Basic Calculus - 1
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Lecture 11 - Part 1
Algebra of Continuous Functions - Part 1

Well, this is lecture 11 of Basic Calculus 1. Let us recall what we had done in the last two lectures. We introduced the notion of continuous functions. We say that a function is continuous at a point c, provided, there is a neighborhood around c, which is contained in the domain of the function and limit of the function at $x = c$ is equal to the functional value at c.

Of course, we had the $\epsilon - \delta$ definition; but it comes to this because we had already discussed the notion of limit. Today we will go on discussing the same thing along with some properties of the continuous functions. Continuous functions are those which are continuous at every point of its domain of definition.

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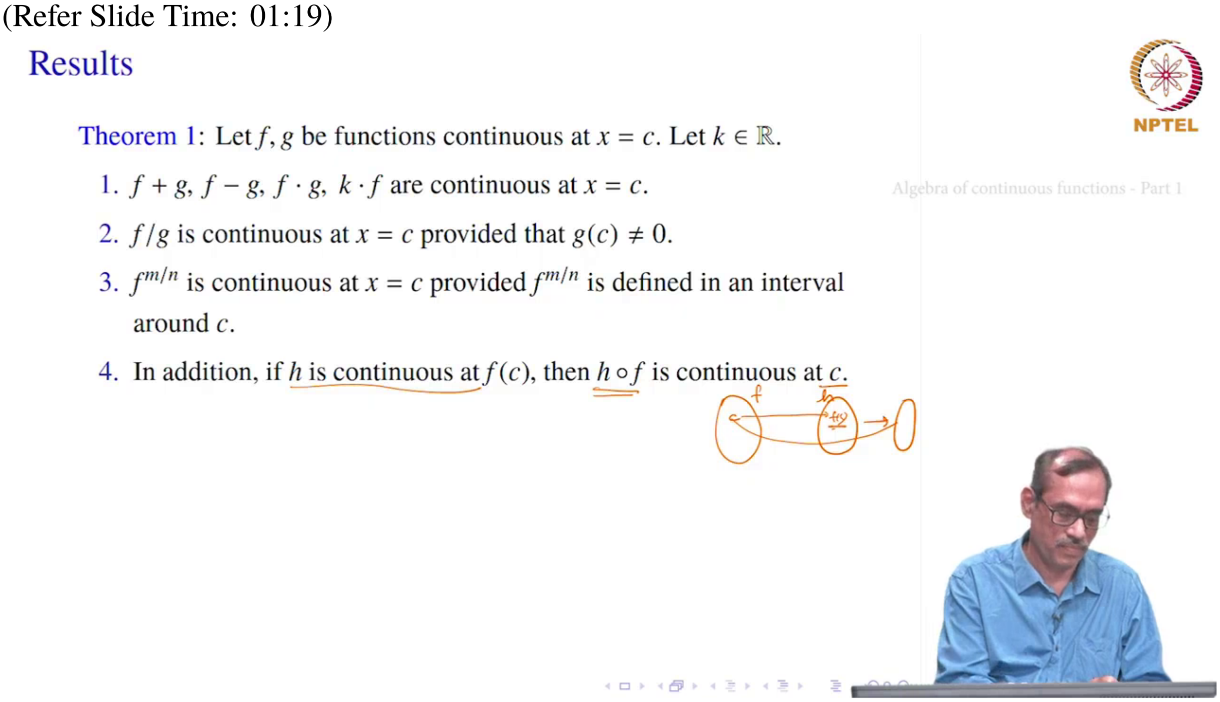

The algebra of continuous function is automatically brought from the algebra of limits. Let us see. Suppose we have two functions f and g defined on some subset of $\mathbb{R}$. In fact, we will take some interval here to make it simpler. But later, we will see that we can go to union of intervals in a similar manner.

We have continuity at $x = c$ or continuous functions in general. We will make use of the $\epsilon - \delta$ definition. It means, for each $\epsilon > 0$, we should be able to find one $\delta > 0$ so that whenever a point x is in your domain, and in the neighborhood $(c - \delta, c + \delta)$, (so it is the intersection of those), then the difference between the corresponding values of x, that is, between $f(x)$ and $f(c)$ should be less than ϵ. That is how we have defined continuity at c.

When it is an interval, we put the added condition that $(c-\delta, c+\delta)$, that is, some some neighborhood of c should be contained inside our interval. We will be going slowly to a union of intervals, and that will be very straightforward. First, let us see for an interval.

Suppose f and g are continuous functions defined on subsets, which you may assume to be intervals now. Once they are continuous at $x = c$, the point c must be a point inside the domain of definition of f so that $f(c)$ is well defined. Now, suppose k is any real number. Then what we get is, the new function kf is also continuous at $x = c$. That follows, of course, directly from the limit. Because, you know that $\lim_{x\to c}(kf)(x) = k \lim_{x\to c} f(x)$. Therefore, $k f$ is also continuous at $x = c$.

Similarly, all the others such as $f+g$, $f-g$ and fg are also continuous. Recall their definitions. $f+g$ is a new function whose value at x is equal to $f(x)+g(x)$, which is the value of f at x plus the value of g at x. That is how $f+g$ was defined. Similarly, $f-g$ and fg have been defined. These are also continuous at $x = c$. They come from the algebra of limits. And also, though we have separated this case, it also goes along with that. That is, the ratio f/g is continuous at $x = c$. Of course, we have a condition here that it should be well defined. So, $g(c)$ should not be equal to 0. And, if f/g is well defined, that is, in a neighborhood of c, $g(x) \neq 0$. That is also required for this limit definition. With those conditions, we will see that f/g is also continuous at $x = c$.

Similarly, if you take some rational power of f, $f^{m/n}$, its value at x is defined as $[f(x)]^{m/n}$. Now, this is continuous at $x = c$ provided, $[f(x)]^{m/n}$ is defined in an interval around c, in a neighborhood of c. We put a condition that it should be well defined because all these powers are not well defined, right? For instance, $(-1)^1/2$ is not defined. With that constraint, $f^{m/n}$ should be continuous. These are pretty straightforward from the limit since similar things hold for the limit.

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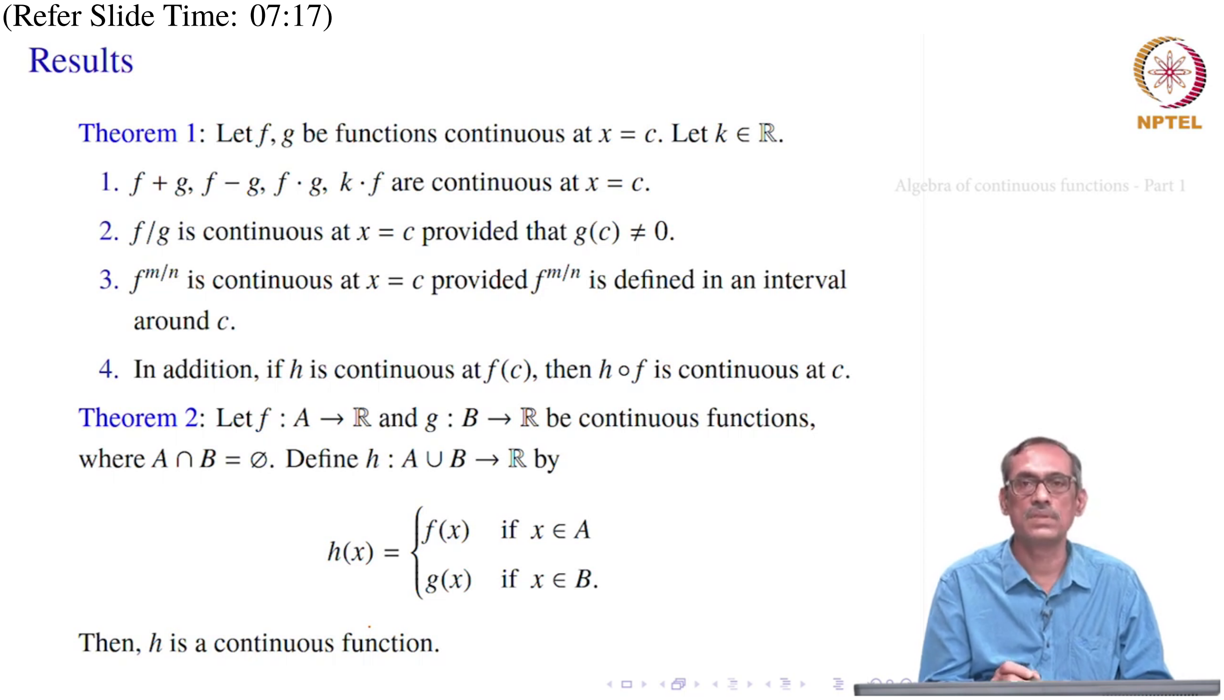

We also have the composition of functions. Suppose f is a function continuous at $x = c$ and we have another function h. This is something like your f is defined from a set and there is another set where h is defined, right? This h is defined on a set. Now, c is taken to $f(c)$ by f. We know that $f(x)$ is continuous at $x = c$. Now, we assume that h is a new function, which is defined from the range of f, at least which contains a set which contains the range of f. Suppose h is continuous at $f(c)$. Then, their composition $h \circ f$, which is taken as g has the value $g(c) = g(f(c))$ at $x = c$. We will say that g is continuous at $x = c$, because $h \circ f$ is defined on the first set, when the conditions of neighborhoods should be satisfied that there is a neighborhood around $f(c)$, there is a neighborhood around c and so on. These conditions should be satisfied. And then, we would say that $g = h \circ f$ is continuous. Informally, we say that the composition of two continuous functions is continuous.

We will go to the next result. It says something about continuous functions on union of intervals. Earlier we have defined continuity of functions which are given on intervals. When a function is defined on a subset of $\mathbb{R}$, not necessarily an interval, we define through the condition that x must belongs to the intersection of $(c - \delta, c + \delta)$ and that set. For all those x, $|f(x) - f(c)|$ must be less than ϵ. We then restricted to intervals. Suppose f is defined from some interval, say A, to $\mathbb{R}$ and g is defined from some interval, say, B to $\mathbb{R}$, where these two intervals do not intersect. It is something like $f : [-1, 1] \to \mathbb{R}$, $g : [2, 5) \to \mathbb{R}$, or even, $g : (1, 5] \to \mathbb{R}$. Now, the union of these intervals is the new domain of definition. We are defining another function on this new domain, that is, on $A \cup B$. That is why we require that $A \cap B = \emptyset$; otherwise, there will be confusion as to whether the new function will be taken via f or via g. There will be a problem.

So, we assume that $A \cap B = \emptyset$. Using this condition we define the new function h on $A \cup B$ taking $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$. It is something like join of two functions. Now, if f and g are both continuous, then we would see that h is also continuous, except perhaps at a point which is an endpoint of both A and B. Because anyway, whenever $x \in A \cup B$, and x is not an endpoint of both A and B, then it is in exactly one of them. So, that function really takes over for your inequalities to be satisfied in the definition of continuity. So, this h also will be continuous. We will require this sometimes when you define a fnction conditionally. Basing on these properties, we will try to solve some problems; but first some examples.

Let us take the first one; it is the easiest. You have a polynomial function $f(x) = a_0 + a_1 x + \cdots a_n x^n$, where $a_0, a_1, \ldots, a_n$ are fixed real numbers. And n also a fixed natural number. So, you have a polynomial $f(x)$ of degree n. Once you say its degree is n, $a_n \neq 0$; that is assumed. Such a polynomial is a continuous function.

Why is it continuous? Because $f_1(x) = a_0$, the constant function is continuous, $f_2(x) = x$ is continuous, $f_3(x) = x^2$ is continuous, and so on, $f_{n+1}(x) = x^n$ is also continuous. These are all continuous functions. Multiplication by constants give continuous functions; then addition gives a continuous function.

Now, if you start with any polynomial, this is of course defined on the whole of $\mathbb{R}$. Its domain is the whole of $\mathbb{R}$. So, it is continuous on $\mathbb{R}$. We say that $f(x)$ is continuous or $f(x)$ is continuous on $\mathbb{R}$. And it follows from the algebra of limits (our Theorem 1).

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Examples

Algebra of continuous functions - Part 1

1. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree n.

Then $f(x)$ is continuous on $\mathbb{R}$.

2. The function $f(x) = \dfrac{\sin x}{x}$ is not continuous at $x = 0$ since it is not defined at $x = 0$.

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

However, the function $g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

is continuous everywhere.

Well, let us consider the other one. Next one has the function as $f(x) = (\sin x)/x$. For this function, the question itself says that it is not continuous at $x = 0$. Why? The reason is it is not defined at $x = 0$. Once you take $x = 0$, its denominator becomes 0, and numerator also 0, but that does not matter here. So, this is not defined at $x = 0$; and we cannot talk of continuity at $x = 0$.

In fact, it should be continuous means it should be well defined at that point and the limit must be equal to the functional value. It is not defined at all; so, it is not continuous. But then we can define another related function, because we know that the limit of $(\sin x)/x$ as x goes to 0 is 1, both sides, whether $x \to 0+$ or $x \to 0-$, the limit as x goes to 0 of $(\sin x)/x$ is equal to 1. That is, if you define another function $g(x)$, which is $(\sin x)/x$ for $x \neq 0$, and $g(0) = 1$. Now, this function is anyway continuous at $x = 0$, because the limit is equal to 1 and that is the functional value. At every other point, $(\sin x)/x$ is well defined, $\sin x$ is a continuous function, and x is a continuous function, so, their ratio is also continuous. Therefore, this function $g(x)$ is continuous everywhere. It is defined on the whole of $\mathbb{R}$ and it is continuous. Sometimes we say continuous everywhere to emphasize that 'yes it is continuous'.

Let us take another example. We have the function given as $f(x) = |(x \sin x)/(x^2 + 2)|$. We want to decide whether it is continuous or not. Well, first we will look at this function and see how it has been formed by using plus, minus, multiplication, division, or by composition, because we know continuity of the simpler ones. That is, we know that $f(x) = x$ is continuous, $f(x) = \sin x$ is continuous, and $f(x) = x^2 + 2$ is continuous.

The first thing we ask whether this function is well defined everywhere. If you take any real number x, whether this expression makes sense. Yes, it is, because the denominator can never be equal to 0 as x^2 is always greater than or equal to 0. Of course, $x^2 + 2 \geq 0$. Now, it is not only greater than or equal to 0; $x^2 \geq 0$, so $x^2 + 2 \geq 2$. So, $(x \sin x)/(x^2 + 2)$ is well defined. And you can find out what is f at any particular number c.

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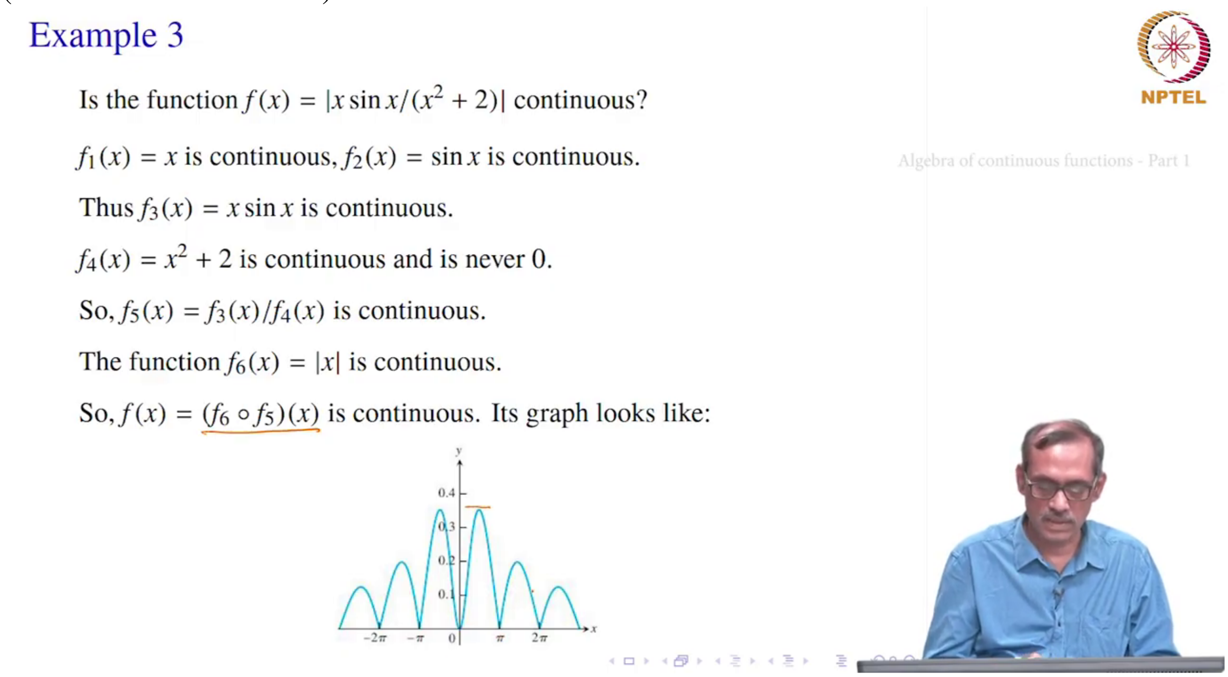

Let us see how to use the algebra of continuous functions to show that it is continuous. It looks that it will be continuous. The first one is $f_1(x) = x$, which is which is the identity function; we know that it is continuous. Another function $f_2(x) = \sin x$; that also we know to be continuous. Then $f_3(x) = f_1(x)f_2(x)$, the product of f_1 and f_2. That is, $x \sin x$ is also continuous due to our theorem. Next, let us look at the denominator: $x^2 + 2$, call it $f_4(x)$. Now, $f_4(x) = x^2 + 2$ is continuous, and it is never 0. It can be smaller or smaller but always it is greater than or equal to 2; it is never 0. So, $f_5(x) = f_3(x)/f_4(x)$ is also continuous. It is well defined, it is continuous. Fine?

But you want the modulus of that. Let us define another function which is just modulus, that is, $f_6(x) = |x|$, absolute value of x. That is also a continuous function. You remember its graph? Now, our function $f(x)$ can be thought of as a composition of f_5 and f_6, right? Which composition? It is absolute value of f_5, that is, f is equal to f_6 of f_5 of x; it is this composition: $f(x) = (f_6 \circ f_5)(x) = f_6(f_5(x))$. And that has to be continuous, due to again Theorem 1. In fact, you can plot the graph. It would look something like this. It never goes beyond this point, which is smaller than 0.4, and it goes in this funny manner, symmetric about y-axis. And you see there is no break for it. But sometimes graphs are illusory, so this analysis would be better, and it looks something like this. It is a continuous function.