Basic Calculus - 1

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Lecture 10 - Part 2

Continuity - Part 2

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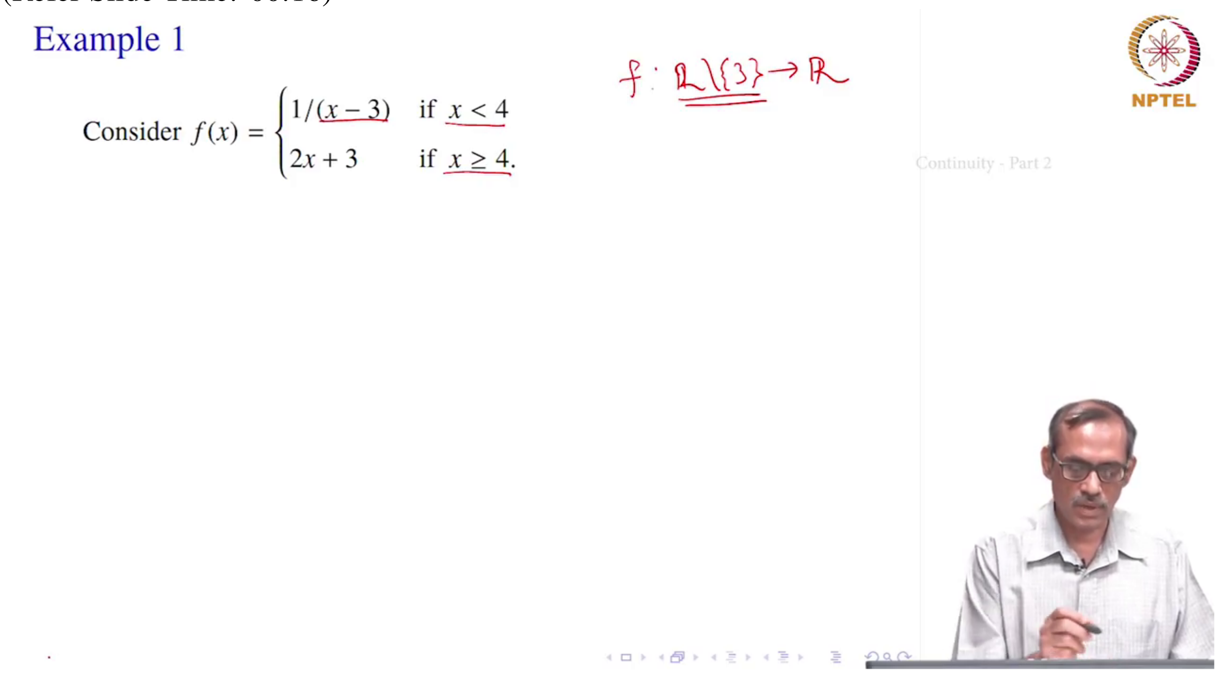

Let us see an example. We have a function given here. It is defined on the whole real line. For $x < 4$, it is given this way; and for $x \geq 4$, it is given this way. Looks like, it is defined on the whole of $\mathbb{R}$. But let us see the definition: $f(x) = 1/(x-3)$ if $x < 4$. It says that at $x = 3$ the function is not defined. So, from the domain of this function you have to exclude 3. Now, for $x \geq 4$, it is $2x+3$; here it is well defined. That means this function is defined from $\mathbb{R} - \{3\}$ to $\mathbb{R}$. The domain of the function is $\mathbb{R} - \{3\}$, that is, excluding the point 3. It is same as $(-\infty, 3) \cup (3, \infty)$.

So, suppose this is our function. Now. you want to see whether it is continuous at all these points or not. Let us take a point c, which is less than 3. It belongs to this interval: $(-\infty, 3)$. That means the first line, the first condition holds; so this definition will be applicable. Notice that 3 is a right endpoint of the interval $(-\infty, 3)$. But there is a right neighborhood, which is inside the other set: $(3, \infty)$. So, on both the sides a deleted neighborhood is there. If you take deleted neighborhood of 3, then that is contained inside this domain. So, not only one sided but also both sided limits are possible. That is, the notion of the limit as $x \to c$ is meaningful here. Not only for $c < 3$ but also at $x = 3$ the limit is meaningful.

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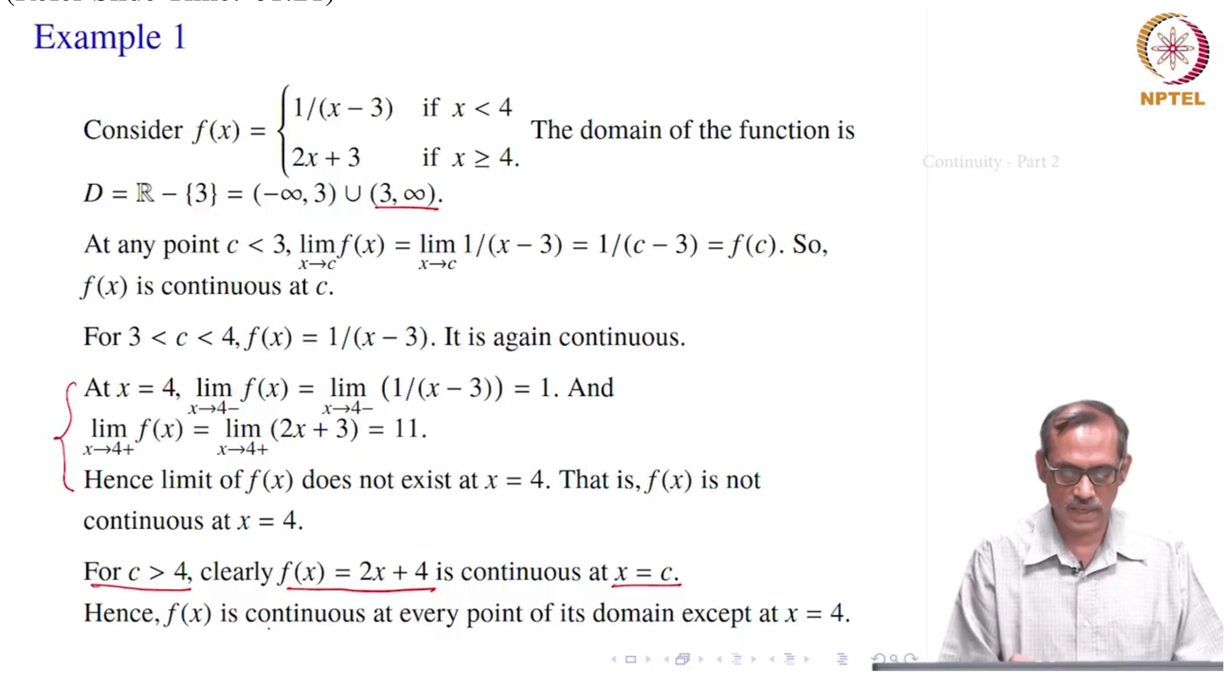

Once you take $c < 3$, the first one is applicable. Now, c is an interior point of the domain; and we consider taking the limit of $f(x)$ as $x \to c$. As first one is applicable, using the algebra of limits, we find that the limit of $1/(x-3)$ as x goes to c is $1/(c-3)$. Notice that $1/(x-3)$ is meaningful since x is near c, $c < 3$ and $x \neq 3$ because x is in the domain of $f(x)$. Now, $f(c) = 1/(c-3)$. Therefore, $f(x)$ is continuous at $x = c$. You see that using the limit is easier than reverting back to epsilon-delta.

Suppose, c is a point between 3 and 4, that is, $3 < c < 4$. Because the definition of the function involves $x < 4$, we are considering this case first. Here again, the first one is applicable; so $f(x) = 1/(x-3)$. What happens for the limit? You see that the limit of $f(x)$ as $x \to c$ is similarly, $1/(c-3)$. But $f(c) = 1/(c-3)$. Therefore, $f(x)$ is continuous at $x = c$.

Now what happens at $c = 3$ or at $c = 4$? As $f(3)$ is not defined, the question of its continuity at 3 is meaningless. We do not consider it. We go to the other possible break point, that is, 4. Here, we have two different definitions of the function. One to the left of 4, where $f(x) = 1/(x-3)$ and the other on the right of 4, which is $f(x) = 2x + 3$.

We consider this special point 4. Suppose we take the point $c = 4$. At this point we cannot go to the limit directly, because on the left it is defined one way, and on the right it is defined another way. So, let us consider the left side limit. We are interested in $x \to 4-$. Here, the limit of $f(x)$ as x goes to $4-$ will be the limit of $1/(x-3)$ as x goes to $4-$. From the limit computation as we did earlier this x will be replaced by 4 (from the algebra of limits), and that should give $1/(4-3) = 1$. So, this left side that limit at 4 is 1.

What about the right sided limit? The limit of $f(x)$ as $x \to 4+$ is the limit of $(2x+3)$ as $x \to 4+$. Again, from the algebra of limits it is $2 \times 4 + 3 = 11$. That means the limit of $f(x)$ as x goes to 4 does not exist. Since the limit does not exist, it is not continuous at $x = 4$.

What about at any point which is bigger than 4? At such a point $f(x) = 2x + 3$. We try to find its limit at any point c, which is an interior point in that interval. Here, both sides limits exist, and the algebra of limit says that it will be equal to $2 \times c + 3 = f(c)$. Therefore, it is continuous at $x = c$.

In summary, what we see here is that $f(x)$ is continuous at every point of its domain except at $x = 4$, where the limit of the function does not exist. Is that clear? I think this an exhaustive example which gives all kinds of possibilities.

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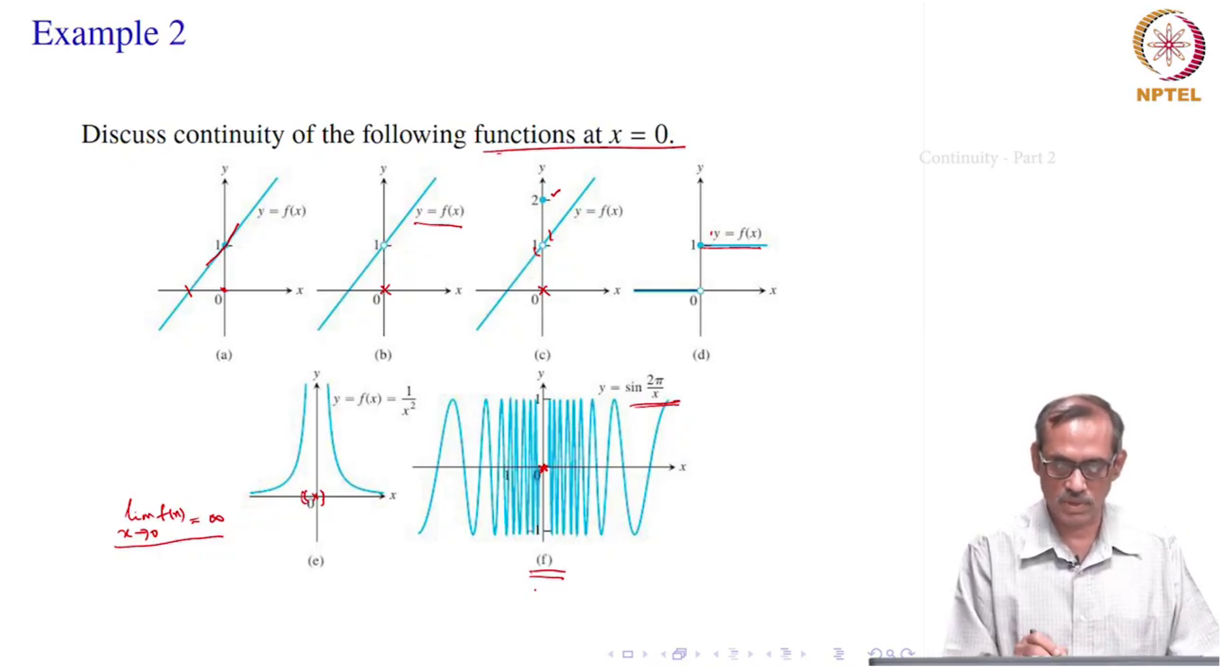

Let us go to the second example. Here, there are various questions and not one. Here are given six functions. We want to find whether all these functions are continuous at $x = 0$. How do we proceed? It looks like a straight line. This distance on the y-axis is 1. This distance on the x-axis looks smaller than 1. Had the distance on the x-axis been 1, it would have been the straight line $y = x + 1$. But it is not. But it should go here; so $y = x+$ something, that something is this distance on the x-axis. Obviously, it is not equal to 1, it is something else.

For this function we want to find out whether it is continuous at $x = 0$. As it looks you do not have to lift your pen while you draw the curve, including that point $x = 0$, where $f(0) = 1$. So, it should be continuous; that is how it looks. We can see that the Left side limit is equal to the right side limit, which is equal to the functional value $f(0)$. So, $f(x)$ is continuous at $x = 0$.

Now we go to the second function. It is same as earlier except at $x = 0$, there is a hollow circle. It means $f(0)$ is not defined. Once at 0 it is not defined, it is not continuous at 0; that is clear. Because we need the functional value, functions should be defined at that point. However, here the limit of $f(x)$ as x goes to 0 exist and it is equal to 1. Both left side and right side limits are 1, but $f(x)$ is not defined at that point. So, it cannot be continuous. So, that is what we write here. We see that in (b), the function is not defined at $x = 0$; so, it is discontinuous at $x = 0$.

Let us look at (c). Here, it is the same function as in (b) except that at the point 0 its value is

2. So, what happens here? The left side limit of the function as x goes to 0 is 1. And right side limit is also 1. So, the limit of $f(x)$ as x goes to 0 is equal to 1. But the limit is not equal to the functional value which is given as 2. So, it is not continuous at 0.

Now what about the fourth one in (d)? Here, when $x < 0$, the function is given as 0 itself, this value is 0. And when $x \geq 0$, its value is given as 1. The function has value 1 at 0. Now you see that the left side limit at $x = 0$ exists, and that is equal to 0. The right side limit at $x = 0$ exists, and that is equal to 1, which is also the functional value at 0. So, the right side limit is equal to the function value, but the left side limit is not equal to the functional value. So, we see that the limit of $f(x)$ as x goes 0, does not exist. Because the left side and right side limits are different. Therefore, the function is discontinuous at $x = 0$.

In the first case, it is continuous; in the second case, it is not continuous because the function value is not given; in the third case, it is not continuous because the function value is different from the limit value; in the fourth case, it is not continuous because the limit does not exist. Let us come to the fifth one.

In the fifth case, at $x = 0$ if you take the left side limit, it does not exist; it blows up to ∞. Let say that the limit is equal to infinity. The right side limit at $x = 0$ is also equal to infinity. So, the limit as x goes to 0 is equal to ∞ here. You say that the limit of $f(x)$ as x goes to 0 is ∞; we do not say here that the limit exists. The limit exist means it should be a real number but it is not. Both ∞ and $-\infty$ are special cases of nonexistence of limit though we write in symbols as $\lim f(x) = \pm\infty$. Notice that in the fifth case, $f(x)$ is not even defined at $x = 0$. It is the function $f(x) = 1/x^2$. The function is discontinuous at $x = 0$. Of course you cannot redefine it to make it continuous. You cannot say that $f(0) = 5$ so that the function would become continuous; because the limit at $x = 0$ is equal to infinity. And we cannot say that $f(0) = \infty$. That will not define a function because ∞ is not a real number. So, we cannot do anything; we cannot redefine at 0 to make it continuous. Because limit is equal to infinity, the function becomes unbounded in a neighborhood of 0; that is what the fifth case is.

Now come back to the picture, and consider the sixth one. Here the function is $f(x) = \sin(2\pi/x)$. Look at its graph. In the neighborhood of 0, you see that there is oscillation of the function. It takes value from -1 to 1; it is taking all these values near 0. As we see, the function is not defined at $x = 0$; so it is discontinuous. But can we redefine it in such a way that it will become continuous? We cannot, because it is oscillating. Though it is bounded unlike the last case, where the function was unbounded in a neighborhood of 0. Here it remains bounded between -1 to 1. However, the limit does not exist and also the function value is not specified at $x = 0$. Therefore it is discontinuous there.

You can see various possibilities of discontinuity here. We write that the function is discontinuous at $x = 0$ though the types of discontinuity are different. In (b) it is not defined at $x = 0$; in (c) the limit is not equal to $f(0)$; in (d) the right hand limit is not equal to the left hand right hand limit at $x = 0$, so the limit does not exist, and in (e), the limit is ∞ as $x \to 0$ and also $f(0)$ is not defined. Further, in (e), we cannot redefine $f(0)$ to make it continuous. In (f), neither the left hand limit nor the right hand limit exists at $x = 0$. This limit does not exist because the $f(x)$ oscillates when x is near 0. So, $f(x)$ is not continuous at $x = 0$. Is that fine?

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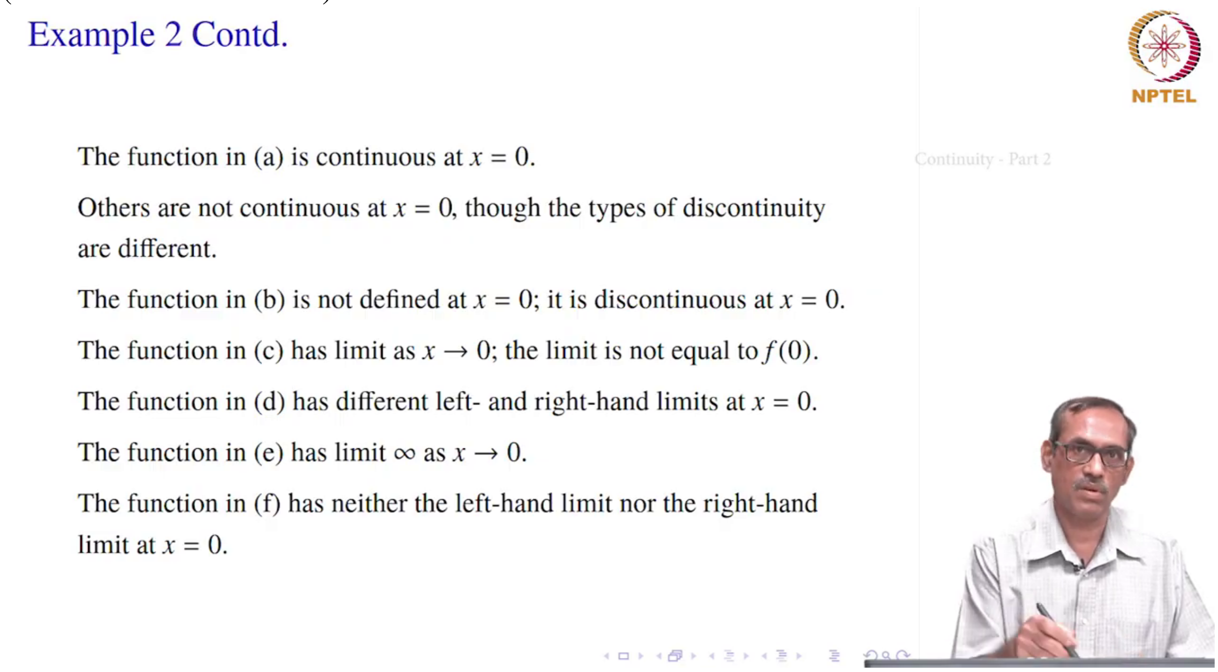

Let us continue solving some problems basing on this notion. We have a function given as $f(x) = 1 - \sqrt{1 - x^2}$. This is defined in the interval $[-1, 0)$, and also the same way when it is in the interval $(0, 1)$. Then, $f(x) = 0$ when x is in the interval $[1, 2]$. So, we have now three kinds of things: $[-1, 0)$, then we have $(0, 1)$, and then $[1, 2]$. All that we have left is f at 0. $f(0)$ is defined separately as 1. That is how it is defined. So, in the interval -1 to 1 except at 0 it is $1 - \sqrt{1 - x^2}$, in the interval 1 to 2 it is the constant 0, and $f(0) = 1$. That is how it looks. We are to find the points where $f(x)$ is discontinuous and if possible, redefine at those points to make it continuous.

It looks that the possible points are $-1, 0, 1$ and 2. Let us see how it goes. Now, $f(x)$ is defined on the closed interval $[-1, 2]$. It starts from -1 and goes to 2. It is constant, the value 0, between 1 to 2. So, it is continuous there. We do not know what happens at 1; everywhere else it is continuous there. Then, it is defined from -1 to 0; there is a possibility of discontinuity at 0; except at 0, it is continuous there. So, the only possibility of discontinuity are 0 and 1. At the end points, of course, we can say that this function is continuous because one side limits exist. So, only 0 and 1 are possibile points of discontinuity. But we have to determine whether they are or they are not exactly. It is just a guess work till now.

Let us take $x = 0$. For the limit of $f(x)$ as $x \longrightarrow 0$, we find that in a neighborhood of 0, $f(x)$ is defined as this $1 - \sqrt{1 - x^2}$. We find its limit at x goes to $0-$. The algebra of limits says that it should be $1 - \sqrt{1 - 0} = 0$. As x goes to $0+$, again the same thing happens; we see that the algebra of limits gives the limit as $1 - \sqrt{1 - 0} = 0$. So, the limit exists and it is equal to 0. What about $f(0)$? It is given to be 1. But the limit is equal to 0. So, that is why it is discontinuous at $x = 0$.

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Exercise 1

Let $f(x) = 1 - \sqrt{1-x^2}$ for $-1 \le x < 0$ and $0 < x < 1$; $f(x) = 0$ for $1 \le x \le 2$ and $f(0) = 1$. Find the points where $f(x)$ is discontinuous; and redefine $f(x)$ at some of the points so that $f(x)$ would be continuous there.

Ans: $f(x)$ is defined on $[-1, 2]$. It is continuous at all points except possibly at $x = 0,\ 1$.

Now, $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (1 - \sqrt{1-x^2}) = 0.$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (1 - \sqrt{1-x^2}) = 0.$

Thus we redefine $f(0)$ to be 0 so that $f(x)$ will be continuous at $x = 0$.

Similarly, $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (1 - \sqrt{1-x^2}) = 1$ and

$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} (0) = 0.$

Thus the limit of $f(x)$ at $x = 1$ does not exist.

We cannot redefine $f(1)$ to make $f(x)$ continuous at $x = 1$.

In this case, we can redefine our function. We will define another function g which is just like f everywhere except at 0. At 0 we give the value $g(0) = 0$, which is its limiting value. Then that would make the new function g to be continuous at 0. This is the meaning of redefining f.

Now, we will discuss at the point 1. At 1 again we go in a similar way. As x goes to $1-$, $f(x) = 1 - \sqrt{1-x^2}$ again. The limit is $1 - \sqrt{1-1} = 1$. When x goes to $1+$, that is from the right hand side, this one is applicable; so $f(x)$ is the constant 0. Then, the limit should be equal to 0. That is, the left side limit at 1 is not equal to the right side limit, though both of them exist. That means, the limit of $f(x)$ as x goes to 1 does not exist.

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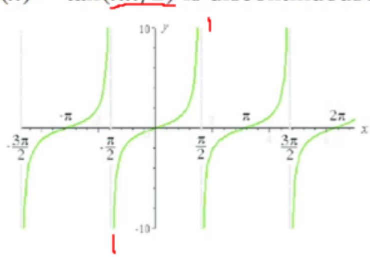
Exercise 2

At which points $f(x) = \tan(\pi x/2)$ is discontinuous?



$$\frac{\pi x}{2} = (2n+1)\frac{\pi}{2}$$

Ans:

$$\lim_{t\to(2n+1)\pi/2} \tan t = \pm\infty.$$

$\tan t$ is continuous everywhere else.

So, when $\pi x/2 = (2n+1)\pi/2$, that is, when $x = 2n+1$, an odd integer, $f(x)$ is discontinuous.

So, whatever way we define f at 1, that is, forgetting this f and giving a new definition at 1 some other way, we cannot make it continuous. Because this limit as such does not exist, that is, the limit of $f(x)$ at $x = 1$ does not exist. So, we cannot redefine $f(x)$ to make it continuous at $x = 1$.

Let us take another problem. At which points $f(x) = \tan(\pi x/2)$ is discontinuous? That should be quicker because we know the behavior of tan. It looks like this, it is $\pi x/2$ instead of $\tan x$. So, there will be a scaling in the independent variable, $\pi/2$ is multiplied with x here. This is how $\tan t$ looks. So, what we see is: when t goes to $(2n + 1)\pi/2$, $\tan t$ goes to $\pm\infty$; that is what we know.

So, at those points it is discontinuous. This conclusion is for the $\tan t$ function. That is $\pi x/2 = (2n + 1)\pi/2$. Whenever this is satisfied, at those points x, we have the discontinuity. One side blows up to ∞ and the other side blows down to $-\infty$. We say that at these points x the function is discontinuous. That gives $x = 2n+1$. These are the only points of discontinuity; everywhere else it will be continuous. That is, when $x = 2n + 1$, an odd integer, $f(x) = tan(\pi x/2)$ is discontinuous. At all the other points this function $\tan(\pi x/2)$ is continuous. We stop here.