Basic Calculus - 1

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Lecture 8 - Part 2

Limits at Infinity - Part 2

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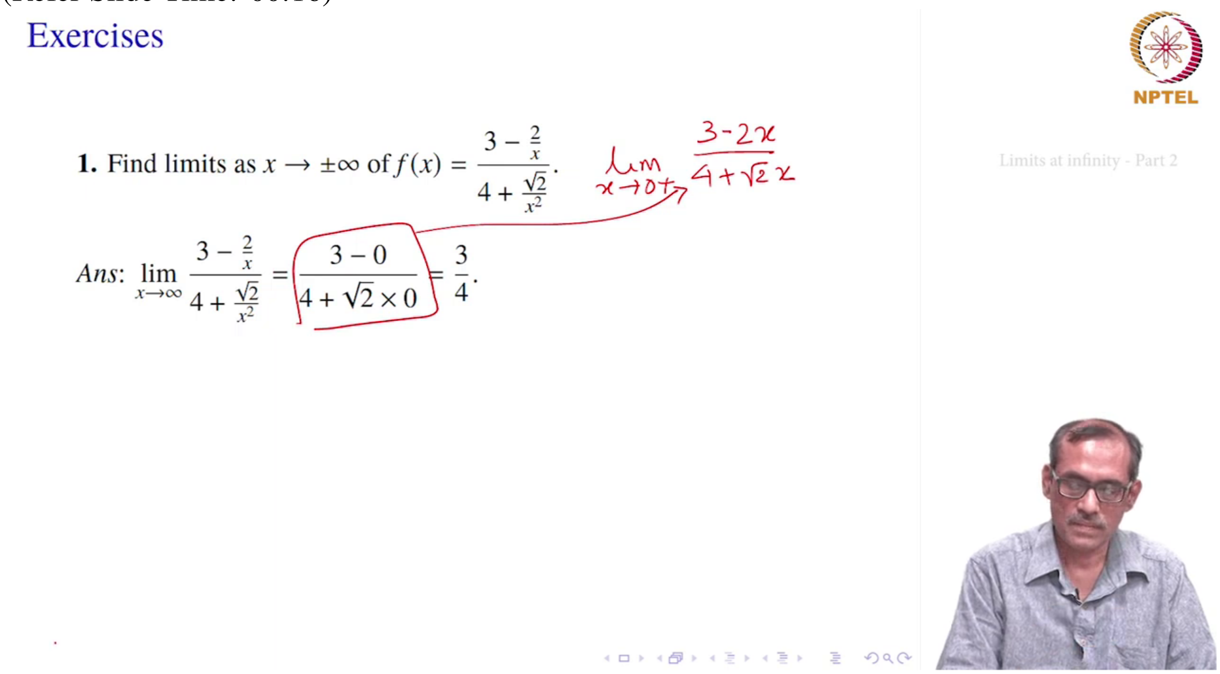

Now, let us solve some problems. The first problem says, find limits as x goes to plus infinity, or x goes to minus infinity of $f(x) = [3 - (2/x)]/[4 + \sqrt{2}/x^2]$. As you see intuitively, when x goes to infinity, $1/x$ goes to 0. With the algebra of limits; all the algebra of limit laws hold also for limits at infinity; this factor becomes 0, this also becomes 0. Finally, it should give 3/4. And that is what we write:

$$\lim_{x\to\infty} \frac{3 - 2/x}{4 + \sqrt{2}/x^2} = \frac{3}{4}.$$

If you use the earlier idea, you may write it as the limit of $f(1/x)$ as $x \to 0+$. Now, $f(1/x) = (3 - 2x)/(4 - \sqrt{2}x^2)$. When x goes to 0 plus, directly it gives 3/4 because this becomes 0, this becomes 0. So, this expression is still valid, right. That is how you can use the earlier comment.

So, what about x goes to minus infinity? Again, it is similar. As x goes to minus infinity, this would give you limit x goes 0− of $(3 - 2x)/(4 - \sqrt{2}x^2)$. That makes it 0, makes it 0 and you get 3/4. That is easy.

Let us take the second problem. It asks us to compute the limit as x goes to $-\infty$ of $(\cos x)/(3x)$. Here, when x goes to $-\infty$, this denominator goes to minus infinity; 3 times minus infinity goes to minus infinity. On the top, we do not have any such thing, it is $\cos x$. $\cos x$ will not have a limit as x goes to infinity, because it cannot selectively take multiples of π. Depending on what kind of multiples of π x is, you will get different values of $\cos x$ from 0 to 1. Of course, everything between 0 to 1 can come. Even -1 also can come. So, they will vary from -1 to 1. Well, that may help. Since $\cos x$ is bounded by -1 and 1, you can have an estimate for this and then finish it. Let us see that. $\cos x$ is always bounded.

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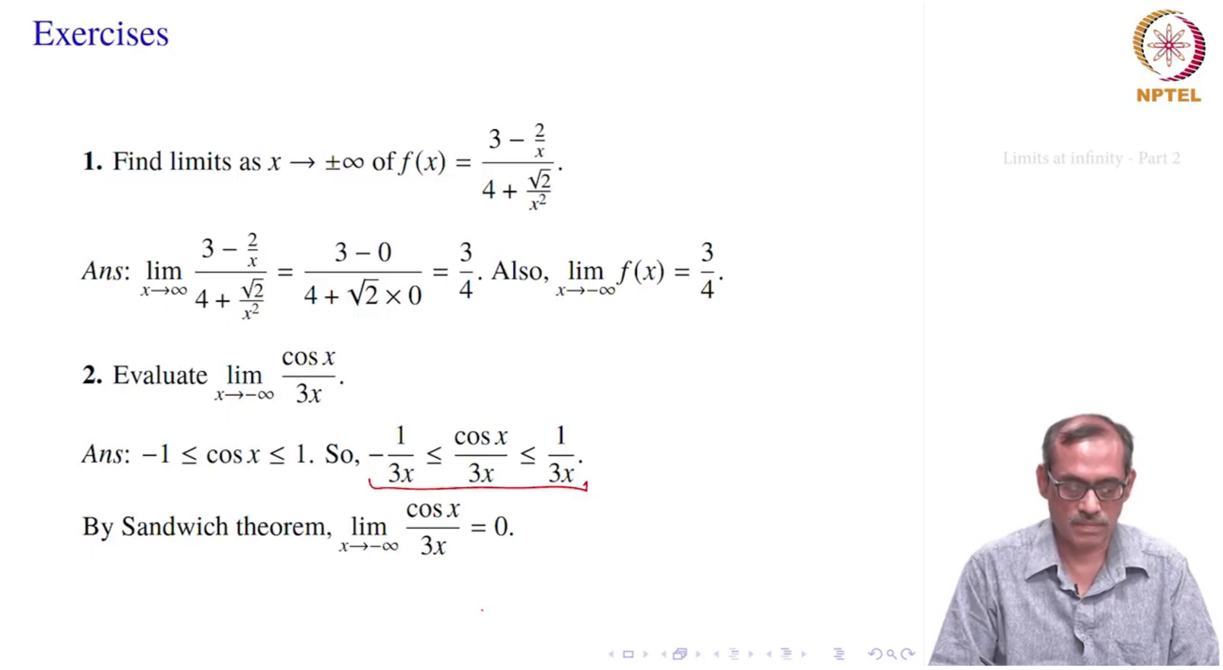

Suppose x goes to $-\infty$, then what happens to $3x$? If we take x to be negative, then this will change the inequalities. If x is negative, then $1/(3x)$ is also negative. So, $-1) \times (1/(3x))$ will become positive. This will be greater than or equal to $\cos x \times (1/(3x))$. This will be greater than or equal to $1/(3x))$ for x negative. If x is positive, of course, this inequality is correct. But we are interested in x goes to $-\infty$. So, you would say that for $x < 0$, these inequalities will be replaced by greater than or equal to. Is that right? So, we will have $-1/(3x) \ge (\cos x)/(3x) \ge 1/(3x)$ for $x < 0$. Then, of course, you apply Sandwich theorem to get this goes to 0. This goes to 0 implies $(\cos x)/(3x)$ goes to 0.

Now what happens when x goes to infinity? In that case, this inequality will be correct; as x goes to infinity will give you this inequality. And again, that will also become equal to 0. In fact, you can do both the things at a time by taking $|\cos x)/(3x)|$. That is also another approach.

Let us take another problem. This asks to evaluate limit x goes to infinity of $(2 - x + \sin x)/(x + \cos x)$. Look at this, whether this is all right. $\cos x$ can vary between -1 to 1, but x can be very large. So, $x + \cos x$ can vary between $x - 1$ to $x + 1$. That is also large. So, this makes sense i not going to 0 of course, it is becoming very large. On the top similarly, it becomes very large, $-x + \sin x$, but in the negative direction, right? $-x$ will dominate $\sin x$. So, it will behave just like $-x$. That means it is – in red on the top and infinity on the bottom. That may happen to be -1; that can be minus of something. That can be 0. It can be anything, but it cannot be positive. That is what it says initially. When you just inspect it, it looks something like that.

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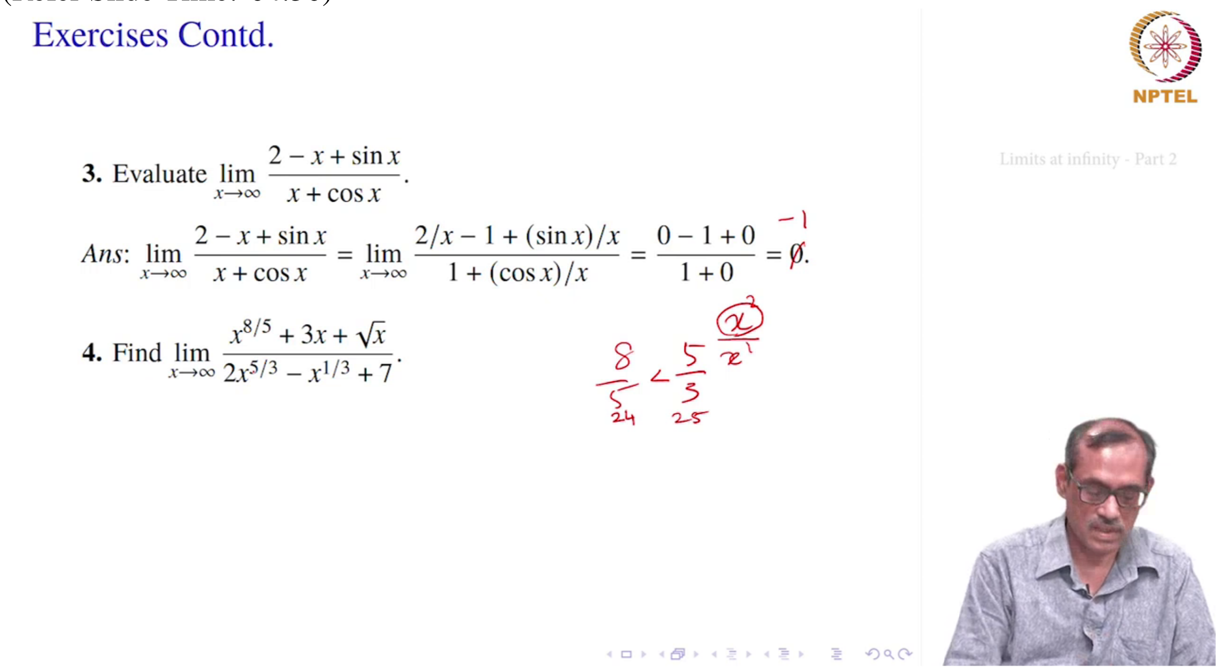

Let us see how to proceed. What do we do? We now use our $(\sin x)/x$ and $\cos x$, which we know when $x \to \infty$. So, we divide by x. That gives us $2/x$ on the top. This becomes -1 plus $(\sin x)/x$. On the down we have 1 and then $(\cos x)/x$. Now, what about limits of $(\sin x)/x$ and of $(\cos x)/x$? Already we have computed them to be 0 as x goes to infinity; and $2/x$ also goes to 0. All that remains is -1 and 1. You get -1 and this is 1. So, this becomes -1, equal to -1. That is what our guess is. Because this becomes 0, this also goes to 0. All that remain is -1 and 1; and we have the limit as -1.

Let us go to the next problem.

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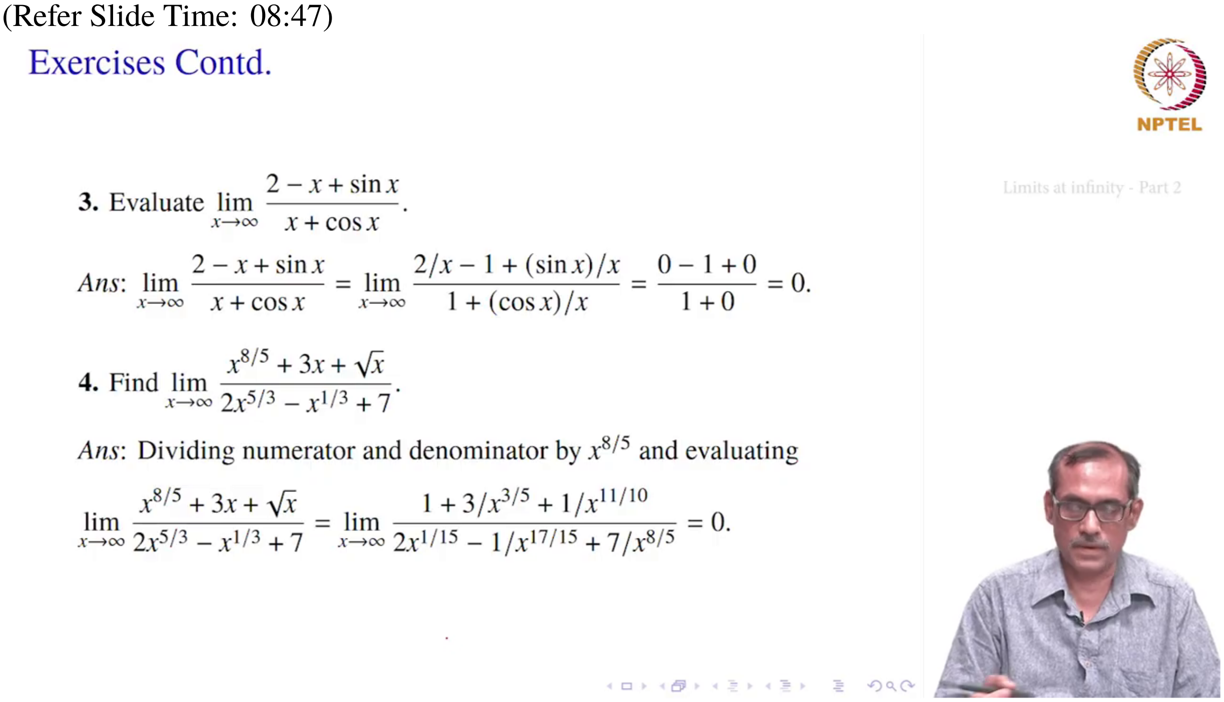

It asks to find the limit as x goes to infinity of $(x^{8/5} + 3x + \sqrt{x})/(2x^{5/3} - x^{1/3} + 7)$. How do we go? It is something like limit of $(x^2)/(x^1)$. If this happens, then we will say that x cancels and we have limit of x as infinity. This works when you have a rational function. But it is not really rational, some square root is there. But that is okay. We will find out what is the biggest power of x. On the top it is happening or in the bottom it is happening? If it is on the top, then it would go to infinity. If it is on the bottom, it would go to 0. That is our usual notion by inspection. So, which one is bigger, $8/5$ or $5/3$? Of course, we will justify it; but we are just having a guesswork now. If you see it is, which one is bigger, $8/5$ or $5/3$? If you multiply by 15, the left side would give 24, and the right side would give you 25. This is what happens. Then the downsides should dominate. And let us try to see what happens.

We divide by $x^{8/5}$. We can divide any one of them. We could have divided by $x^{5/3}$, that would be an an alternative way. Well, divide the numerator and denominator by $x^{8/5}$. Now it is $x^{8/5}$; it becomes 1. And this is $x^{1-8/5}$, which is $x^{-3/5}$, which we write as $x/x^{3/5}$. This is $x^{1/2}$; so $1/2 - 8/5$ would give $-11/10$; so it is $1/x^{11/10}$. Next one is $5/3$, it is bigger; and it should be $5/3 - 8/5 = 1/15$. And this is $1/3 - 8/5 = 17/15$. And this is, of course, $-8/5$ or $7/x^{8/5}$. This is $1/3 - 8/5$. This is $-17/15$, $1/x^{17/15}$. Now, we can take limit. Once the bigger ones are there, this would go to 0. This factor also would go to 0. This factor also goes to 0. This also goes to 0.

So, now, it is $1/(2x^{1/15})$. It is something like the constant half times $1/x)^{1/15}$. When x goes to infinity, x becomes larger, $x^{1/15}$ becomes larger. Then, 1 divided by that becomes smaller, that goes to 0. So, answer should be 0. Is that right? That is how it goes. Whenever it is some powers of x divided by some other powers of x, we can proceed this way.

We will take another problem, of slightly different type. Suppose $f(x)$ and $g(x)$ are polynomials in x. That means they would look like $a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$.

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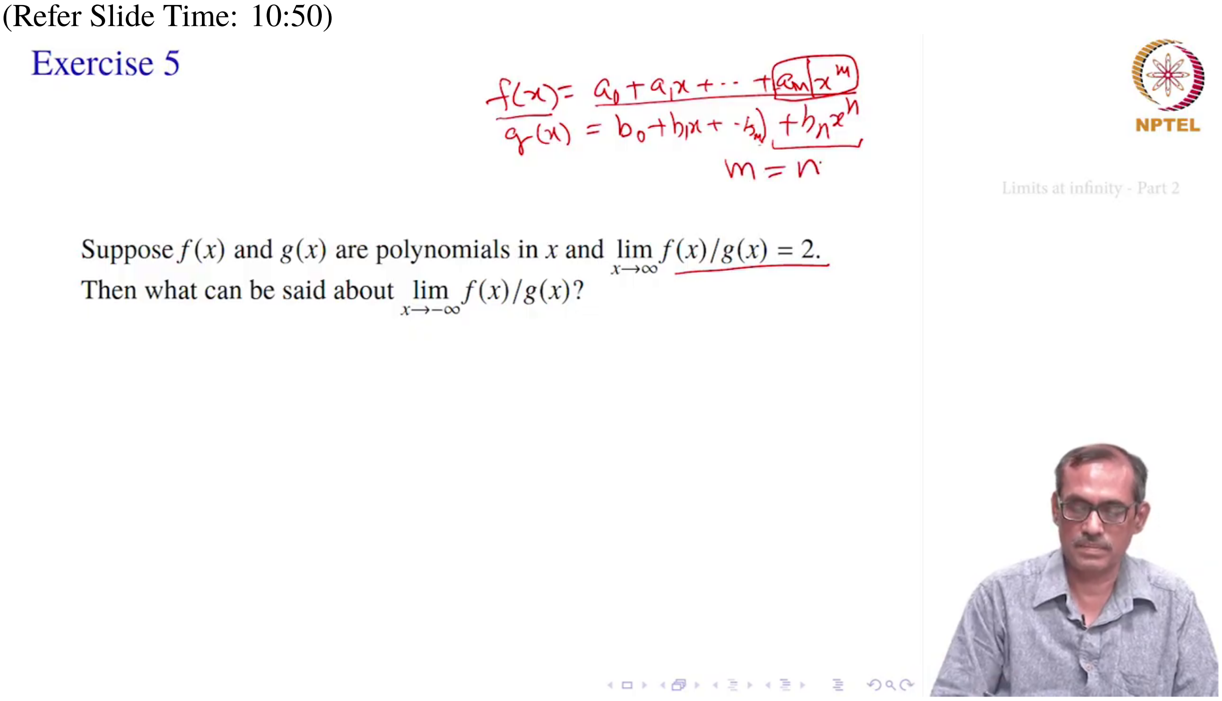

Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$. And we have $g(x)$ is also a polynomial in x. That will look something like $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$. We do not know whether m and n are connected or not. That is how it looks. So, $f(x)/g(x)$ would look something like this. It is given that the limit of $f(x)/g(x)$ as x goes to infinity, is equal to 2. So, suppose $m < n$. Then, if you divide x^m, the top one becomes $a_0 x^{-m} + a_1 x^{1-m} + \cdots$. If $m > 1$, then everything goes to 0 except the last one, right? So, there there is no x to the power something, it becomes a_m. That is the limit of the top one as x goes to infinity when you divide it by x^m. As x^m is divided, and $m < n$, in the down side, there will be some powers of x. That is, upto the mth power we may have negative powers of x, after that the powers of x will be positive. So, in the limit, it becomes infinite. Then, $f(x)/g(x)$ goes to 0. That is, when $m < n$, this limit will become 0.

And if $m > n$, then similarly the limit will become infinity because some powers of x will live on the numerator, and denominator will be constant after you divide x^n But it is given that it is something which is nonzero real number. So, this limit exists, and this idea of 'equal to infinity' does not arise. We will talk about that later. Now, it is given that $f(x)/g(x)$ has the limit equal to 2. If $m < n$, the limit becomes 0. If $m > n$, then the limit does not exist. But what we have is limit exists and it is nonzero. Therefore, $m = n$.

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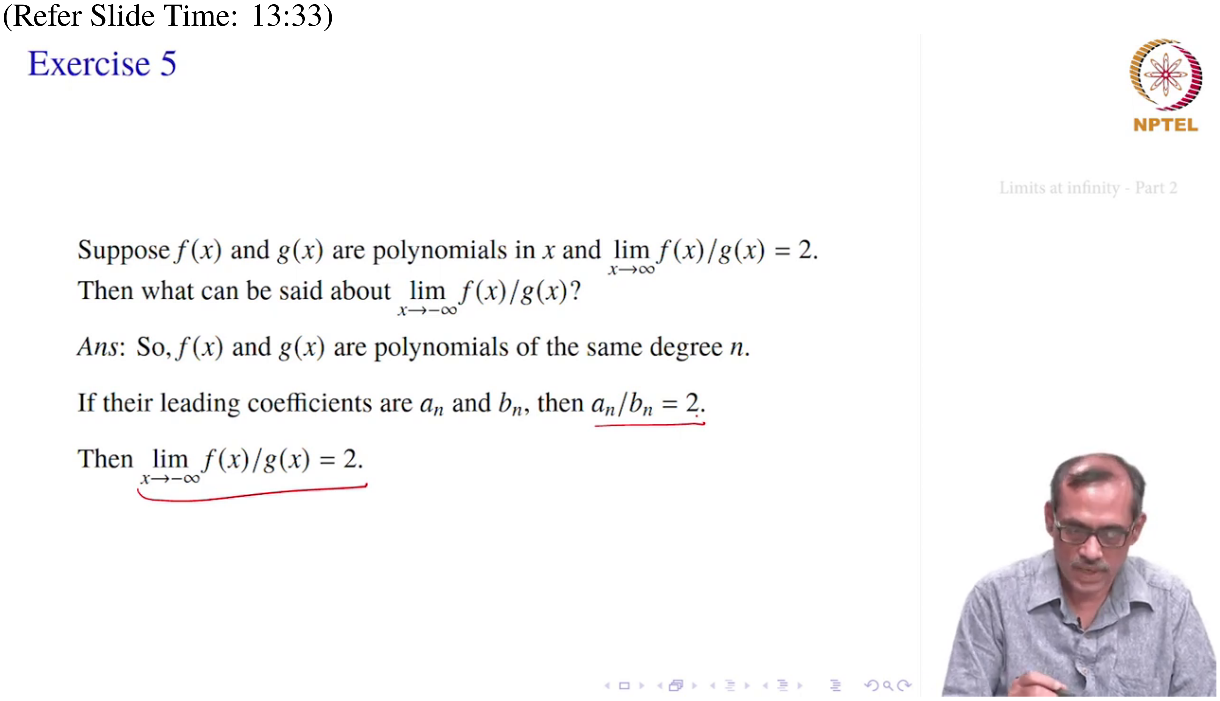

That is the first thing we find. So, the polynomials must be of the same degree. Once $m = n$, then one has to divide x^n. Now, everything becomes 0 except the last one. So, you would get a_n/b_n as the limit. That is how it will look. All the other terms will have negative powers of x and they will go to 0 as x goes to infinity. So, you must have $a_n/b_n = 2$, that is what this condition says.

What can be said about $f(x)/g(x)$ when x goes to minus infinity? Similar thing will happen when $x \to -\infty$. All terms will go to 0 after division by x^m. Only the last two will remain: a_n/b_n again. That is a constant and that will be equal to 2. That is what we write. It should be equal to 2 also. You can also try substituting x equal to $-x$; and then, $(-1)^n$ will be on the top, $(-1)^n$ will be on the bottom; they will cancel and you get again $a_n/b_n = 2$. That is an alternative way of doing it. So, let us stop here.