Basic Calculus - 1

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Lecture 9 - Part 2

Infinite Limits - Part 2

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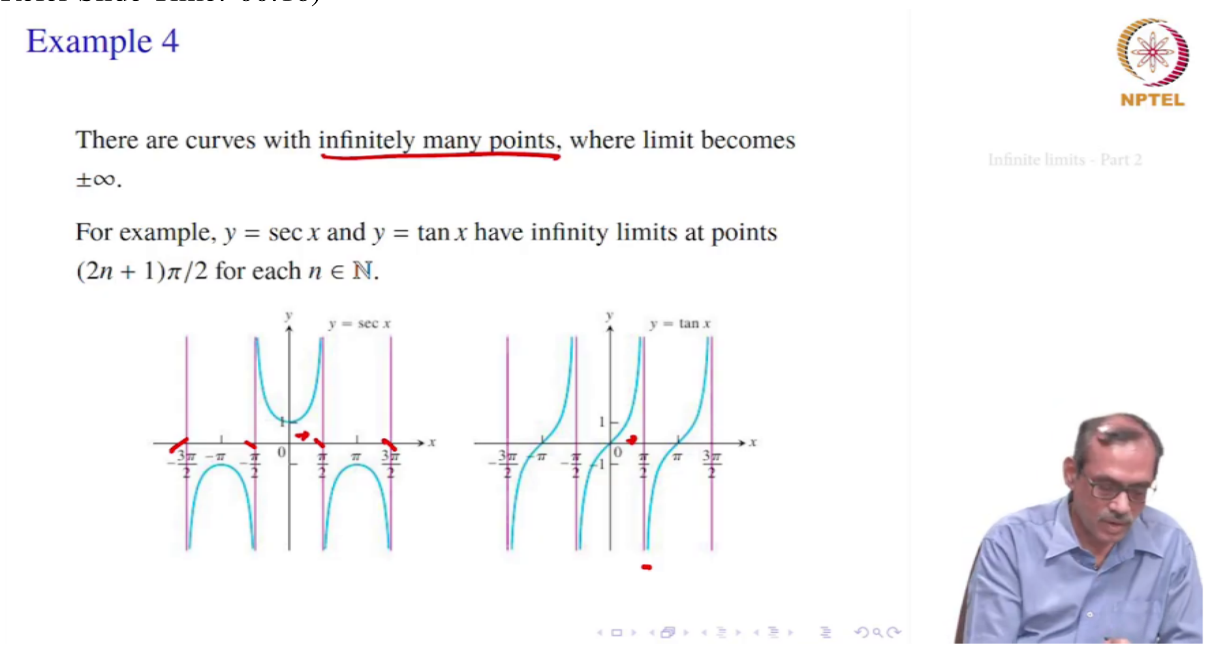

Let us take the next example. It says that there are curves with infinitely many parts where limit becomes $\pm\infty$. It does not happen only at the end points, it becomes $-\infty$ ot $+\infty$ as it happend at -2 in the earlier example. There, it was a single point, but there can be infinitely many points where it becomes infinite.

Usually they are the trigonometric functions; they serve as easy examples. Let us take $y = \sec x$. Look at all these points $x = -\pi/2$,, $x = \pi/2$, $x = 3\pi/2$, on the other side minus $x = -3\pi/2$, all multiples of $\pi/2$. The line $x = \pi/2$ becomes an asymptote. That means as x goes to $\pi/2$ from the left, the line touches the curve. But it touches the curve at different parts of the curve. When x goes to $\pi/2$ from the left side, it touches this curve on the top one. When x goes to $\pi/2$ from the right side, it touches this curve; similarly at all the multiples of $\pi/2$.

When you take $y = \tan x$, these are really not all multiples $\pi/2$; these are all odd multiples of $\pi/2$, the even multiples of $\pi/2$ are not so. For $y = \tan x$, at those points exactly again, you get the asymptotes. That is, as x goes to $\pi/2$ from the left side, y becomes infinity and near $-\pi/2$, the other portion of the curve becomes $-\infty$. That is how there can be infinitely many points where limit becomes $\pm\infty$. So, all sorts of things can happen.

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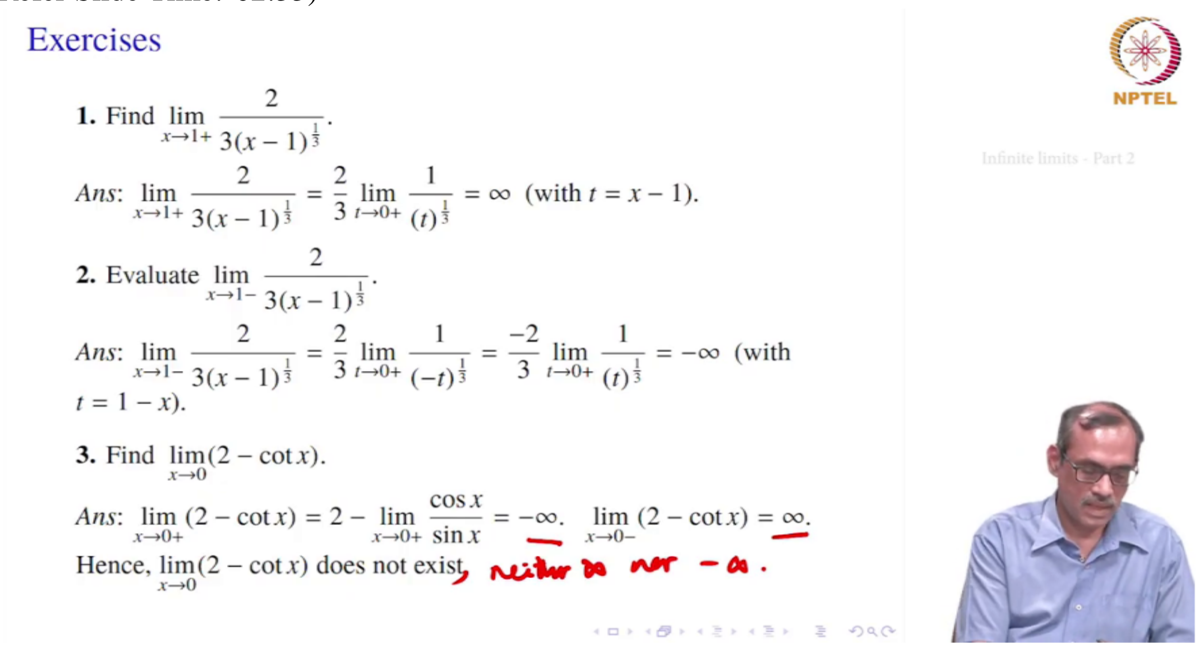

Maybe we should solve some problems now. Let us see. This is the first problem. We have the function given as $2/(3(x-1)^{1/3})$. We are asked to find the limit as x goes to 1+. You see the denominator at 1 becomes 0, but it is 1+, not 0. It is well defined in a neighbourhood where $x > 1$, so we are asked to find the limit as $x \to 1+$. Now that $x > 1$, $(x-1)^{1/3}$ remains positive. What we expect is, this is positive but becomes near 0 so these 2 by 3 times that should be infinite, positive infinite; it should give us ∞. Let us see how we proceed.

For the limit as x goes to +, and 2 divided by this, we will substitute $t = x - 1$. Because x is staying bigger than 1, t is positive here; and it is 2 by 3; so the limit as t goes to 0 plus, or t approaches 0 from the positive side, is required. This is $1/t^{1/3}$ and we know that $1/t$ goes to ∞ as t goes to 0+; we are using that here. So, that gives us the results as ∞.

In the second exercise; you want to evaluate the limit as x goes to 1− of the same thing, where $3(x-1)^{1/3}$ is in the denominator. It is the same function. But now we want to find the limitas $x \to 1-$. When x goes to 1−, x is smaller than 1. So, $x - 1$ becomes negative. Again, if you substitute something of that form with $t = 1 - x$, we want to keep it positive, then t will go to 0+. Then, this is 1 divided by $(-t)^{1/3}$, which will be $-t^{1/3}$; that minus sign comes here; it is $-2/(3t^{1/3})$. That we know goes to infinity. So, this is $-\infty$. Obviously you can see this from the function itself. If x remains smaller than 1, this becomes negative and this is closer to 0 from the negative side; so, the limits would be $-\infty$.

In the third problem, we are required to find the limit of (2 − $\cot x$ as x goes to 0. What do you expect here? It is $\cot x$, which is $\cos x/\sin x$. Near 0, $\cos x$ remains near 1, but $\sin x$ becomes 0. So, it might go to infinity. But then from the negative side and positive side, $\sin x$ will have different signs. So, let us see how to proceed.

Let us find out first what happens when x goes to 0+, that is, as x remains positive and near 0. We get 2 minus the limit as x goes to 0+ of $\cos x/\sin x$. Here, $\sin x$ is positive and $\cos x$ is near 1.

So, this factor becomes near infinity and it is a negative sign, 2 gets absorbed, so, it becomes $-\infty$. As x goes to 0+, we get the limit as $-\infty$. If x goes to 0−, then this is $\cos x$, it is near 1; and $\sin x$ is now negative. The whole thing becomes negative, and there is another negative sign; so, that becomes ∞; 2 gets absorbed. We should get $+\infty$. Therefore, we would say that the limit does not exist; the limit of $2 - \cot x$ as x goes to 0 does not exist.

In fact, when you say the limit is equal to infinity, that also means that the limit does not exist. Recall what we have done. When the limits become infinity or minus infinity, we do not say that the limit exists. 'Exists' means it should be real number. But, from the non-existence of limit, we have separated two cases; they are specifically interesting for us. Here, the limit does not exist. But not only that, here find that the limit does not exist and the limit is neither infinity or minus infinity; that also does not happen here. Also, we would say that the limit is neither ∞ nor $-\infty$. One side it is infinity, and the other side it is minus infinity. So, the limit does not exist, it is neither infinity not minus infinity; not even these two interesting cases.

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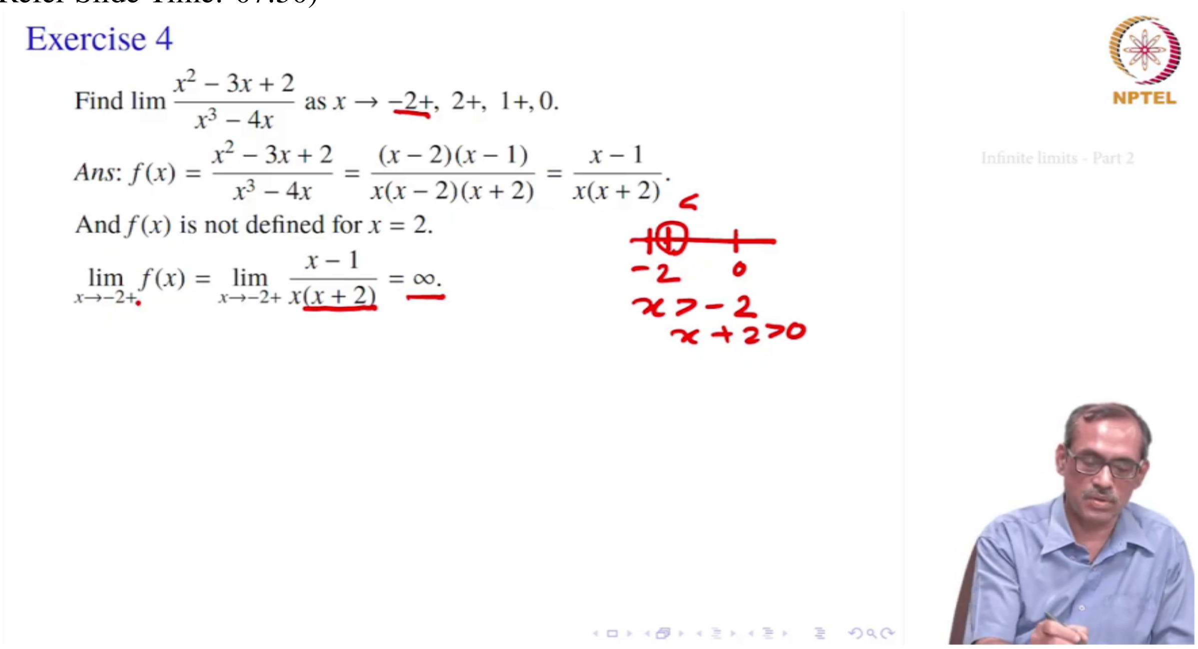

Let us go to the next problem. Here we are required to find the limit of this function: $(x^2 - 3x + 2)/(x^3 - 4x)$. It is a rational function, a polynomial divided by a polynomial. But we are given really four problems here. You want to find the limit of this as x goes to −2+, 2+, 1+, and 0. We will do it slowly.

What do you see from this? It is a rational function. On the top, the leading coefficient or the leading term is x^2, and on the bottom it is x^3. As x goes to 0, it should go to infinity or minus infinity. Of course, you can find that out immediately. What happen when x goes to −2+? We have to see what really happens to the denominator. Let us rewrite it first. Linear things will be easier to tackle than the quadratic or cubic. Fortunately, you can factorize it here easily. You se that $x^2 - 3x + 2 = (x - 2)(x - 1)$, and on the down side, we have, taking x common, it is $x^2 - 4$, which is $(x + 2)(x - 2)$. This is how it looks. Look at this $x - 2$. See that $x \neq 2$ since 2 is not in the domain of the function because this function will not be defined if $x = 2$. As $x = 2$ is not in the domain; we cancel this factor. Then, we get $(x-1)/(x(x+2)$.

That is how the function looks like now. Initially it was given in a trickier way. We are supposed to do this much work; some simplification might help us to tackle the function more easily. Now, $f(x) = (x-1)/(x(x+2)$. Think about the point -2. Here, when $x = -2$, the denominator becomes 0. So, it is not defined for $x = -2$. It is also not defined for $x = 0$. At these two points such as -2 and 0, the function is not defined; and everywhere else, at other two points, it is defined. You are required to find the limit at these points. From the domain we have excluded $x = 0$ and $x = -2$, and $x = 2$ also. The function is defined everywhere else except at these three points: 0, -2 and 2. Of course, this last expression does not show that, but our function is given this way. That means we cannot define it at $x = 2$ also. Except at the three points $0, \pm 2$, it is defined everywhere else.

Let us try to find the limits. At $-2+$, x is remaining bigger than -2 and it is near -2. So, $x - 1 > -3$ and near -3. On the bottom, it is x, which is bigger than -2. This factor is $x + 2 > 0$ as $x > -2$. So, it is somewhere here. When you take $x + 2$, it is this, and minus of this point, that distance is always positive. Is that correct? I can find out algebraically. Now, this is positive and x is remaining near -2 and bigger than -2. It is negative here. So, x is remaining negative, $x - 1$ is also remaining negative; it is near -3 and bigger than that which is negative. So, $(x-1)/(x(x+2))$ shows that it remains positive when $x > -2$ but near -2. Since $x + 2$ is in the denominator, this goes to 0, this is near 0. As the total is ∞ or $-\infty$, and it is positive; it has to be ∞. That is how we have to take care of the signs and whether it is bigger or smaller. So, the limit as x goes to $-2+$ is $+\infty$.

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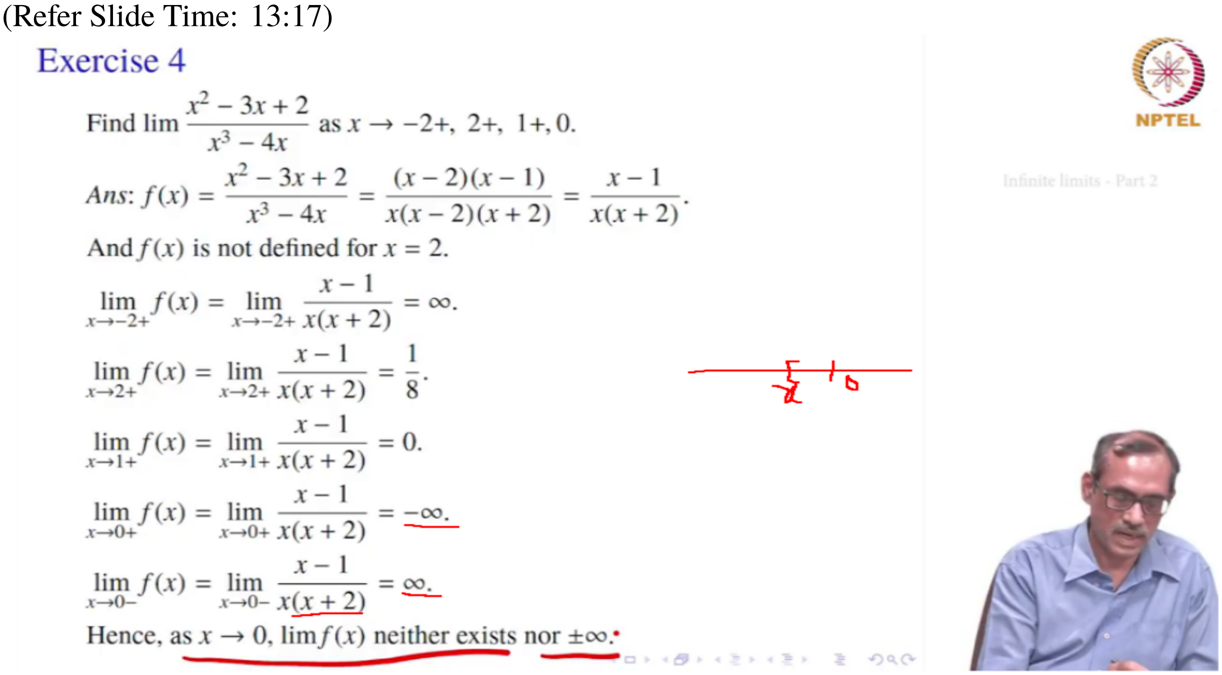

Now, what about the second one? When x goes to $2+$, it means $x > 2$ and also x is near 2. This is bigger than 2 but near 2. So, this is bigger then 4 and near 4; this is near 8; as x is bigger than 2, $x - 1$ is bigger than 1 but near 1. So, the limit sould be $1/8$. But it has to be justified formally.

That is not difficult, because you take the limit separately. This limit is 2, this top limit is 1, this bottom limit is 4, so 1 divided by 2 into 4 gives you $1/8$, using the algebra of limits.

Let us go to third one: to find the limit of the same function as x goes to 1+. As x goes to 1+, there is no problem in the denominator. This is near 1; this is near 3; so it is really 3. The limit of the denominator is 3. The limit of the numerator is $1 - 1$, which is 0. Therefore, the limit is 0.

Now what about the limit at 0+? We are supposed to find the limit at 0 so we are dividing it into two cases: finding at 0+ and at 0−; that may be easier.

So, let us say that x is remaining positive but near 0. It is $x - 1$; it is positive but near 0. That is where x is. So $x - 1$ is now negative. As x is positive, $x + 2$ is also positive. So, this becomes negative as a factor is negative; and when x is near 0, this becomes ∞, positive infinity. So, minus of positive infinity gives $-\infty$. That is, the limit of $f(x)$ as x goes to 0+ is $-\infty$.

Now limit as x goes to 0−, in a similar way, let us say x is near 0, but it is towards the negative side. Now, x is near 0 and $x < 0$ gives $x + 2$ remains positive. Since x is negative, on the down side it is negative, on the top it is negative, near 0 and -1; so, that becomes near -1. And, this one is negative. That cancels, we get positive. And it is infinite. So, the limit must be infinite. That is, the limit of $f(x)$ as x goes to 0− is equal to ∞.

So, the limit of $f(x)$ as x goes to 0 is neither infinity nor minus infinity. It does not exist; just like our earlier thing. You would say as x goes to 0, the limit of $f(x)$ neither exists nor equal to $\pm\infty$.

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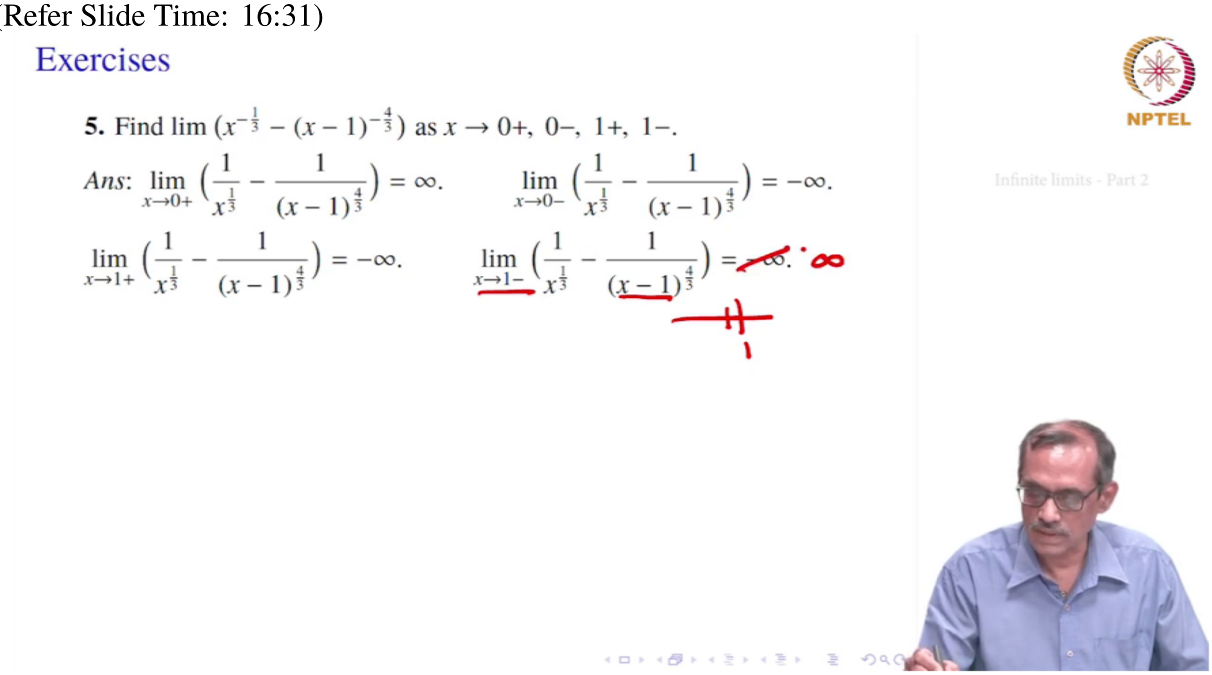

I think we should do one more problem of similar kind. Let us find the limit of this function: $(x^{-1/3} - (x-1)^{-4/3}$ as x goes to 0+, 0−, 1+ and 1−. Let us look at the function. Some rewriting might be useful. We write it as

$$\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}}.$$

As x is remains near 0, but positive, $1/x^{1/3}$ becomes infinity, $(x-1)^{4/3}$ becomes $(-1)^{4/3}$. See, it is some number near 1; this minus also cancels, it is near 1. It gets accommodated in ∞. We would get the limit as ∞.

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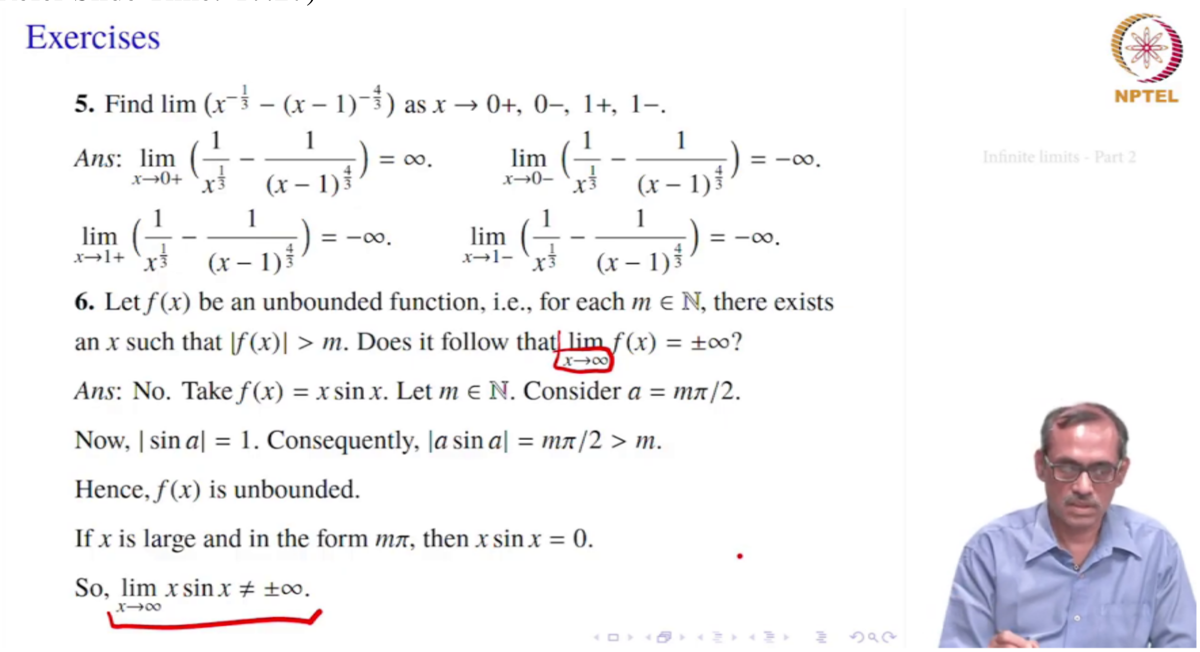

Now, when you go to the limit as x goes to $0-$, x remains negative. This factor becomes negative, and that is the dominating factor. That would give $-\infty$. It is pretty easy here but you have to look at the signs.

Similarly, when you go to $1+$, this factor will be dominating, the other factor will be remaining 1 or -1. Let us see which one. The limit as x goes to $1+$ means $x > 1$ and near 1. So, this one becomes positive, this is $1/x^{1/3}$ and the minus sign is already there. So, it should be $-\infty$. When you go to $1+$, that is $-\infty$.

If you go to $1-$ now, then again this is some number, and it is 1 minus means $x < 1$. but near 1. So, this becomes negative. It is something like $1-$, so it is somewhere here. And, $x - 1$ is negative, then its power $4/3$ becomes negative. This is negative, so you should get $+\infty$. That is, as x goes to $1-$, this becomes ∞. It is just the reversal of signs for $1+$ and $1-$.

Let us see another problem, which is a slightly of different kind. It says $f(x)$ is an unbounded function. It means you cannot say that $f(x) \le m$ for every x. There is no such m. You take any m, which is either a natural number or even a positive real number, then $|f(x) \ge m$ for some x. It implies that you give any real number, then we will be able to find one x such that $|f(x)|$ is bigger than that. That is the meaning of unbounded. It is in absolute value, because it can be unbounded on the other side, towards $-\infty$. Does it follow that when x goes to infinity, we should have the limit of $f(x)$ equal to infinity or minus infinity? That is the question.

That is usually not true. What could be the reason? Why it is not happening to be not true? It is unbounded. That means, for any m you say, I can find one x, where $|f(x)|$ is bigger than that. But when the limit as x goes to infinity, we require another constraint that $|f(x)|$ should be unbounded as x becomes unbound. This condition has to be verified. This unboundedness can happen at a finite point, but when x becomes closer to infinity, that is, x is bigger than some numbers, then unboundedness may not happen.

For example, take $f(x) = x \sin x$. Let $m \in \mathbb{N}$. Consider $a = m\pi/2$. Once you take this, you will see that $\sin a = \sin(m\pi/2)$ has absolute value equal to 1. So, $|f(a)| = |a \sin a| = m\pi/2$. That is certainly bigger than m. That means, for each $m \in \mathbb{N}$, we can find one x, which is $m\pi/2$ such that $|f(x)| > m$. So, f is unbounded, that is clear.

Now, f is unbounded; but as x becomes large, we will find x in various forms. One form, we say, is $m\pi$, which is not this m, some multiple of π. Then, $x \sin x$ at $m\pi$ is 0. So, for every point of that form as you go along, there are points bigger than m where this function becomes 0. See, there are points, where f is unbounded, and there are infinitely many points bigger than any number m, where this function becomes 0. Then, as x goes to infinity $f(x)$ does not approach $\pm\infty$. The limit may or may not turn out to be 0, but certainly it cannot be infinity or minus infinity. Because after some δ, if $x > \delta$, then $f(x)$ should have been bigger than any specified number; then only it can go to infinity. But it becomes 0 for many of the points. Therefore, the limit $f(x)$ cannot be plus infinity. Similarly, it cannot be minus infinity on the other side.

So, you would say that if x is large and in the form $m\pi$, then $x \sin x$ equals 0. If you take minus $-m\pi$, then also 0; so, it cannot be $-m\pi$ either. So, you would say that the limit of $x \sin x$ as x goes to ∞ is not equal to $\pm\infty$. Hence, it can happen that the function is unbounded but its limit as x goes to infinity may not be equal to $\pm\infty$. Let us stop with this.