Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

Lecture 2

The Real Line - Part 2

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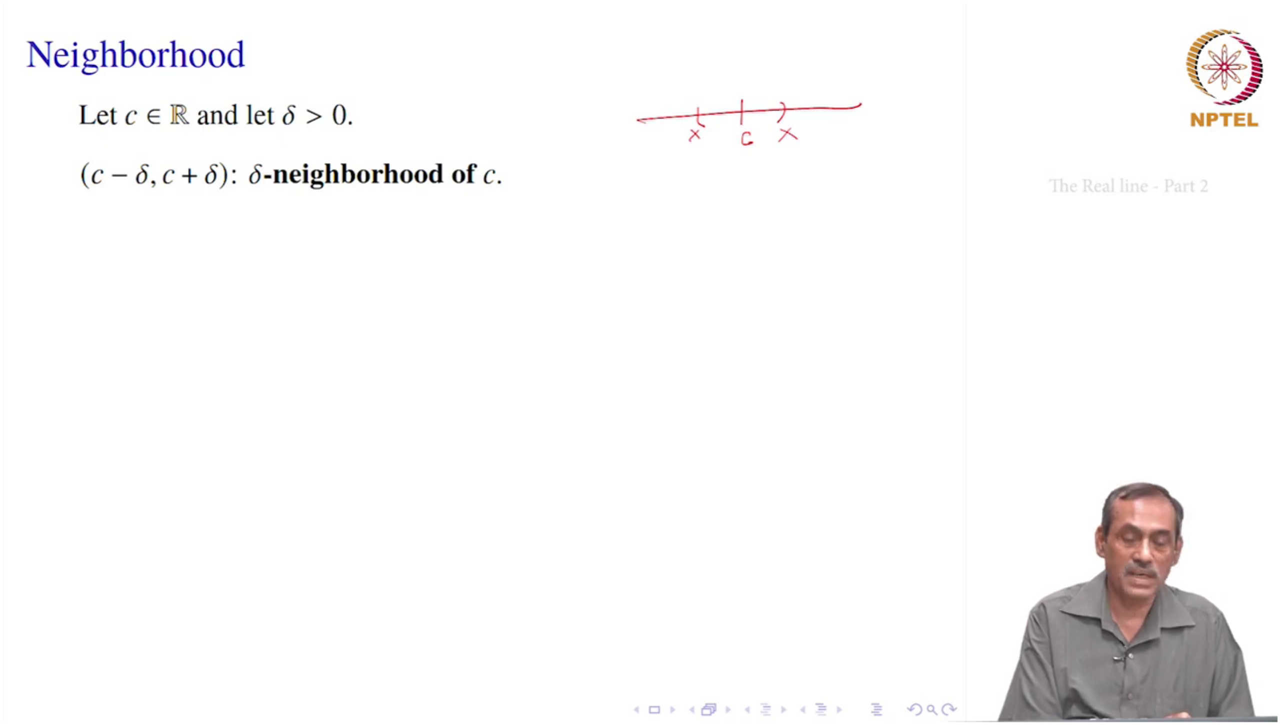

We next discuss the notion of neighborhood. That is very crucial to calculus. The idea of a neighborhood concerns how the points are nearby. Suppose we start with a point c, or an element, a real number c ; and let us take a positive real number which is δ. We are taking two things simultaneously; one real number c and one positive number δ. Then we consider the open interval $(c - \delta, c + \delta)$. Like in the real line we take c, and δ is something; so, $c - \delta$ is somewhere to the left of c and $c + \delta$ is somewhere to the right of c. This says it is open interval: these two points $c - \delta$ and $c + \delta$ are excluded. All the other points inbetween are there. This interval is called the δ-neighborhood of c. It is something like within this 1 kilometer of one's house, it is 1 kilometer neighborhood including anything is in between that. That is why this terminology of δ-neighborhood of c.

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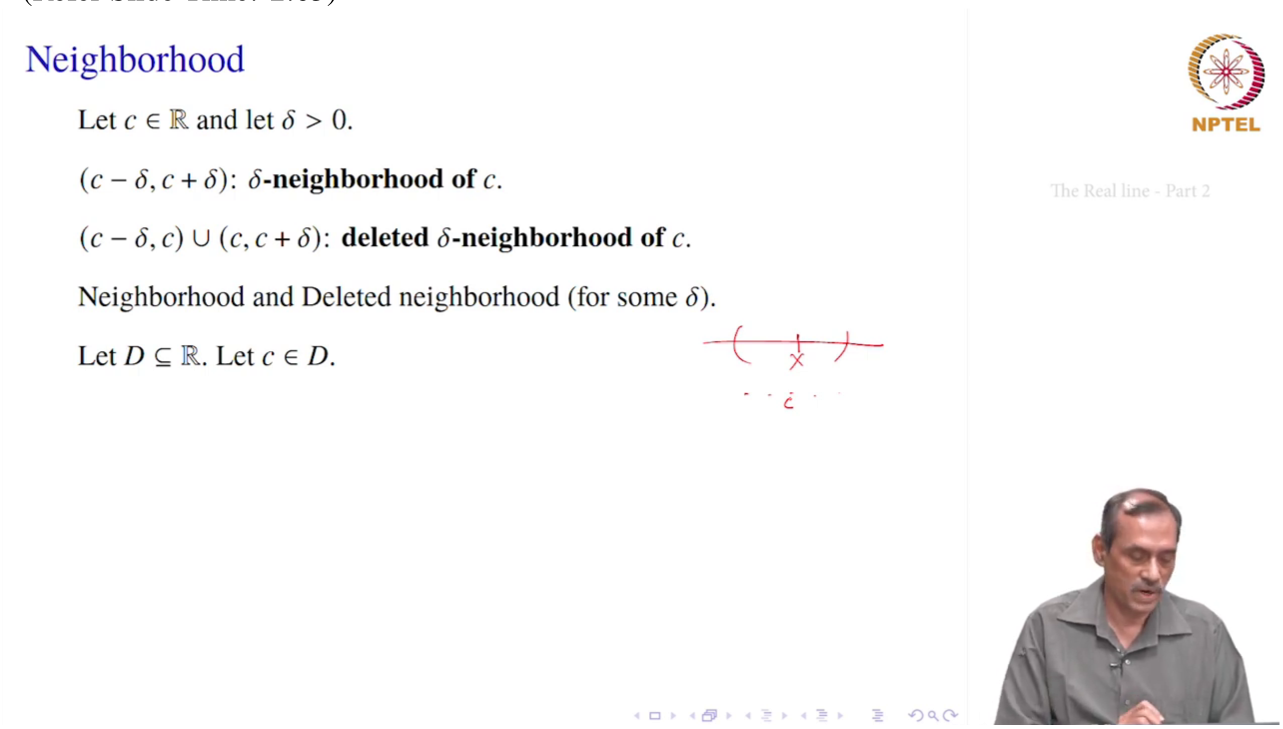

So, the interval $(c - \delta, c + \delta)$ is called as the \$$d$-neighborhood of c. And if you exclude that point c, then we call it the deleted δ-neighborhood of c. It is really the union o two intervals. On the left side are all the points less than c and on the right side are all points greater than c; all those points are in the deleted δ-neighborhood of c .

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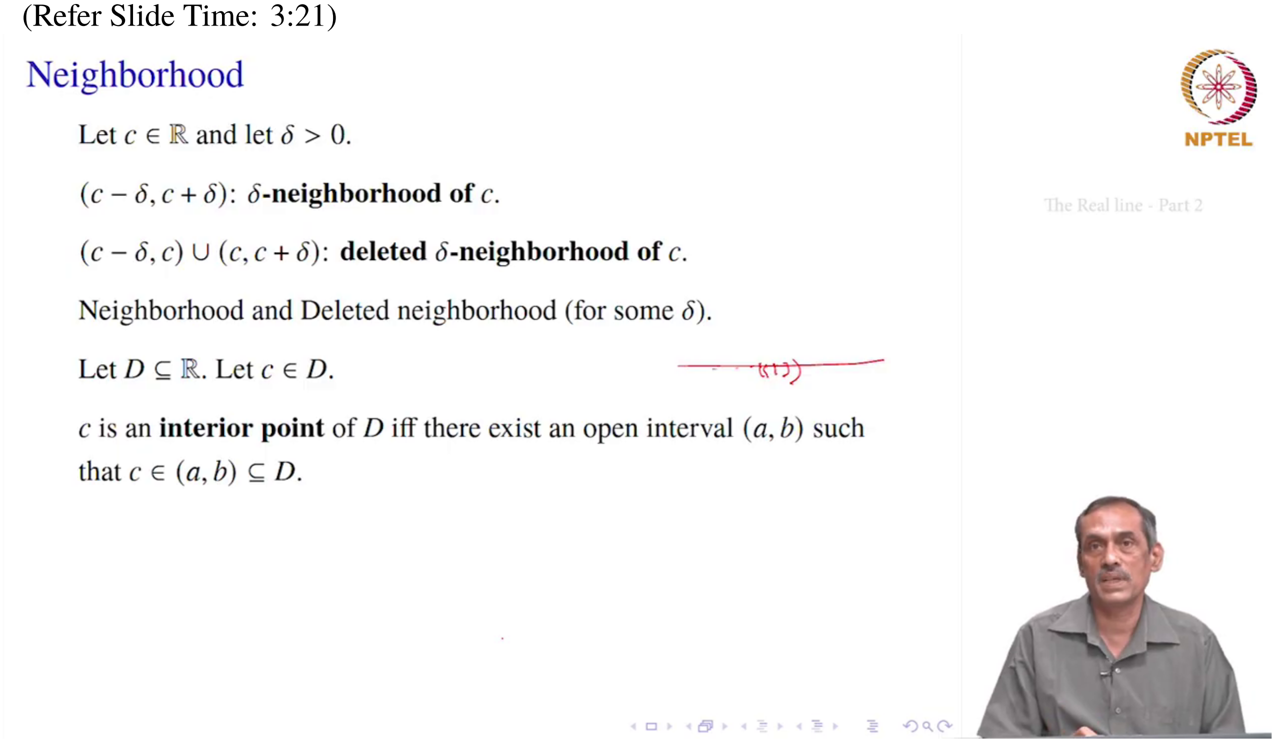

When we say *neighborhood* in abstract, we mean that it is a δ-neighborhood for some δ. When we say a *deleted neighborhood*, you exclude that point c also. That is a deleted neighborhood for some δ. So, this fixes the idea of neighborhood. We are going to use this notion of neighborhood to define something else.

Suppose D is any subset of $\mathbb{R}$. You may take D as an interval or it may not be an interval; it may be some points along the line. That is also possible. Let $c \in D$. So c is inside the set. That means D is non-empty. At least one point is there in D. Then we say that this point c is an interior point of D, if something satisfied. What is it? That there is always an open interval around this c which is contained inside D. That means if c is a point and D is a set, and if you take $(c - \delta, c + \delta)$, where we donot know how big is this δ but some delta exists, so that such a neighborhood is completely contained inside D.

If near everywhere a point is missing from D, then that will not satisfy this constraint. For instance, if all rationals are missing from D, then that will not satisfy this property.

So, we call c is an interior point of D, if there is an open interval around c, or a neighborhood of c that is cpontained in D. We may take something like (a, b) arond c. That is also okay, because inside it again you can find another neighborhood of c which is contained in D. So, if there is an open interval (a, b) such that c is an element of this open interval and that open interval is completely contained inside D, then we say that c is an interior point of D. So, it is the same whether there exists an open interval containing c or a neighborhood of c, namely, $(c - \delta, c + \delta)$ completely contained inside D.

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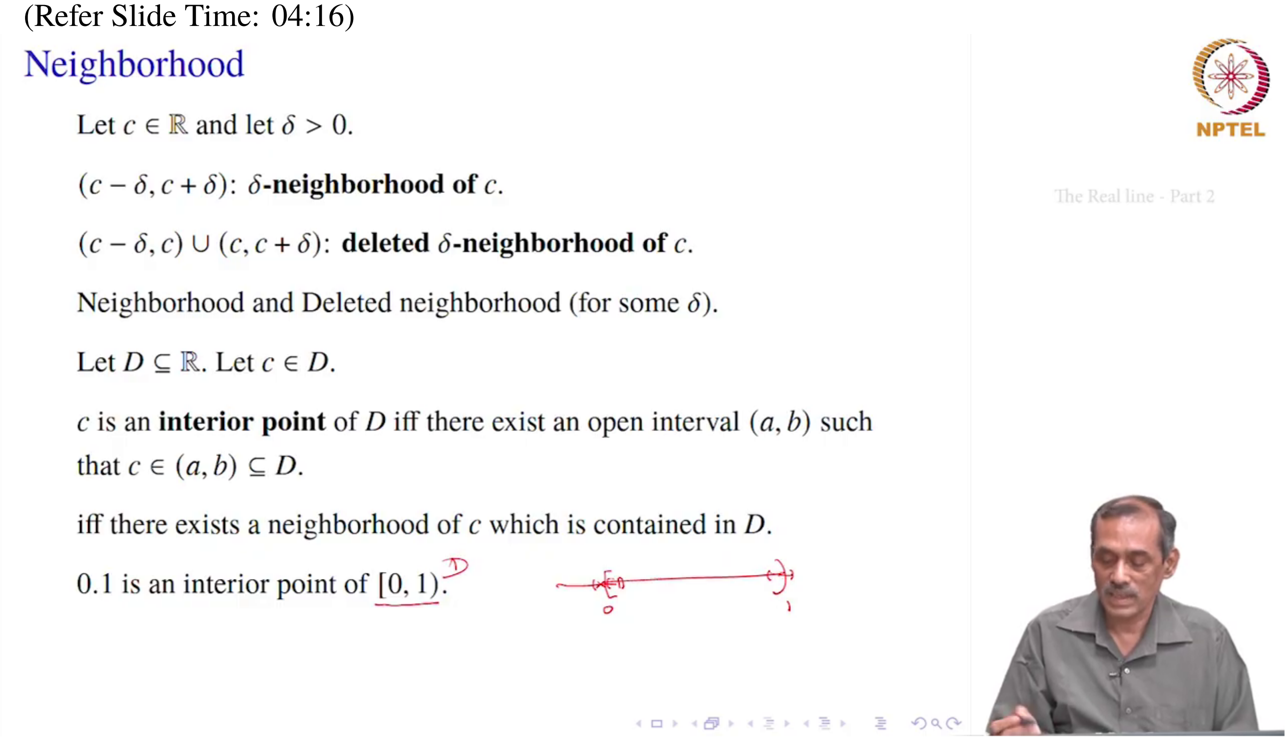

Let us take an example. Say, I take my set D as the interval $[0, 1)$.. That is my D. Then I look at the point 0.1. Obviously 0.1 is there. This is my D, where 1 is excluded and 0 is included. Now, $0.1 \in D$, somewhere here. Also the point is an interior point of D. Why? Because always you can find a smaller neighborhood around that 0.1 or an open interval around that, which is contained inside this interval. We can take of course one δ which is smaller than this 0.1. Then that δ will do.

But 0 is not an interior point. Why? Because once you take any neighborhood of 0, it will contain some negative points, and they are not inside D. So, 0 is not an interior point. But 1 is an interior point. Is it so? So suppose I take 1. Then I take any neighborhood around 1. Is it contained in D? No? Well, it is not. Because it will always contain a point which will be beyond 1. Of course 1 is not inside D. So, that does not satisfy the first property since c must be inside D. So, we will not tell that 1 is an interior point because 1 itself is not in D.

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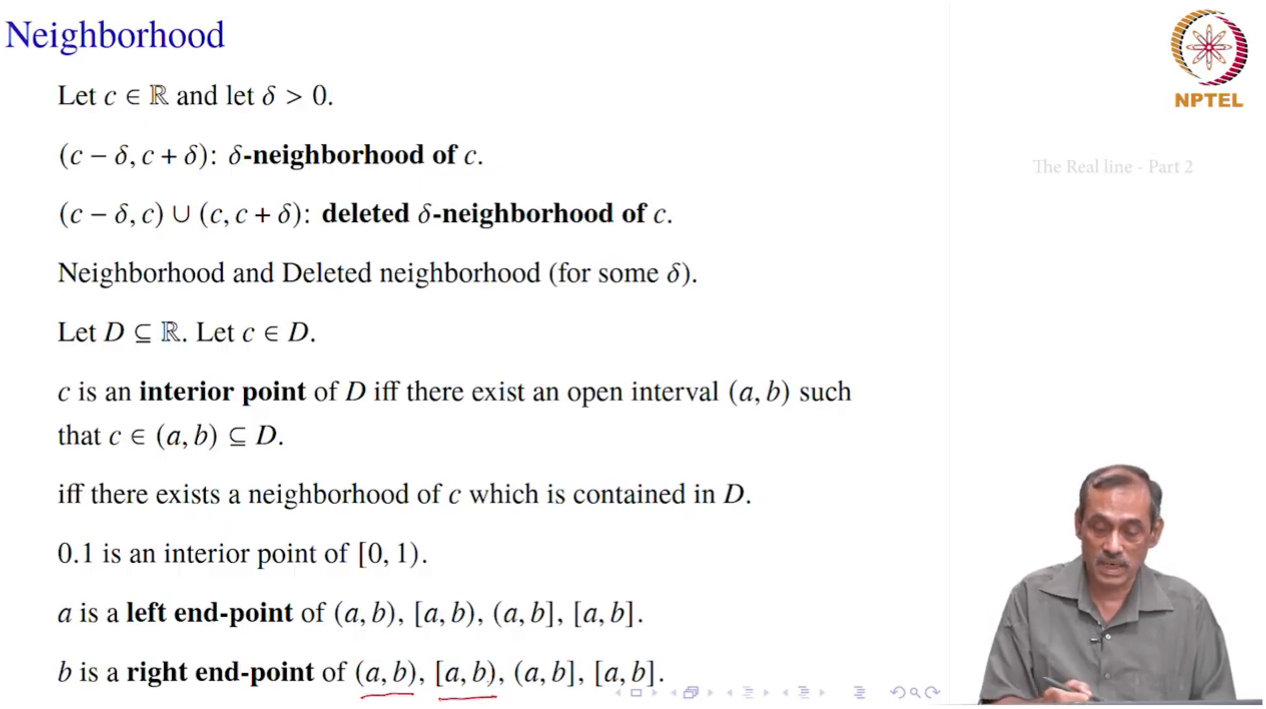

Now the point a is the left endpoint of $[a, b)$. Here, in this interval $[0, 1)$ for example, this 0 is the left endpoint of this interval. You can say a is a left endpoint of all four types of intervals: open, semi-open, and closed. But in some of them the left endpoint is inside, and in some of them left endpoint is not inside. When the interval is open at the left endpoint, it is not there. In other two intervals the left endpoint is there. Similarly you can define the right endpoint which is for the same intervals; we say b is the right endpoint. Again, the right endpoint does not belong to the two intervals (a, b) and $[a, b)$. But it belongs to the other two intervals.

Here we have just covered the notions of neighborhoods, deleted neighborhoods, interior points and endpoints. Let us look at some exercises which may be helpful before we proceed further.

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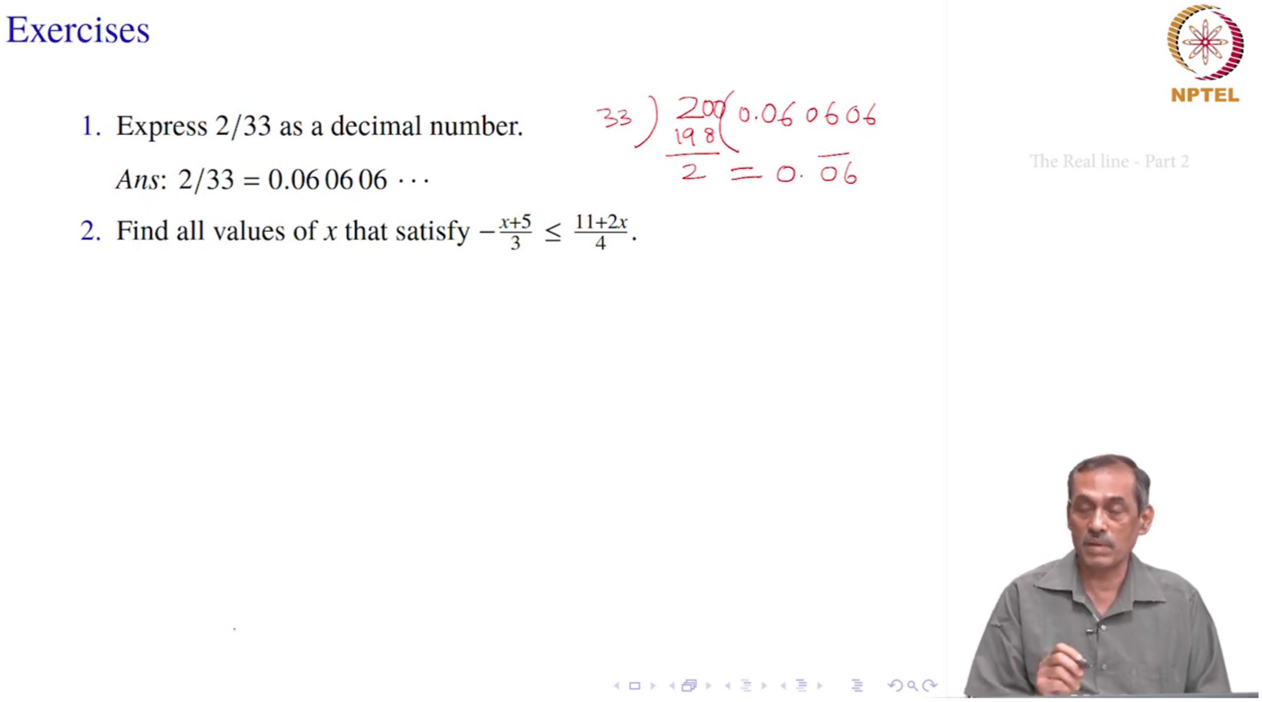

If you have understood the decimal numbers you can think of expressing $2/33$, which is a rational number as a decimal number. So how do you proceed?

Well, it is the usual division algorithm. You take 2, but 2 is smaller than 33. So, there will be a decimal point. You put 20 and start with 0 point in the result. Then you get $20/33$; but that is not enough. This 20 is less than 33. So you put a 0; that gives you 200. Divide by 33. That gives you 6. So, 33 times 6 which is 198; subtracting you get 2. Again it comes back to the earlier thing. So, you take two zeros and put a 0 in the result. Thus you can proceed.

Now you see that this 06 will be repeating. It will go on to giev $0.060606\cdots$. We will write this number in the form of recurring decimals as $\overline{06}$ using a bar. It says that this 06 is getting repeated infinitely many times (recurring) afterwards. That is how it is obtained.

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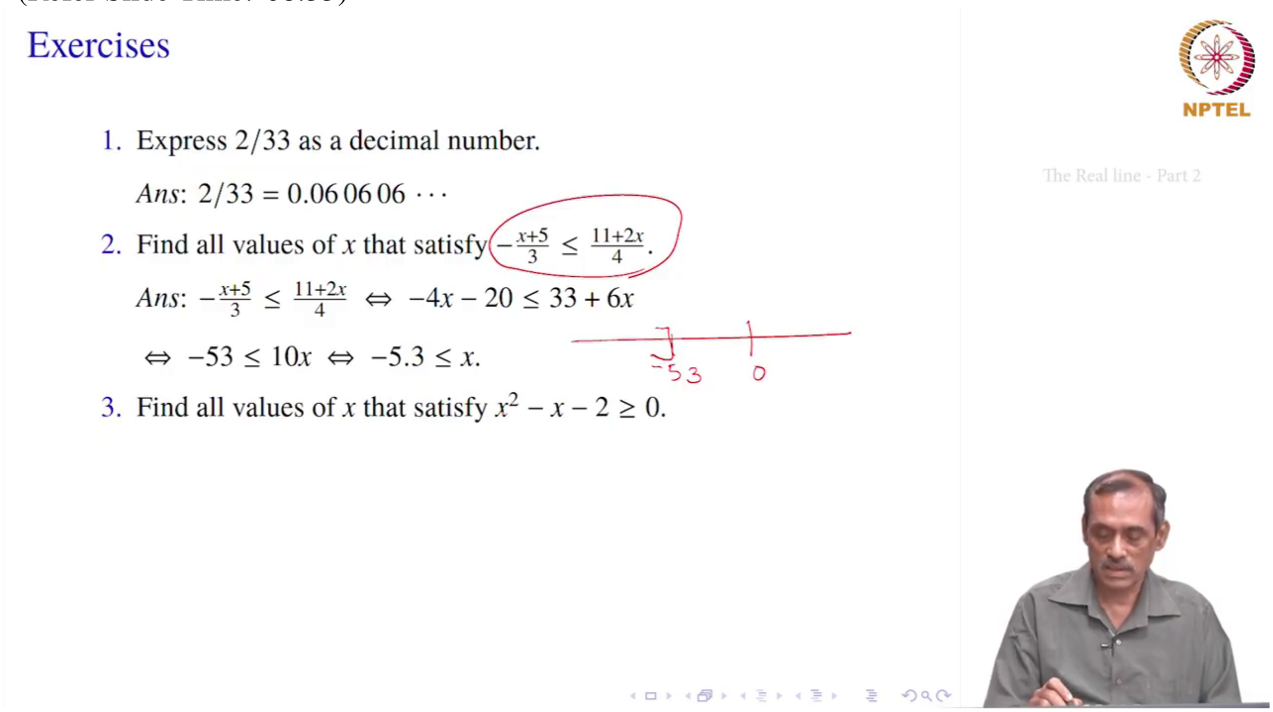

Now let us look at the second problem. It asks to find all values of x that satisfy this inequality: x plus 5 by 3 with a negative sign is less than or equal to 11 plus 2 by 4. So how do we proceed?

Of course, we have to use our laws of addition and multiplication and so on. So, we start with that inequality. It is the same thing as multiplying 12 to both the sides. So multiply 12 with 3, there one 3 is canceled with 12; we get 4. So, $-4x - 20$ on the left side. On the right side you have 11 plus 2x into 3 which is $33 + 6x$. We take away x to one side and numbers to one side. We get $6x$ on the right side; coefficient of x is positive and it will be convenient. I take $-4x$ to the other side. That becomes $10x$, and bring -33 to the left side; that becomes -33. So, result is -53.

Now we can divide both the sides by 10. It is a positive number we are dividing. So inequality is preserved. Same inequality will stay. It is $-53/10$ on the left side, which is -5.3. On the right it is x. So, that is how it looks. That means the set of all real numbers satisfying the given inequality same thing as this interval. What is this interval? It is the set of all numbers less than or equal to -5.3. That is how it looks.

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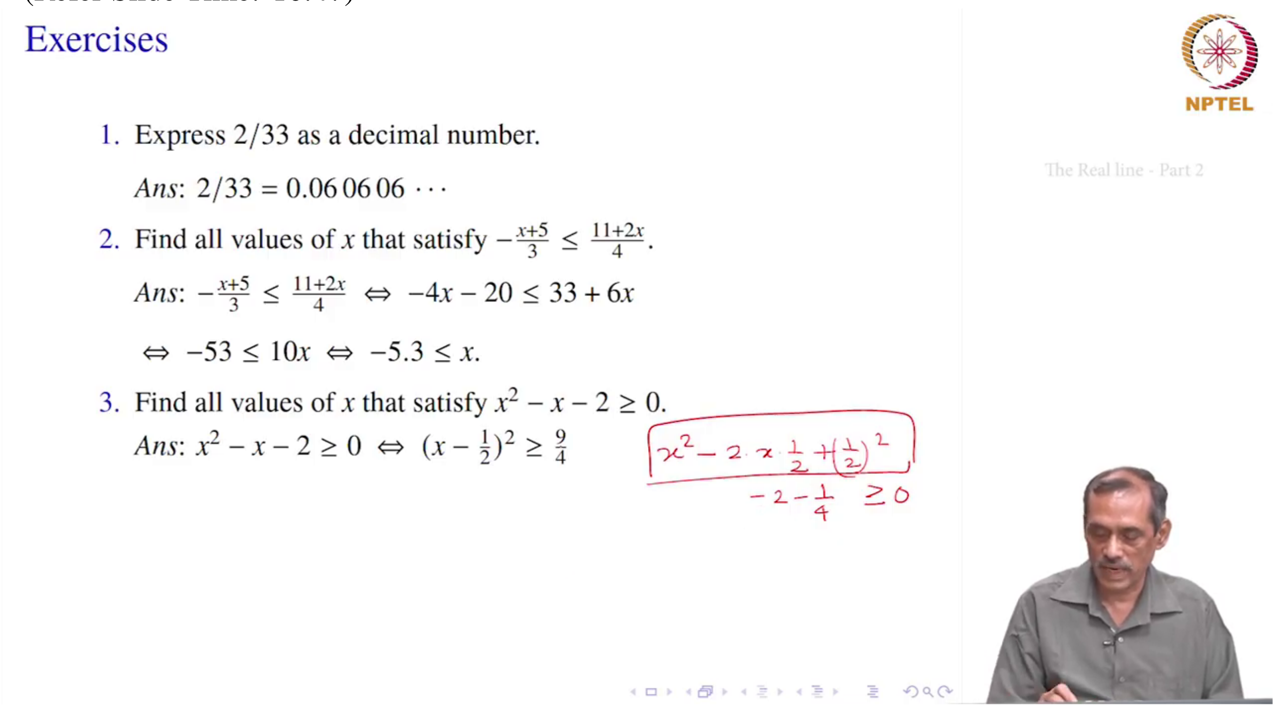

Let us go to next problem. Find all values of x that satisfy $x^2 - x - 2 \geq 0$. It was a linear equation, but now it is a quadratic. There is a power for x, so x^2, which is $x \times x$, then $-x-2$ greater than equal to 0. So what do we do? We start with this inequality and see how it goes. The trick is to make the left side, somehow, a square, complete square, so that you can take the square root. Well, this is x^2 and this is $-x$; so we can say something like: x^2 and this is $-2 \times x$ into $1/2$. We need here $+(1/2)^2$. This is the extra thing we have added, then we should subtract that. This is how the left side looks. Right side of course is staying as it is. Now this expression is simply $(x-1/2)^2$ and this you take to the right side, which becomes $9/4$. This is how the inequality now looks.

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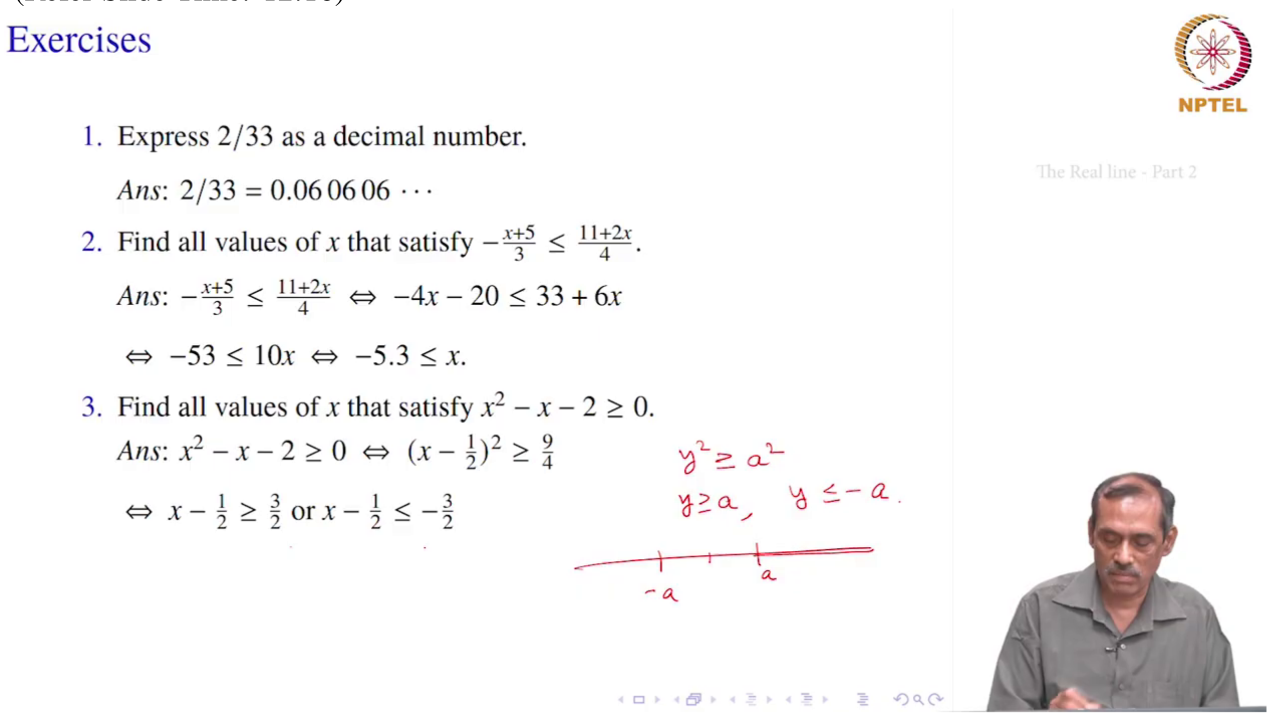

Then from this we take up. Suppose we have $y^2 \geq a^2$ for positive a. Then how this inequality comes? It might be because if y is positive, you would get $y \geq a$ or it is possible that this $y \leq -a$. That is, if a is here and you know $y^2 \geq a^2$; then, if a is positive, then it will look like $y \geq a$ or $y \leq -a$. It is positive, $(3/2)^2 = 9/4$. Here, $3/2$ is positive.

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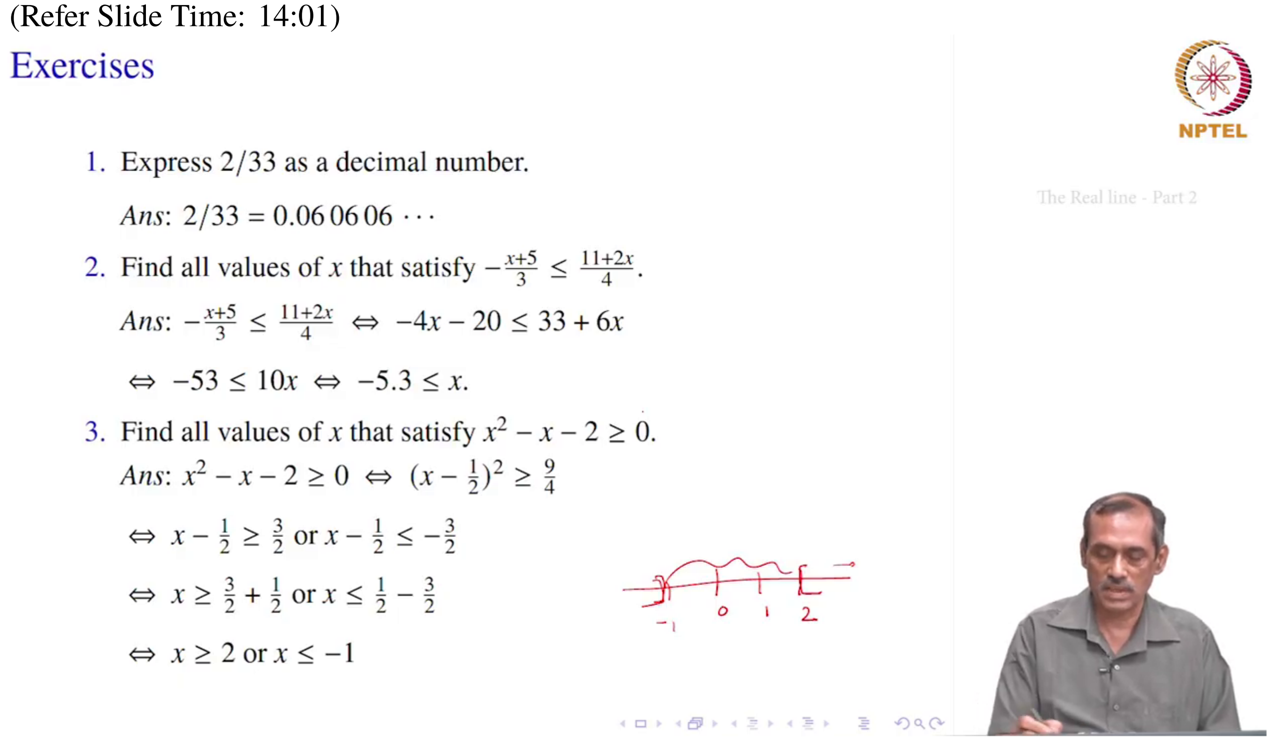

So, you may think of y to be bigger than this, so that $y^2 > a^2$ or it can be y is less here, so $(-y)^2 > (-a)^2$. So, $y^2 \geq a^2$; that is how we do here.

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Exercises

The Real line - Part 2

1. Express $2/33$ as a decimal number.

   Ans: $2/33 = 0.06\,06\,06\cdots$

2. Find all values of x that satisfy $-\frac{x+5}{3} \leq \frac{11+2x}{4}$.

   Ans: $-\frac{x+5}{3} \leq \frac{11+2x}{4} \Leftrightarrow -4x - 20 \leq 33 + 6x$

   $\Leftrightarrow -53 \leq 10x \Leftrightarrow -5.3 \leq x$.

3. Find all values of x that satisfy $x^2 - x - 2 \geq 0$.

   Ans: $x^2 - x - 2 \geq 0 \Leftrightarrow (x - \frac{1}{2})^2 \geq \frac{9}{4}$

   $\Leftrightarrow x - \frac{1}{2} \geq \frac{3}{2}$ or $x - \frac{1}{2} \leq -\frac{3}{2}$

   $\Leftrightarrow x \geq \frac{3}{2} + \frac{1}{2}$ or $x \leq \frac{1}{2} - \frac{3}{2}$

   $\Leftrightarrow x \geq 2$ or $x \leq -1 \Leftrightarrow x \in (-\infty, -1] \cup [2, \infty)$.

See this inequality with the squares. You can write it as $(x - 1/2)^2 \geq (3/2)^2$. Here, $3/2$ is positive. So, you get $(x - 1/2) \geq 3/2$ or $(x - 1/2) \leq -3/2$. In the first one, we can take $1/2$ to the other side; it is $x \geq 1/2 + 3/2 = 1$. So, $x \geq 2$. The other gives $x - 1/2 \leq -3/2$, which simplifies to $x \leq -1$. That means the set of all real numbers which satisfies this inequality with the quadratic will be something like this: it is the union of two intervals $(-\infty, -1]$ and $[2, \infty)$, the portion $(-1, 2)$ is excluded from the set of all real numbers. That is how it looks. So, maybe we stop today.