Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

Lecture 10 - Part 1

Continuity - Part 1

This is lecture 10 of Basic Calculus 1. If you recall, we had just discussed the concept of limit, the limit at a point, a real number c. And then in that connection we had left side limits, and right side limits. Then we moved to limits at infinity, that is, as x goes to ∞ or as x goes to $-\infty$. Also we discussed how a limit of a function can become equal to ∞ or $-\infty$ at a point c or even when x goes to $\pm\infty$. Today we will be extending these notions and discuss the continuity of a function.

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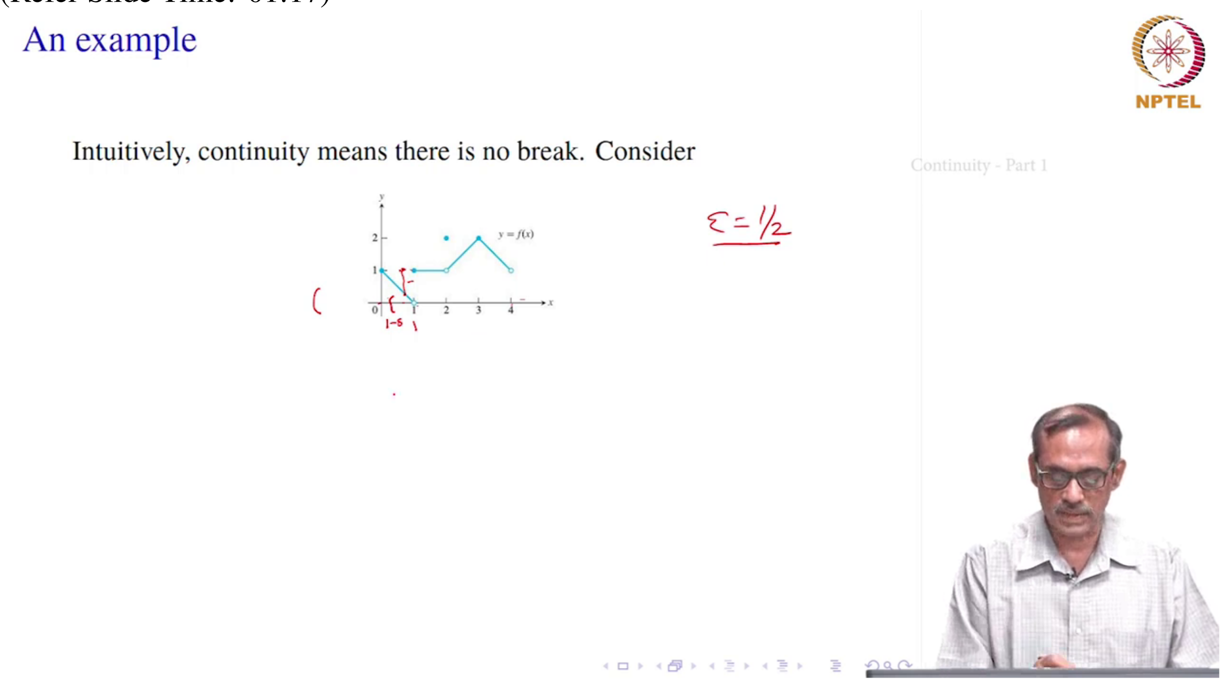

When we think of continuity, we think something like continuity over an interval. It is a process; and continuous process means it continues throughout an interval of time. Here we do not have time, we have a function which would be continuous throughout an interval. In mathematics, there is another thing. Instead of just telling continuity over an interval, we will also be talking about continuity at a point. It looks a bit ridiculous linguistically, for, what is the meaning of continuity at a point? It will be over an interval. However, the negative concept of it that 'it is not continuous at a point' is meaningful and that is what we take up; continuity means, there is no break. You would say that a function is continuous at a point if it is not having a break at that point.

What is the meaning of not having a break? Look at the graph of a function. You say at this point, last one, there is a break. That means, you cannot continue as you draw it, you may have to jump and do something like this or something this way. That is what we mean that there is a break. This is what we are going to define; when a function is not having a break. You should understand what is the meaning of 'it is having a break'.

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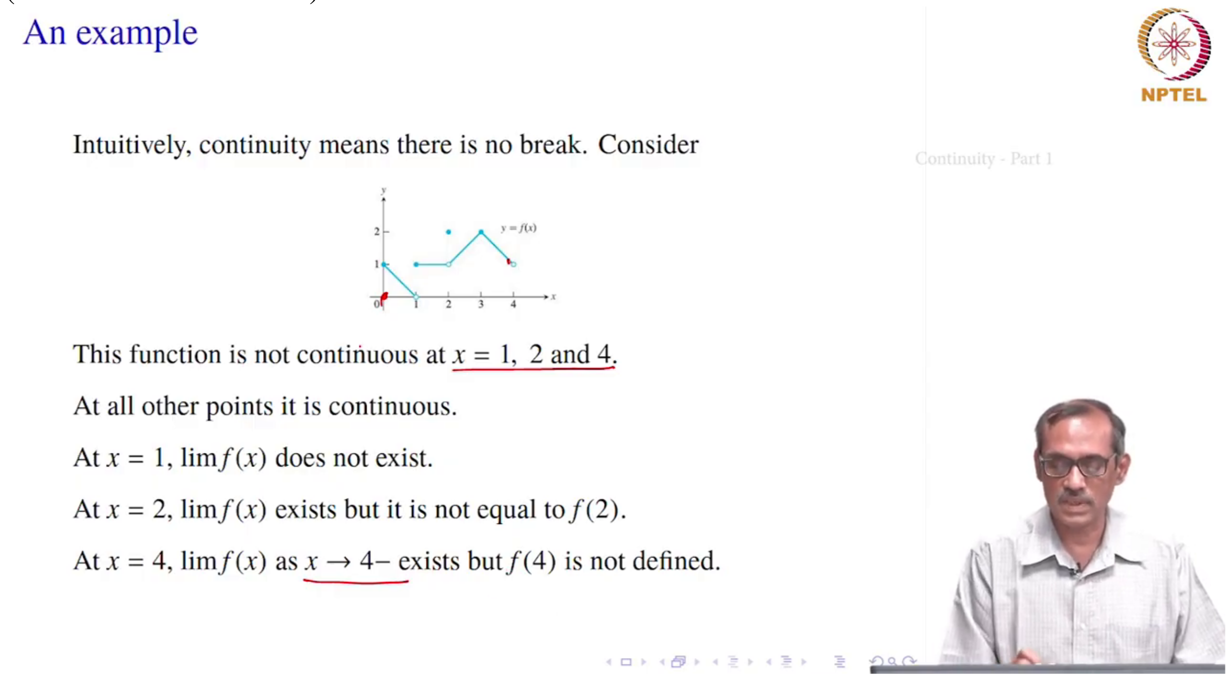

Let us look at this function which is drawn here. Its graph is given. It is defined on $[0, 4]$ and its values are given in a graphical way. Let us look at the point $x = 1$. Obviously, if we start drawing it, the value at 0 is 1. From 1 it decreases to 0 at $x = 1$, than at 1, there is a jump; instead of 0, it goes to 1 there. And $f(1)$ is also given as 1. It is not 0. That is what you see as an empty circle, a hollow circle, and on the top there is a solid circle. That means $f(1) = 1$, not that $f(1)$ is equal to 0. So, there is a jump here.

Now, when you see that there is a jump, what is exactly happening? Let us try to connect with the definition of limits. In that style, suppose I choose my $\epsilon = 1/2$. There is a jump; the function values are near 1. Before this 1, if you take x close to 1, $f(x)$ is near 0, and at 1 it is going equal to 1. Since near 1, $f(x)$ is near 0, the function values are near 0. So, the gap is 1 unit. Suppose I choose my $\epsilon = 1/2$ here. Let me choose any δ, and I choose my x, that is, here. We are concerned with the left side, not the right side. Let us take the left neighborhood which is $(1 - \delta, 1)$.

Of course, the values should be inside the domain. If you take bigger delta, it does not matter, but values inside means it will come back to this point 0. So, delta has a maximum to be meaningful here. Take any such delta. Now in this neighborhood, $(1 - \delta, 1)$, I can always find a point x such that $f(x) - f(1)$, which is this distance, is bigger than half. This is what it says.

That is the meaning of 'it is having a break there'. It means I cannot make them as close as I want. There is always something, the functional value is really bigger than this something. The difference between the functional values $f(x)$ and $f(1)$ is always bigger than this. So, I can never make them closer and closer by choosing my δ-neighborhood.

When you take the negation of it, you say that this does not happen. Then what does happen? We have started with 'there exists an epsilon'. It does not happen means we will say now that 'for every $\epsilon > 0$, there exists a δ'. Instead of 'for every delta' now we say 'corresponding to each epsilon greater than 0, there is a delta', and then if I choose any point x between $1 - \delta$ and 1, then, the difference between them is always less than epsilon. That will give us the function having no break at 1. Here the function is having a break; so the opposite is happening.

That is how we go towards the formal definition of continuity. But then as you remember the formal definition of limit, we would say "corresponding to each epsilon greater than 0, there exists a delta greater than 0 such that if you choose any x point in the deleted delta neighborhood of that point c, the difference between the corresponding value $f(x)$ and the limit ℓ will be less than epsilon. This is the notion of limit. Now, instead of limit equal to ℓ, we are comparing $f(x)$ with $f(c)$. So, in the definition of limit, if we just take $\ell = f(c)$, we would see that it is continuous at that point c. So, there is a connection between this and the notion of limit. If you look at this function again, you see that the function is not continuous at 1. It is also not continuous at 2, there is again a jump.

It is also not continuous at 4. We are including the endpoint 4 in the sense that we do not have $f(4)$ defined. For it to be continuous at 4, it should be drawn at 4. But we cannot sih=nce it is not defined there. Since it is not defined, we would say that it is not continuous at that point. So, 'not-continuous' at these points 1, 2, 4 are of different kinds. At all other points, it is continuous.

Of what kind of break or discontinuity is happening at these points? At 1, you can see that the limit of $f(x)$ as x goes to $1-$ (that is from the left side) exists and that is equal to 0. On the right side, you look at the limit of the function $f(x)$ as x goes to 1. The other curve is important here. You will find that its limit is 1. That means limit of $f(x)$ at $x = 1$ does not exist, since they are having two different limits. It can also happen that one of the limits does not exist. Then also, this condition that 'the limit $f(x)$ does not exist' will hold. So, this is of one type that at $x = 1$ the limit of $f(x)$ as x goes to 1 does not exist.

What about 2? At $x = 2$ you find the left side limit. When x is to the left of 2 but near 2, you see that its limit is equal to 1 because all the values of $f(x)$ lie near 1. On the right side all values also lie near 1. So limit exists and it is equal to 2. But $f(2) = 2$. Its limit is not 2, its limit is 1. When it is left side, it remains near 1, when it is right side, it also remains near 1. So, limit of $f(x)$ exists and is equal to 1; whereas the functional value $f(2)$ is equal to 2. The limit exists, but it is not equal to the functional value. This is of a different kind from the earlier one.

Let us see what happens at $x = 4$. At 4, it is simpler. It says that $f(4)$ is not defined. Of course, we can redefine it. Suppose you take a different function, which is defined as $f(x)$ for all the values given here, but at $x = 4$, its value is, say 1. Then it will become continuous at that right end-point $x = 4$. But here, the notion of right side limit does not apply, because in the domain of definition of f, we do not have any point to the right side of 4; only the left side is relevant here. Now, as x goes to $4-$, this limit exists. And all those values of $f(x)$ also lie near 1. That is why limit $f(x)$ as x goes to $4-$ exists and is equal to 1. But $f(4)$ is not defined. Had we defined $f(4) = 1$, it would have been continuous at that endpoint.

So, you have to be cautious about the endpoints, because at those points, one side of the limit may not exist. We can of course define only one of the one-sided limits at the end-points. As for the point $x = 0$, there is no notion of left side limit, it is only right side limit. But at that point, $f(0) = 1$, and the limit as x goes to 0+ is also equal to 1. So, you would say that it is continuous at 0. So, at these 3 points only, we see that there is a possibility of a break. In fact, at these three points only the function is discontinuous, not continuous. At all the other points it is continuous. This is how we will be connecting it with the limits.

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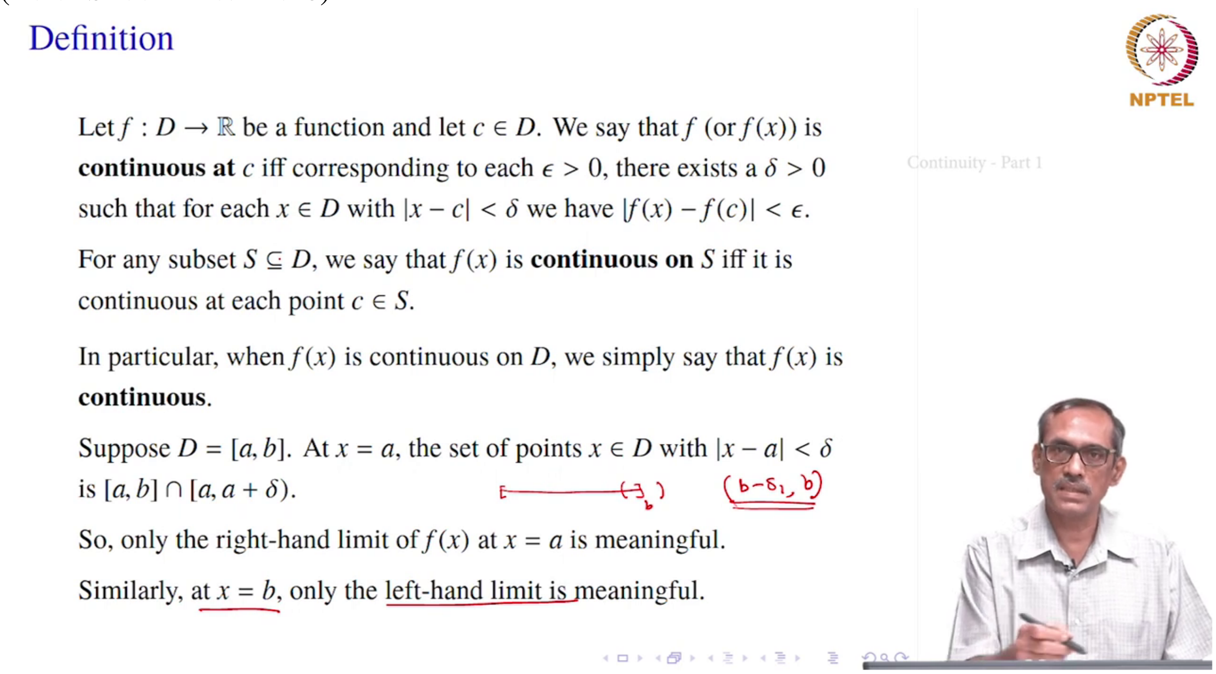

Let us give the formal definition first using the epsilon and delta. Suppose we have a function $f(x)$ from a subset D of $\mathbb{R}$ to $\mathbb{R}$. And we have a point $c \in D$. We want to know what is the meaning of a function being continuous at that point c. Remember, it does not have a break. That is what we have to say.

You say that $f(x)$ is continuous at c, if corresponding to each $\epsilon > 0$, there exists one positive δ such that for all those x inside D for which the inequality $|x - c| < \delta$ is satisfied, (that means, for all $x \in D \cap (c - \delta, c + \delta)$), we should have $|f(x) - f(c)| < \epsilon$. In fact, this will be easier because we will be only considering D as intervals. So, for all those intervals, whenever D is an interval, you will see that $D \cap (c - \delta, c + \delta)$ will be a neighborhood, maybe only left-hand neighborhood or maybe only right neighborhood. It will cover also the endpoints, as we have discussed earlier. If it is an interior point, then you will get a neighborhood of course, but not with this delta, maybe smaller delta. That is fine, because we need only one such delta. So, that is how it will go.

Now, you see that this definition is exactly the same as the limit definition; while instead of $f(c)$ it was ℓ in the definition of limit. So, a function $f(x)$ is continuous at $x = c$ (so that c to is in the domain of f) whenever the limit of $f(x)$ is equal to $f(c)$. It includes many things such as the limits should exist, the function should be defined at c, and then the limit should be equal to $f(c)$. Then, the limit condition implies that the left side limit exists, the right side limit exists, and they must be equal. This is happening for all interior points c. If c is an end-point, then one side of the limit will not be relevant. We will be considering only the relevant one. That limit should be equal to $f(c)$. So, that is how we will be connecting continuity with the limit.

Let us pick some terminology. All that we have done is we have defined continuity at a point c. Now, if you take S as a subset of D having some of the points of D, and you find that f is continuous at every point of that S, then you say that f is continuous on S. Now in particular, if $S = D$, you will say that f is continuous on D. Again, we will have a shortcut. We will not say continuous on D, we will just say that $f(x)$ is continuous, taking care of the domain.

That is how we will be starting from continuity at a point, to a function being continuous over the whole domain. In fact, this is the last one, which is linguistically meaningful, not the first one. The first one is mathematically meaningful and we will continue with that definition. Suppose in particular, because this is what we will be interested in, that the domain of the definition of the function is an interval, maybe $[a, b]$, both side closed, or maybe one of them is excluded, so that is a semi open interval, or maybe both are excluded so you get an open interval. We will be considering these type of sets as domain of our function. At a point $x = c$, as we require in the definition, we will consider the set $[a, b] \cap (c - \delta, c + \delta)$. When $c = a$, this set will look like $[a, a + \delta)$ and the left-side neighborhood points are not included in our domain. Further, this set contains points from $[a, b]$ implies that the δ must be less than or equal to $b - a$. So, at the left endpoint a, we have only the right side limit of (x) is meaningful. Similarly, At $c = b$, it will look like $(b - \delta, b]$ and the right-side neighborhood points are not included in our domain. Again, this δ must be less than or equal to $b - a$. Here, only a left neighborhood is meaningful. These things will be helpful in connecting continuity at a point with the notion of limit as we have done earlier.

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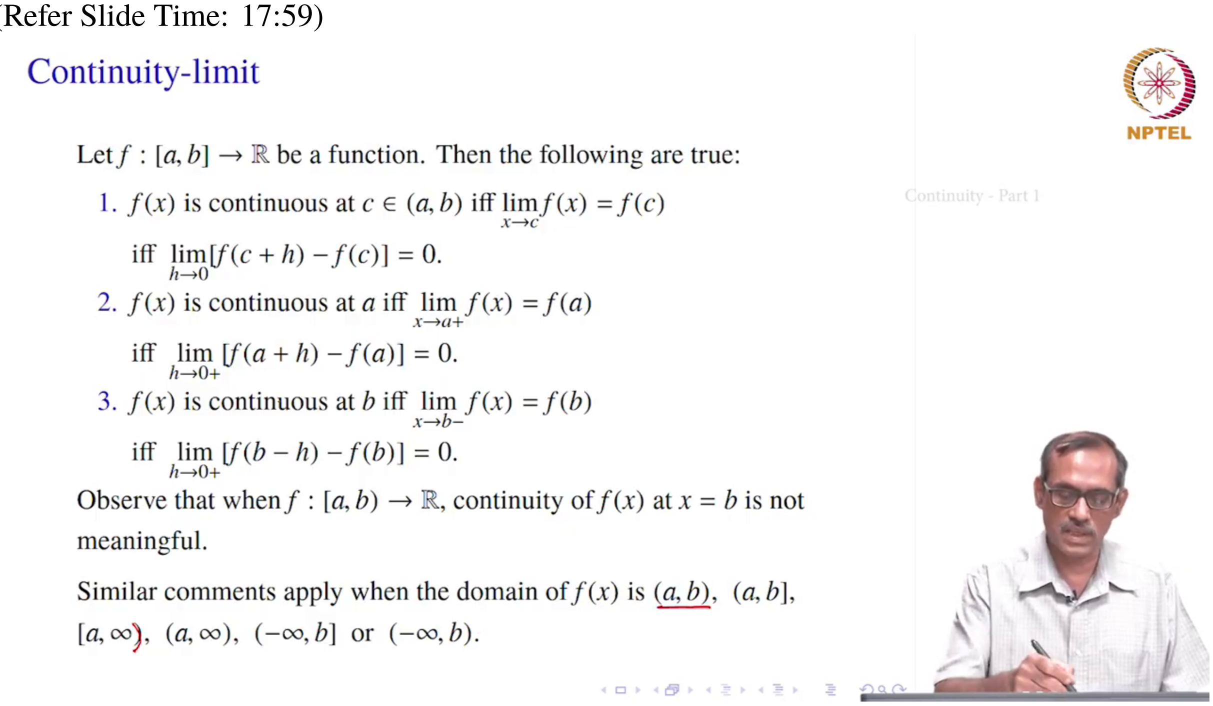

Suppose $f : [a, b] \to \mathbb{R}$ is a function, that is given, and c is an interior point of the domain;

so $c \in (a, b)$. It is something like this. The point c is somewhere, it may be closer to a or b, it does not matter. Now, we say that $f(x)$ is continuous at c if and only if, the limit of $f(x)$ as $x \to c$ is equal to $f(c)$. This limit really includes both left hand and right hand limits, because always you can find a neighborhood which is contained in the interval, in the closed interval $[a, b]$. This happens. Suppose I write $x = c + h$, where h can be positive or negative. Then, $c + h$ is on the right side, and $c - h$ will be on the other side. Then I can express the same limit as the limit of $f(c + h)$ as $h \to 0$. It is only $x - c$, which is our h. Sometimes, this last one is helpful in solving some problems. We will see where. Now you see that $f(x)$ is continuous at c, which is an interior point of $[a, b]$, if and only if, the limit of $f(c + h)$ as h goes to 0 is equal to $f(c)$.

If it is an endpoint, say the left endpoint a, then as we have seen, only the right side limit is meaningful. So, you say that $f(x)$ is continuous at a if and only if the limit of $f(x)$ as x goes to $a+$ is equal to $f(a)$. That c becomes a here, so it is $f(a)$. And that is again the same thing because the left side limit is not meaningful, so h cannot be negative. Writing x as $a + h$, we would see that the limit is as $h \to 0+$. So, as $h \to 0+$, the limit of $f(a + h)$ as h goes to $0+$ must be equal to $f(a)$.

Now, when it is the right endpoint b, similarly the left side limit is meaningful, right side is not. So, we replace that limit $f(x)$ as x goes to c with x goes to $b-$. That is, this is continuous at the right endpoint b, if the limit of $f(x)$ as x goes to $b-$ is equal to $f(b)$. Writing the same way as $x = b + h$, we see that h should be negative, that is, the limit is evaluated as h goes to $0-$. But instead of that we will say h is always positive. Then, we may write $x = b - h$ and require that the limit as h goes to $0+$. We are now writing in both the cases as $h \to 0+$, and require that the limit of $f(b - h)$ as $h \to 0+$ be equal to $f(b)$. So, this is how we will be connecting the continuity at a point c, which is an interior point or a left endpoint or a right endpoint with our limit notion. This will enable us to use our idea of limit to decide whether some function is continuous or not.

But sometimes the $\epsilon - \delta$ notion also helps. Specifically when you are going to use continuity, that becomes of immense help.

We ask whether if f is a function defined on the semi- open interval $[a, b)$, where a is closed b is open, it is continuous at $x = b$ or not? We will say that this question does not arise because b is not a point inside the domain. We can only think of those points where it is continuous or not, if that belongs to the domain. Since b does not belong to the domain, you say that continuity of $f(x)$ at $x = b$ is not meaningful. We will not consider even the question. It is not a question. Similarly, if you have $(a, b]$, where a is open and b is closed, then continuity at a will not be meaningful because $f(a)$ is not defined. If it is both open, that is, (a, b), then continuity at a and continuity at b do not arise. Similarly, on $[a, \infty)$, we do not talk of continuity at ∞; of course, ∞ is not a point in real numbers. That is why we are writing open at ∞. So, at infinity continuity has no meaning. But at a it has a meaning; it is the right side limit. Similarly, all this. These are just guide-lines to connect continue to the limit and that is how we carry over.