one real number, such that $\cos x = x$? Of course, you can see from the graph of $\cos x$, and then take the line $y = x$ and see whether they intersect. But let us see how to do it analytically.

Usually in this kinds of problems, we start like this. We define $f(x) = \cos x - x$, and then see whether there is a point where f of that point is equal to 0 or not. To apply intermediate value theorem, we have to see whether this function takes a negative value or not, and whether this function takes a positive value or not. For this, we do some guesswork to find such points where it may be negative, and where it may be positive.

Here, let us take $f(x) = \cos x - x$. Then, $f(-\pi/2) = \pi/2$. Is that right? At $-\pi/2$, $\cos x$

becomes 0 and x becomes $-\pi/2$. Because this one is $-x$, we get $f(-\pi/2) = \pi/2$. Now, let us take $x = \pi/2$. At this x, again $\cos x$ is 0 and this becomes $-\pi/2$. That is, $f(\pi/2) = -\pi/2$. There is a change in sign.

Alternatively, you can start with the function $x - \cos x$ and those things will be all right.

What we see is that at the point $-\pi/2$, it is positive and the point $\pi/2$, it is negative; it is continuous on $[-\pi/2, \pi/2]$. By IVT, there is a point in $(-\pi/2, \pi/2)$ where it is 0. And that gives a point c where $f(c) = 0$, that is, at that point c, $\cos c = c$.

Let us go to the second problem. It asks us to show that all roots of $x^3 - 15x + 1 = 0$ are between -4 to 4. Again, we will be using the intermediate value theorem. All that we know from our earlier example is that this is a cubic, or degree 3, so it has a root in real numbers. But it says something more. It asks us to show that there is a root between -4 to 4; not only one root, but all roots are there. you have to really find out other points where it may be alternating its sign. That is how we will be proceeding. There will be a guess work.

So, let us start with $f(x) = x^3 - 15x + 1$. It is continuous on $[-4, 4]$. We have to check this first. Next, what we find is $f(-4) = -64 - 15 \times (-4) + 1 = -3$, and $f(4) = 64 - 15 \times 4 + 1 = 5$. Its values lye between -4 to 5. By the intermediate value theorem, there is a root.

Now, we do some other guess work to find other roots. We find $f(-1)$ because this will dominate for the negative for large values; for small values, we do not know. So you have to try. Now, $f(-1) = -1 - 15 \times (-1) + 1 = 15$, and $f(1) = 1 - 15 \times 1 + 1 = -13$. So, $f(-1)$ is positive and $f(1)$ is negative. So, there is a root in between and -1 and 1.

And, $f(1)$ is negative, $f(4)$ is positive, so there is again a root in between 1 and 4 because of intermediate value theorem. So, you have got now 3 roots between -4 to 4; one, within -4 to -1, another within -1 to 1, and another between 1 to 4. This is a cubic equation; so it has 3 roots at the maximum. It follows from the fundamental theorem of algebra, which we are assuming here. It has maximum of three roots. Therefore, all the roots are inside the interval -4 to 4.

Let us go to the next exercise. Show that that $f(x) = x^3 - 8x + 10$ achieves the values -3, π and 800000. Here, 'achieves the values' means there is a point a such that $f(a) = -3$; similarly, there is a point b such that $f(b) = \pi$ and there is a point c such that $f(c) = 800000$.

I think it is a polynomial of odd degree. If x goes to $-\infty$, then it goes to $-\infty$ in the limit, and its limit as x goes to ∞ is equal to ∞. So, its range will be the whole of real numbers. Therefore, -3, π and 800000 are there in its range.

However, we want to make a better argument, we give the point or say at least where it is. We can do that. Let us see. The number 800000 has 6 digits. Let us start with 100, as 100^3 is bigger than that. If I evaluate $f(-100)$, then I would get this one. Just use your calculator and see. If you take $f(100)$, then it will be this number. And all these numbers such as -3, π and 800000 lie between these two numbers. Therefore, by intermediate value theorem, there are points $a,\ b,\ c \in (-100, 100)$ such that $f(a) = -3$, $f(b) = \pi$ and $f(c) = 800000$, because f is continuous.

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Exercises 3-4

3. Show that $f(x) = x^3 - 8x + 10$ achieves the values -3, π and 800000.

Ans: $f(-100) = -999190$ and $f(100) = 999210$.

$f(x)$ is continuous.

It achieves all values between -999190 and 999210.

4. Give examples of $f(x)$ and $g(x)$ such that at $x = 0$, both f and g are continuous but $g \circ f$ is not continuous.

Ans: Take $f(x) = x + 1$ and $g(x) = 1/(x-1)$.

Both are continuous at $x = 0$.

$(g \circ f)(x) = g(x+1) = 1/(x+1-1) = 1/x$.

It is not continuous at $x = 0$.

Notice that g is not continuous at $f(0) = 1$.

Let us go to the next problem. Here we are asked to find some examples of functions such that they are continuous at $x = 0$, but their composition is not continuous, say, at $x = 0$. If you recall, this g should be continuous at $f(0)$, then only $g \circ f$ will become continuous. But g is given to be continuous at 0. That means we have to construct a function where $f(0)$ will not be 0. And then finally, we have to show $g \circ f$ is not continuous.

Well, let us try one; say, $f(x) = x + 1$ and $g(x) = 1/(x-1)$. Where are the defined? Both are continuous at $x = 0$. Let us take some neighborhood of 0 and see. So, both are continuous at $x = 0$. We find that $(g \circ f)(x) = g(f(x)) = g(x+1) = 1/(x+1-1) = 1/x$. And, this is not continuous at $x = 0$. That is it. Notice that g is not continuous at $f(0) = 1$ also. That is an extra information we are getting.

This is our fifth problem. We want to discuss the continuity of the function given in this way, with three conditions that if x is negative, then it is $1 + x$, if x lies between 0 to $\pi/2$, both inclusive, then it is $1 + \lfloor x \rfloor + \sin x$. What is this $\lfloor x \rfloor$? It is the largest integer less than or equal to x. So, it will assume the value 0 for $x \in [0, 1)$. From 1 to 2 it will be 1. But where is $\pi/2$? It is less than 2. So, there we stop. It will not go even up to 2. It assumes the value 1 when $x \in [1, \pi/2)$. And for $x \geq \pi/2$, it is 3.

So, this is given in three different domains differently. We had discussed some results like this that if A and B are sets with $A \cap B = \varnothing$, f and g are continuous functions, then you can define like this, but we give a caveat there that it is not for all sets, it is for open intervals. But here they are not open. One of them is closed also, and there is a point included inside the interval.

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Exercise 5

Results on continuity - Part 2

Discuss the continuity of $f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 1 + \lfloor x \rfloor + \sin x & \text{if } 0 \leq x \leq \pi/2 \\ 3 & \text{if } x \geq \pi/2. \end{cases}$

Recall: Theorem

Let I and J be intervals such that $I \cap J = \emptyset$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be continuous functions. Define $h : I \cup J \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in I \\ g(x) & \text{if } x \in J. \end{cases}$$

Then, h is continuous at all $x \in I \cup J$ except possibly at the common end-points of I and J.

Ans: For $x < 0$, $f(x) = 1 + x$, which is a continuous function. At $x = 0$,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (1+x) = 1, \ \lim_{x\to 0+} f(x) = \lim_{x\to 0+} (1+\lfloor x \rfloor+\sin x) = 1, \ f(0) = 1.$$

So, what happens we need a modification it, we may do like this. We have done a problem also similar to this, which says that "Let I and J be intervals such that $I \cap J = \emptyset$, and f, g are continuous functions defined on I, J, respectively. Now, you can define the new function $h(x)$ by $h(x) = f(x)$ if $x \in I$, and $h(x) = g(x)$ if $x \in J$. Notice that They are not intersecting, but they can share endpoints. So, h is continuous at all points of $I \cup J$ except possibly for the endpoints which are common to both of them, that is except the common endpoints.

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Exercise 5 Contd.

Results on continuity - Part 2

$$f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 1 + \lfloor x \rfloor + \sin x & \text{if } 0 \leq x \leq \pi/2 \\ 3 & \text{if } x \geq \pi/2. \end{cases}$$

For $0 < x < 1$ and $1 < x < \pi/2, f(x) = 1 + \lfloor x \rfloor + \sin x$ is continuous.

At $x = 1$, $\lim_{x\to 1-} (1 + \lfloor x \rfloor + \sin x) = \lim_{x\to 1-} (1 + 0 + \sin x) = 1 + \sin 1$.

$$\lim_{x\to 1+} (1 + \lfloor x \rfloor + \sin x) = \lim_{x\to 1-} (1 + 1 + \sin x) = 2 + \sin 1 = f(1).$$

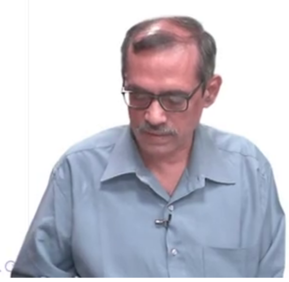
So, $f(x)$ is not continuous at $x = 1$; it is only right-continuous at $x = 1$.

For $x > \pi/2$, $f(x) = 3$. It is continuous. At $x = \pi/2$,

$$\lim_{x\to \pi/2-} f(x) = \lim_{x\to \pi/2-} (1 + \lfloor x \rfloor + \sin x) = 3 = \lim_{x\to \pi/2+} f(x) = f(\pi/2) = 3.$$

So, $f(x)$ is continuous at $x = \pi/2$.

In summary, $f(x)$ is continuous everywhere except at $x = 1$.



Like here, $x < 0$ and $0 \leq x < \pi/2$; so, 0 is an endpoint of the first one and also of the second one. Similarly, $\pi/2$ is an endpoint of the second one and also of the third one. So, except at 0 and at $\pi/2$, everywhere else the function will be continuous provided the individual functions are continuous.

But, in this case this may not be continuous because floor function is discontinuous at the integers. So, you have to discuss that also. And, there is a point in between, which is 1. At 1 you have to discuss, that is, we have to discuss possible discontinuities at 0, at 1 and at $\pi/2$. Everywhere else, they will be continuous.

Let us see. For $x < 0$, $f(x) = 1 + x$; it is a continuous function. At 0, we find their limits. The limit of $f(x)$ as $x \to 0-$ is 1, and the limit of $f(x)$ as $x \to 0+$ is also 1. Is it so? When you take $x \to 0+$, this one is not applicable, the second one is applicable. So you have to find it out, not immediately. For $x \to 0+$, $f(x) = 1 + \lfloor x \rfloor + \sin x$. As $x \to 0+$, $\sin x$ gives 0, $\lfloor x \rfloor$ gives 0, so $f(x)$ goes to 1. And what is $f(0)$? For $f(0)$ again, the first one is not applicable, the second one is applicable; where $f(0) = 1$. Therefore, it is continuous at 0. That is fine?

Let us see what happens for the other points, that is, intervals and also the points. For $0 < x < 1$ and $1 < x < \pi/2$, this is continuous because it invloves $\lfloor x \rfloor$, which is discontinuous only at integers, everywhere else it is continuous. So, you have to think about 1. At $x = 1$, let us take the left side limit. It gives $1 + \lfloor x \rfloor + \sin x$, which is $1 + \sin 1$. And, if x goes to 1+, $\lfloor x \rfloor$ gives the floor of 'something bigger than 1', which gives the value 1. That is what it is. Then, $\sin x$ gives $\sin 1$. So, the limit becomes $2 + \sin 1$. Of course, this one is applicable for f at 1, and that is equal to $f(1)$. Now, both the limits exist; one of them is equal to functional value, the other is not. Therefore, the limit does not exist at $x = 1$, and the function is not continuous at $x = 1$.

It is only 'right continuous' as we say. Anyway, we have not introduced this notion. We have to think about the point $\pi/2$. For $x > \pi/2$, it is a constant function, it is continuous. And at $\pi/2$, we take the left side limit. As $x \to \pi/2-$, we get $1 + \lfloor \pi/2 \rfloor + \sin(\pi/2)$; it gives 3. And if you take the limit as $x \to \pi/2+$, the third one applicable, that is, directly 3. Also, $f(\pi/2) = 3$. Hence, $f(x)$ is continuous at $\pi/2$. Fine?

So, what do you see? It is continuous everywhere except at $x = 1$. That is what we see. In such problems we have to really go in detail to find out what happens at those possible break points. With this, let us stop today.