Basic Calculus - 1
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Lecture 11 - Part 2
Algebra of Continuous Functions - Part 2

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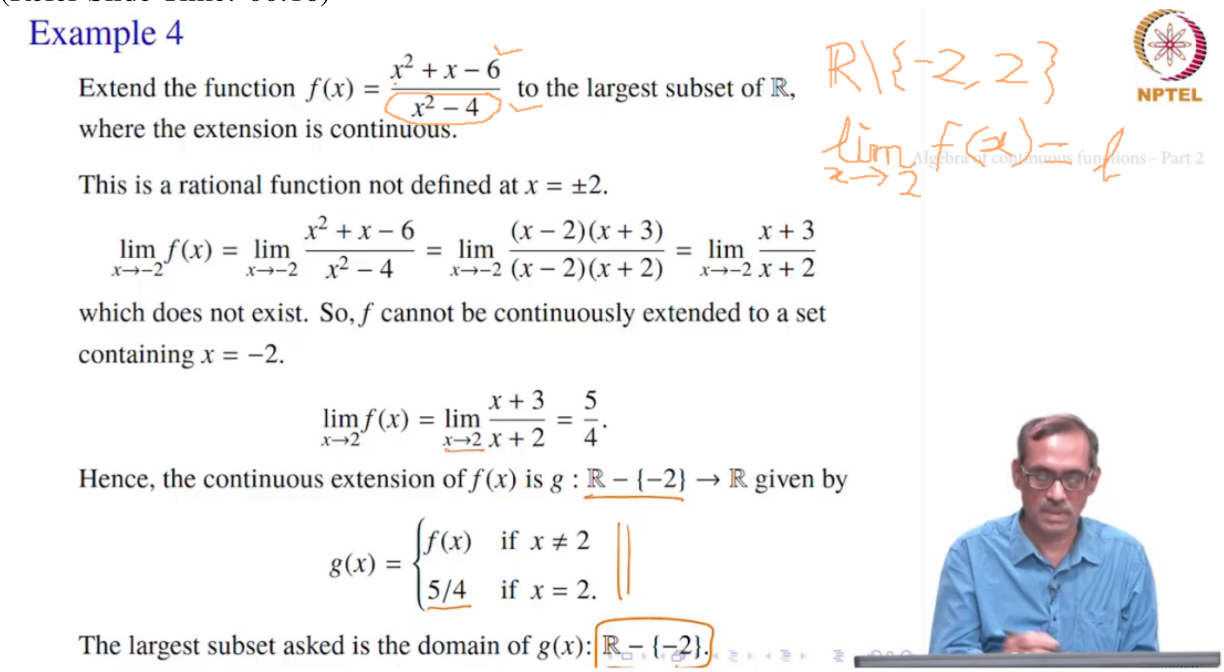

Let us take another example. We want to extend this function $f(x) = (x^2 + x - 6)/(x^2 - 4)$. Why do you want to extend? It says you extend the function so that the extension is continuous. And where do you extend? The largest possible subset of $\mathbb{R}$. It is something like: "find the domain of the extended one which will be continuous". So, how do we go about it?

First, let us see where the function is well defined. It is a rational function. It will be defined provided its denominator is nonzero. The denominator is $x^2 - 4$, which is 0, when $x = \pm 2$. Except at ± 2, this function is well defined. So its domain is $\mathbb{R} - \{-2, 2\}$. This is the domain now. In this domain is it a continuous function? Yes it is, because the numerator is a polynomial function, denominator is a polynomial function; so, it is a rational function and it is continuous everywhere in its domain. Then when you want to extend it, that means, you want to define the function at -2 and at 2, whichever is possible (or both might be possible) so that the extended function, the new one which is f everywhere else and at -2 and at 2 we are going to define, that extension should be continuous.

That means we should have continuity at -2, and we should have continuity at 2, if possible. Then, we must know the limits of $f(x)$ at ± 2. If the limit of $f(x)$ at 2 exists and is equal to is ℓ, then we would like to define our $g(x) = f(x)$ everywhere except at ± 2, and $g(2) = \ell$. In this case, that g will become continuous at 2. Similar thing might be required for -2 also. So, you must first determine whether this limit at ± 2 exist and what are those limits. This is what we require for the extension.

Let us see what we can do. This is a rational function not defined at $x = \pm 2$, and we want to find its limit. First, consider the limit at -2. The limit of $f(x)$ as x goes to -2 will be the limit of $(x^2+x-6)/(x^2-4)$ as x goes to -2. Now, you can see that we have a factorization. The numerator can be written as $(x-2)(x+3)$ and the denominator is $(x-2)(x+2)$. As x goes to -2, it is not equal to -2, (right, in the limit), $x+2$ is nonzero. And, $x-2$ is a factor which we can cancel because $x-2$ is nonzero for x in a neighborhood of -2. So, we just cancel it and we get the limit of $(x+3)/(x+2)$ as x goes to -2.

Now, what about this limit of $(x+3)/(x+2)$ as x goes to -2? We see that as x goes to -2, the numerator remains near $-2+3 = 1$, but the denominator is really near 0. So, this limit does not exist because it becomes very large. That is what it says. Where is it large? Say, x is to the right of -2; $x > -2$. Then, $x+2 = x-(-2) > 0$ and then $(x+3)/(x+2)$ will go to infinity. This will be increasing to ∞. If $x < -2$, then $x+2 < 0$, and this will blow up to $-\infty$. It will go on reducing to (on the negative side) $-\infty$. So, the limit of $(x+3)/(x+2)$ as x goes to $-2-$ is $-\infty$, and the limit of the same as x goes to $-2+$, is ∞. So, the limit does not exist.

We remark that since the left side limit is $-\infty$, and the right side limit is ∞, we cannot say that the limit is either of ∞ or $-\infty$. But even if we are able to conclude that (in some case) the limit s definitely one of ∞ or $-\infty$, we cannot say that the limit exists, because ∞ or $-\infty$ are not real numbers. Anyway limit does not exist here.

Since the limit does not exist, f cannot be continuously extended to include the point -2. Because we needed the limit to exist, and then we could redefine $f(-2)$ as that limit. As the limit does not exist, we cannot continuously extend it to this point $x = -2$.

Now, we should think about the point $x = 2$. What happens for $x = 2$? For the limit of $f(x)$ as x goes to 2, $x \neq 2$. So, we can cancel this $x+2$ again. After canceling, you would get the same thing as $(x+3)/(x+2)$. Now, as $x \to 2$, $x+3$ remains near 5, and $x+2$ remains near 4. So, $f(x)$ remains near $5/4$. Since this limit exists and it is equal to $5/4$, we define the new value of $f(x)$ at $x = 2$ as $5/4$.

Of course, we cannot include -2 in our domain. Our domain which will be $\mathbb{R} - \{-2\}$, every real number except -2. On that domain, we define the new function as $f(x)$ when $x \neq 2$, and at $x = 2$, we define its value at 2 as this limit $5/4$. This new function g is the extended one obtained from f by including this 2 in our domain of definition of g, so that this becomes continuous now; it is continuous everywhere else, it is continuous at 2 also.

So, this is the largest subset asked, which is the domain of g; it is $\mathbb{R} - \{-2\}$. On this only we can define g. Sometimes, we need to consider this domain and it is a reverse way of, reverse way of solving it.

Let us solve some problems. The first problem asks: "at which points $f(x) = (x+3)/(x^2-4x+3)$ is discontinuous, that is, not continuous?" Obviously, the first thing is if the denominator becomes 0, then the function will not be defined at those points and those will be discontinuous points. But let us see what happens. It is not defined when $x^2 - 4x + 3 = 0$. Can we simplify that? It is a quadratic; so you can factor it.

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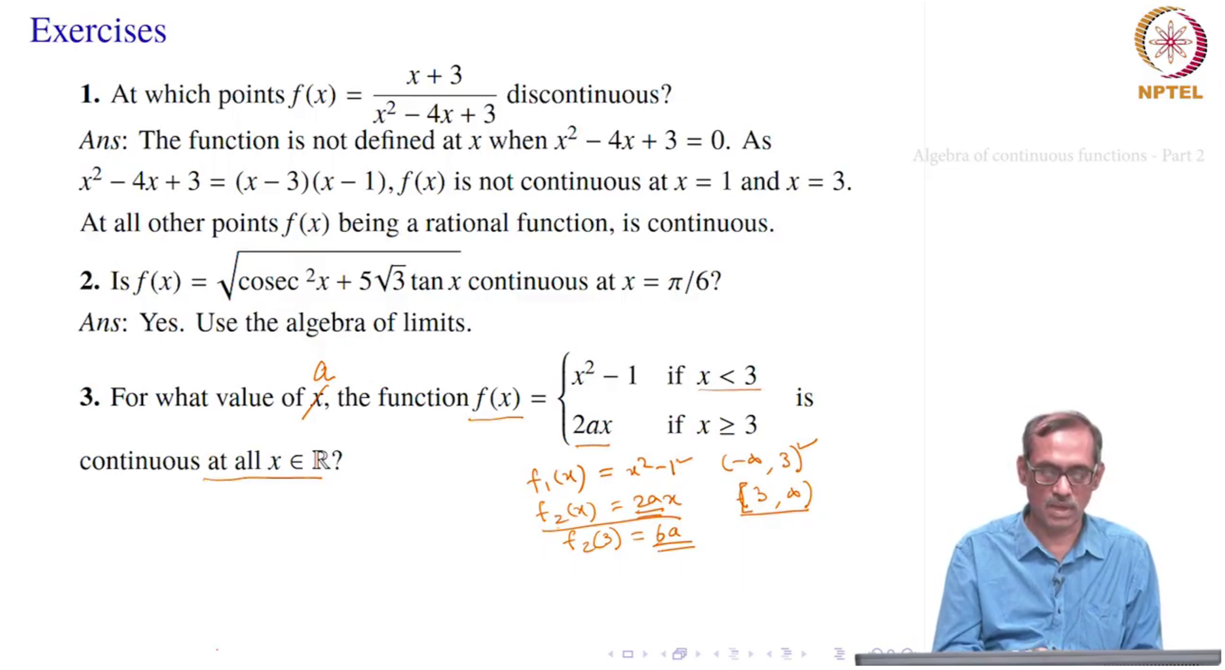

The factorization says $x^2 - 4x + 3 = (x-3)(x-1)$. Just verify. It is $-3 - 1 = -4$ and $(-3)(-1) = 3$; that is correct. Then, at $x = 3$ and at $x = 1$, the denominator becomes 0. Therefore, $f(x)$ is not defined at 3, and it is not defined at 1. The domain of f is the set of all real numbers except 3 and 1. Obviously, at these points it is not continuous. As these are polynomials, it is a rational function; it is continuous everywhere else; that is, wherever it is defined. That is how we get $f(x)$ is not continuous at $x = 1$, and at $x = 3$. Everywhere else it is continuous.

Let us see the second problem. It gives a function $f(x) = \sqrt{\operatorname{cosec}^2 x + 5\sqrt{3}\tan x}$. The question is, "is it continuous at $x = \pi/6$?" How do we go about it? The function $\operatorname{cosec} x$ is not defined at $x = 0$ or wherever $\sin x$ is 0, at all multiples of π; $\tan x$ is not defined at odd multiples of $\pi/2$. As $\pi/6$ is neither of these, $f(x)$ is well defined at $\pi/6$.

Now, $\tan x$ is continuous, $\operatorname{cosec} x$ is also continuous, $5\sqrt{3}$ times $\tan x$ is continuous, $\operatorname{cosec}^2 x$ is also continuous because it is $\operatorname{cosec} x$ into $\operatorname{cosec} x$; then their sum is continuous; and it is square root, and it is composition of taking square root of that; that also should be continuous, just to use the algebra of limits. This is how we use. It is the composition of the square root function with addition of two functions. Those two functions are multiplication of $\operatorname{cosec} x$ with $\operatorname{cosec} x$; another is a constant times $\tan x$. Therefore, it is continuous at $x = \pi/6$.

Let us see the third problem. For what value or values of x, the function $f(x)$ as given is continuous at all $x \in \mathbb{R}$? Now, there is some problem in the formulation, because we are asking for all $x \in \mathbb{R}$. We correct it and ask: "for what values of a", because a is involved here and a is not specified. So, the function is defined as $f(x)$ is equal to $x^2 - 1$ if $x < 3$, and $f(x)$ is equal to $2ax$ if $x \geq 3$; fine?

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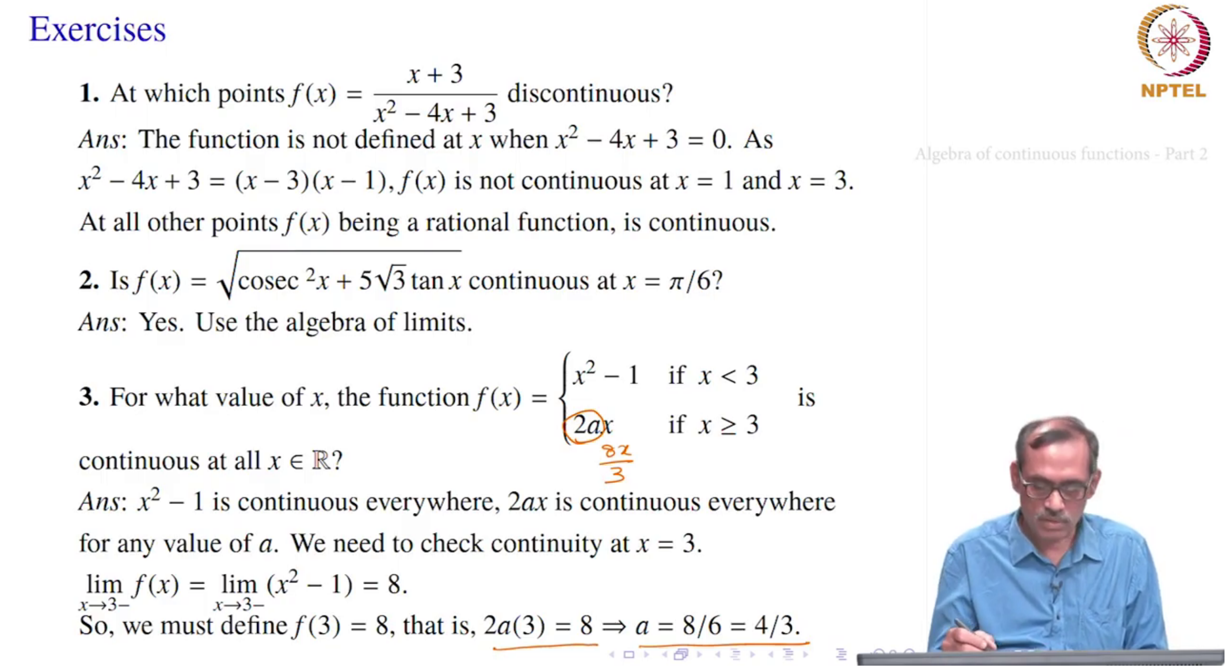

Our aim is to show that $f(x)$ will be continuous, but we do not know this unknown a. This unknown a can be defined so that it will become continuous at all x. What is this a?

You may think of $f(x)$ is defined as $f_1(x) = x^2 - 1$ given on the interval $x < 3$, that is, on the interval $(-\infty, 3)$. And, you may think of $f_2(x) = 2ax$ defined on the interval $[3, \infty)$. These two intervals do not have any point in common, their intersection is empty. And, $f_1(x)$ is continuous. If a is any real number, then $f_2(x)$ is also continuous, because it is the identity function multiplied by a constant. So, if you join them together this way as we have done in our Theorem 2, this will also be continuous, except perhaps at the common endpoint $x = 3$. Notice that 3 is the endpoint of both the intervals. So, at that point only, we do not know whether it will be continuous or not continuous. It may require some more study.

We must find out the left side limit, the right side limit and the functional value of $f(x)$ at $x = 3$. For the left side limit at 3, we consider all $x < 3$ so that $f(x) = f_1(x)$; for the right side limit at 3, we consider all points $x > 3$ so that $f(x) = f_2(x)$, and $f(3) = f_2(3)$. Now, the left side limit of $f(x)$ at 3 is the limit of $f(x) = f_1(x) = x^2 - 1$ as x goes to $3-$. This is $3^2 - 1 = 8$. The right side limit of $f(x)$ at 3 is the limit of $f(x) = f_2(x) = 2ax$ as x goes to $3+$, which is $2a \times 3 = 6a$. The functional value $f(3) = f_2(3) = 2a \times 3 = 6a$. Since continuity demands that all these three are same, we have $8 = 6a = 6a$. It gives $a = 8/6 = 4/3$. With this value of a, the function $f(a)$ is continuous at $x = 3$. At every other point it is already continuous.

That is how we are going to fix the value of a. So, the new function will have $f_2(x) = 2 \times (4/3)x = (8/3)x$. Then this function $f(x)$, which is equal to $x^2 - 1$ for $x > 3$, and $(8/3)x$ for $x \geq 3$, will become continuous. That is how we are going to solve the problems basing on the algebra of limits. Let us stop here.