Basic Calculus - 1

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Lecture 6 - Part 2

Algebra of limits - Part 2

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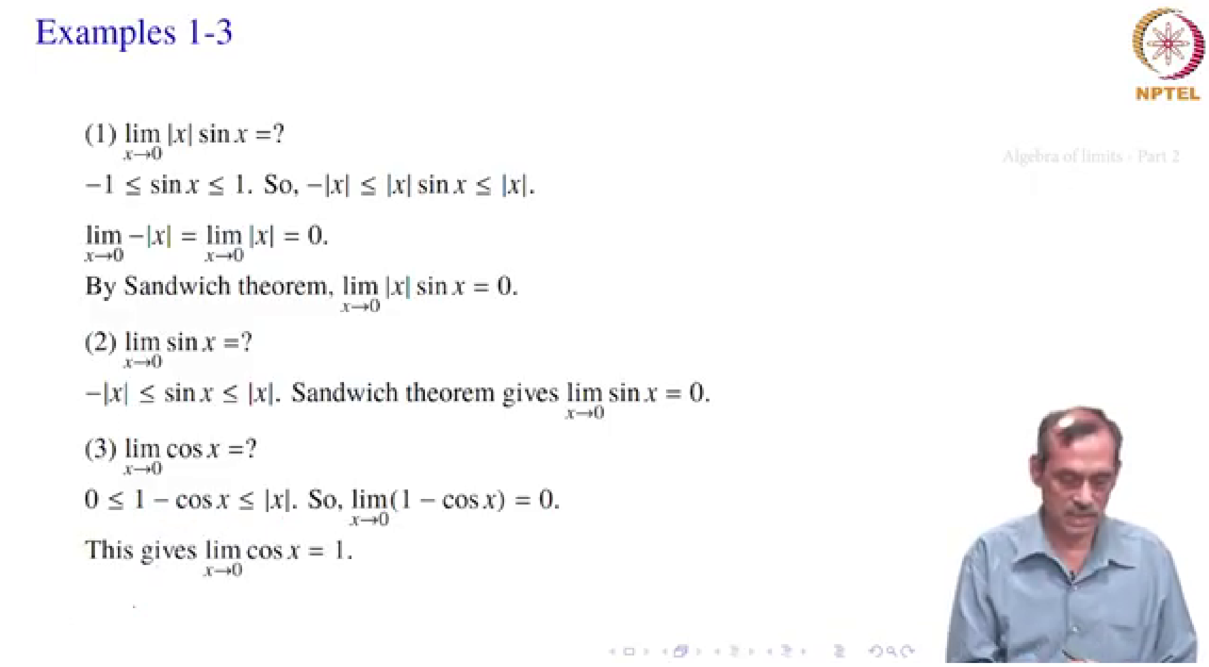

Here, we are required to find out the limit of $|x| \sin x$ as x goes to 0. As such, we cannot apply it because the we had results for polynomials, x, constants, polynomials, and so on. But you have here $|x| \sin x$. We try to estimate this from both the sides. So, think of some inequality. What it says, $\sin x$ can have maximum 1 and minimum minus 1, wherever x maybe; $-1 \le \sin x \le 1$. I multiply $|x|$. As x goes to 0, $|x| \ge 0$. So it can be multiplied. Multiplying $|x|$, we get $-|x| \le |x| \sin x \le |x|$. Now, look at the both the sides of this function. As x goes to 0, both of them go to 0. That is clear. The limit of $|x|$ and of $-|x|$ are 0. Use the Sandwich theorem. It tells you that the limit of this sandwiched function should also be equal to the same as those two limits. That is, the limit as x goes to 0 of $|x| \sin x$ is equal to 0.

Now, we ask for $\sin x$. You think of $\sin x$. It is bounded by -1 and 1; but that will not yield any limit. So, how to go about limit of $\sin x$ as x goes to 0? We use again the same inequality whatever we have done earlier, but in a different way. We have the inequality that $\sin x$ is always less than or equal to $|x|$. So, it is not -1 to 1, but it is something else. That is, $\sin x \le |x|$. Similarly, $-|x| \le \sin x$ for any x around 0. Then you use sandwich theorem. because these two limits are 0, that gives you $\lim_{x\to 0} \sin x = 0$.

These inequalities are really helpful. We have really mentioned them earlier, such as $x \le |x|$ and others. Similarly for cos x. But we do not have $\cos x \le |x|$. But something else is there. What is it? If you take $1 - \cos x$, that is less than or equal to $|x|$. This is the inequality we have discussed earlier. And we know that $\cos x$ is always less than or equal to 1. So, we can say $0 \leq 1 - \cos x \leq |x|$. Once you have this estimate, then you just see that as x goes to 0, this $|x|$ goes to 0, and this is already 0. So, by sandwich theorem again, the limit as x goes to 0 of $1 - \cos x$ is equal to 0. Then by algebra of limits, limit of $\cos x$ should be equal to 1. Because in the limit if $1 - \cos x$, the limit of 1 is 1 and total is 0. So, limit of $\cos x$ must be equal to 1. That is what it says. So, you keep in mind this kind of estimates; they will come off handy.

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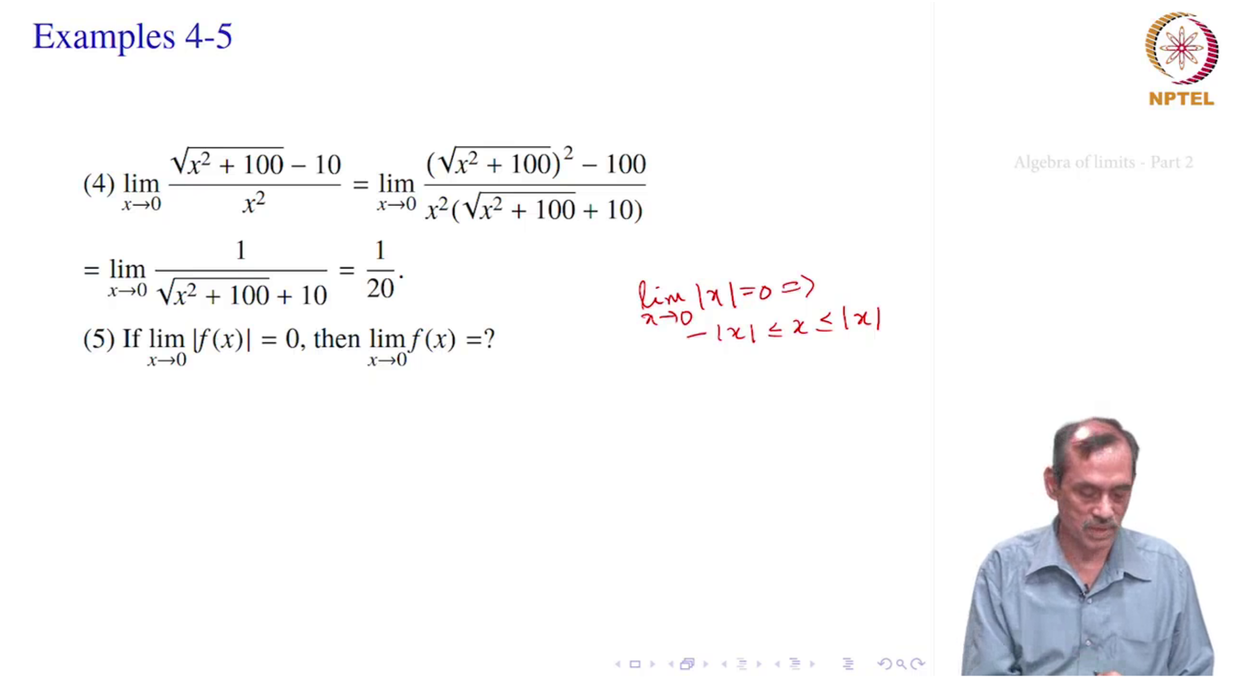

Now, let us find out another. It asks for finding the limit as x goes to 0 of this function, which is a ratio of two functions; on the top we have the square root of $x^2 + 100$ minus 10, and on the bottom we have x^2. When $x \neq 0$, these are well defined. But when x goes to 0, we cannot just apply the limit formula such as the earlier limit of $f(x)$ divided by the limit of $g(x)$, because the limit of the denominator is equal to 0. So, what do we do? We try to rewrite just like your $x^2 - 9$ divided by $x + 3$. We rewrite it the same way. We multiply square root of $\sqrt{x^2 + 100} + 10$. Earlier the denominator was 0, now it will not be 0. Our purpose is to bring it to some form where denominator may not be 0. Let us see how to proceed. This is some guesswork because you want to take away the square root also.

Once you multiply $\sqrt{x^2 + 100} + 10$, on the top, it becomes $[\sqrt{x^2 + 100}]^2 - 10^2$. That is simplified to $x^2 + 100 - 100 = x^2$. Now, this x^2 and this x^2 get canceled. That is the main idea of making the denominator non-zero. Once this x^2 gets canceled, all that remains is $\sqrt{x^2 + 100} + 10$. On the top there is 1, because this is numerator was x^2. Now, you can take the limit and the denominator is non-zero. It is the limit of square root of x^r; we apply it now. So, this is the limit of $x^2 + 100$ to the power half. The limit of $x^2 + 100$ will be equal to square root of 100; that is 10. So, 1 divided by 10 plus 10, which is 1 divided by 20.

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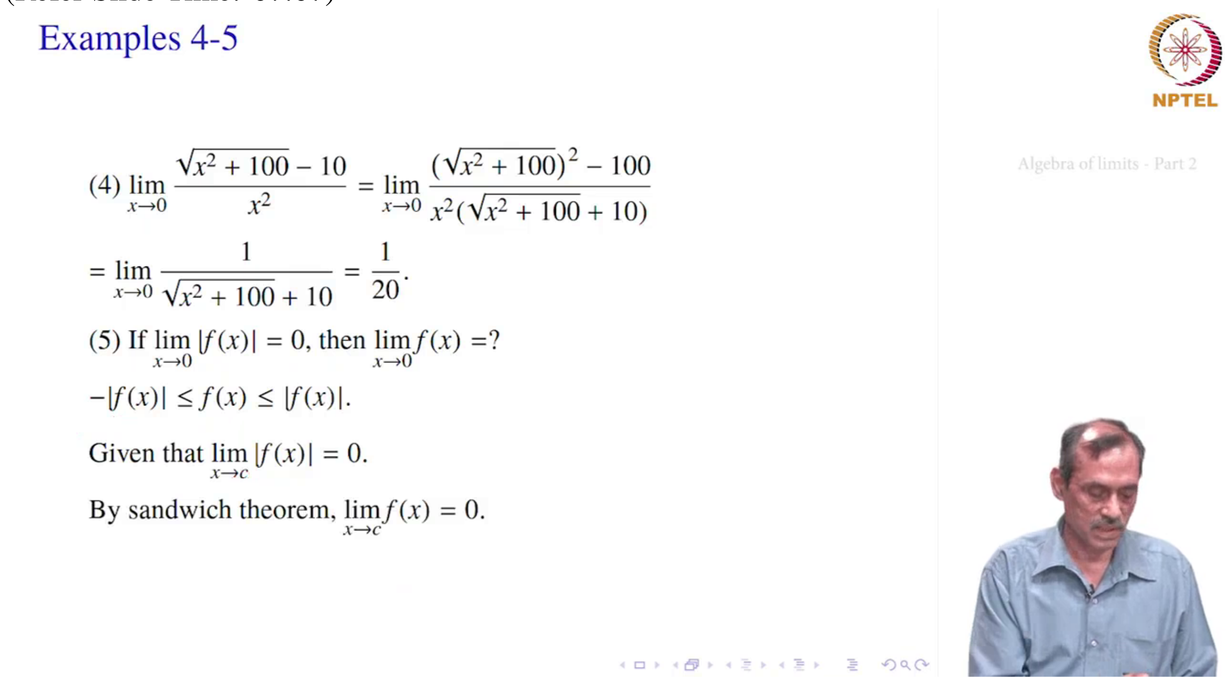

Now it is given that the limit of $|f(x)| = 0$. What can you say about the limit of $f(x)$? Does it really exist or it does not? Suppose, you take the limit of $|x|$. What will happen to the limit of $|x|$ as x goes to 0? We know that this is equal to 0. And we know that the limit of x is also equal to 0. But the question is: does it imply that the limit of x equal to 0? Well, this is correct; so, it implies. But then, how to bring x from $|x|$? We have learnt an inequality. We can see that $-|x| \le x \le |x|$. Now you can use sandwich. So we proceed in a similar way.

The limit of $|f(x)|$ is given. We can have an estimate for $f(x)$, which looks like $-|f(x)| \le f(x) \le |f(x)|$. Now, use sandwich theorem. The left side gives 0, the right side also goes to 0 as x goes to 0; therefore, the sandwiched function. So, the limit as x goes to c of $f(x)$; c is 0 here; must be 0. The limit as x goes to 0 of $f(x)$ must be 0; that is what it says.

Let us apply these techniques, whatever we have learned, to some more problems. Here, it is not given that the limits exists; it says, if possible find the following limits. If not, then you have to say that a limit does not exist.

The first one has $x^2 - 2x - 3$ in the numerator and $x^2 - 4x + 3$ in the denominator. So, first thing we have to check is when x goes to 3, what happens to the denominator. That will become really $9 - 12 + 3$, which is 0. That means $x - 3$ is a factor. So, first question we should ask, is $x - 3$ a factor of the numerator? Then we can cancel it. Let us see what happens.

So, it is not defined at $x = 3$; but it is okay because we need the limit, where x may not be equal to 3. Now, the numerator can be factored as $(x-3)(x+1)$. Just check it, with $x^2 - 3x + x - 3$, that is $x^2 - 2x - 3$. And the denominator has a factor $x - 3$. It is $(x-3)(x-1)$. After canceling; we can cancel because $x \ne 3$ in the limiting process; so, this limit as x goes to 3 is $(x+1)/(x-1)$. As x goes to 3, on the top, since limit of x goes to 3, is 3. So, this is $(3+1)/(3-1)$, which is equal to 2.

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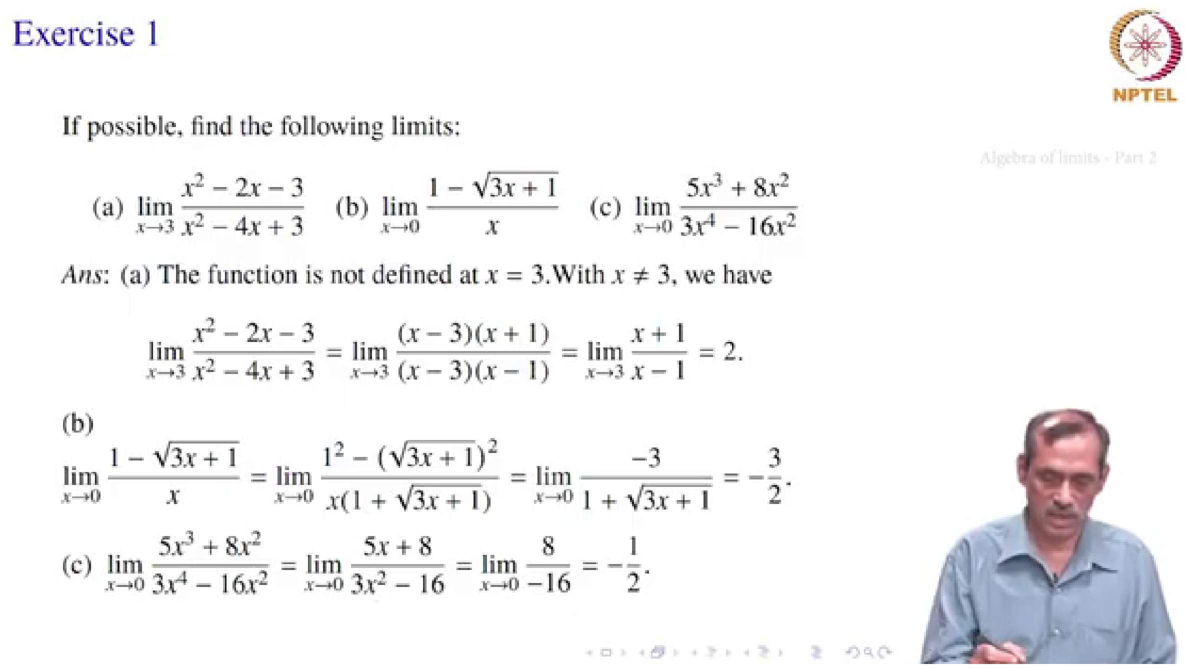

Let us see the second one, Part (b). Here, we have to find the limit of $[1 - \sqrt{3x+1}]/x$ as x goes to 0. You have to take away this x somehow. What do we do? Usually, this is the case in such problems that we have to rationalize the square root. So, we multiply $1 - \sqrt{3x+1}$. It was 1 minus on the top, now we multiply 1 plus. The numerator becomes 1 square minus this, which is $[\sqrt{3x+1}]^2$. It is really $-3x$ and this x; they get canceled. You get -3 on the numerator, and the denominator is $1 - \sqrt{3x+1}$. Now you can take the limit. Under this square root the limit of $3x$, as x goes to 0 becomes 0. So, this is square root of 1; it is really the non-negative square root, which is 1 again. So, 1 plus 1 becomes 2 on the top anyway. It is minus 3; so the limit is $-3/2$.

Let us see the third one, Part (c). It is the limit of $5x^3 + 8x^2$ divided by $3x^4 - 16x^2$, as x goes to 0. Again, the denominator goes to 0 as x goes to 0. Is any factoring possible? That is our question. We see that it is x^2. We can take it out from here. It is x^2 times $3x^2 - 16$. And the top is x^2 times $5x + 8$. So, we can cancel this x^2 and we get $(5x + 8)/(3x^2 - 16)$. And then we can find the limit of the numerator and denominator separately and divide them. That gives rise to 8; $5x + 8$ as x goes to 0 gives 8; and this is -16. So, it is $-1/2$. I think till now it is alright.

Let us go to another exercise. It is given that $f(x)$ satisfies $\sqrt{5 - x^2} < f(x) < \sqrt{5 + 2x^2}$. And this is so for x in the open interval minus 1 to 1. There only we have $f(x)$ defined. Now, $g(x)$ is another function, which satisfies $2 - x^2 < g(x) < 2\cos x$. But this x is in the interval minus half to 1 by 3. So, one is minus 1 to 1, another is minus half to 1 by 3. So, both are defined in $(-1/2, 1/3)$ anyway. And they will satisfy the same inequalities. You want to find the limit as x goes to 0 of $f(x)g(x)$. Since x goes to 0, we can even consider a smaller interval where it is defined. Both are defined in $(-1/2, 1/3)$. So, $f(x)g(x)$ is also defined there. At least we have these inequalities satisfied there.

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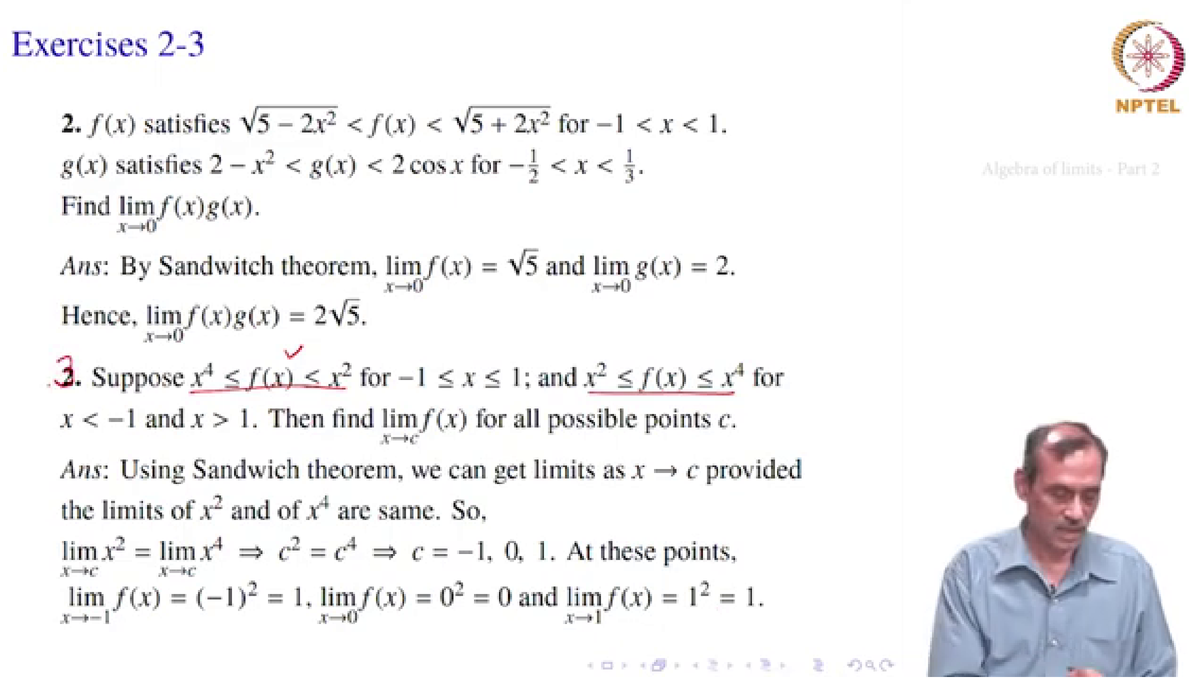

Exploiting these two inequalities, we have to do something for $f(x)g(x)$, which will be valid inside this neighborhood around 0. That δ should not exceed $1/3$ anyway. That neighborhood should be inside this, and we can always bound it that way, because limit needs only nearness, which we can make smaller and smaller. So, assume that both the inequalities hold for x near 0 and try to find out $f(x)g(x)$; that is our question.

What we see is: since both the conditions hold, the limit of $f(x)$, we can see from the sandwich theorem, as x goes to 0 is $\sqrt{5}$. So, the limit of $f(x)$ is $\sqrt{5}$. Similarly, the limit of $g(x)$, as x goes to 0, is 2 on the left side, 2 on the right-side, since $\cos x$ goes to 1. So, this becomes 2. Then, the limit of $f(x)g(x)$ must be equal to $2\sqrt{5}$. If you attempt it at a time there might be some difficulties, but this will be easier. Here it is easier to find the limit of $f(x)$ and the limit of $g(x)$ separately.

Take the third problem. Suppose $x^4 \le f(x) \le 2x^2$ in the interval $[-1, 1]$. But now it is closed interval: $[-1, 1]$. The inequality is satisfied in this interval. And similarly, $x^2 \le f(x) \le x^4$. It is the other way around now. This holds for $x < -1$, and $x^2 \le f(x) \le x^4$ for $x > 1$. That means, -1 to 1, this inequality, within that this inequality, and elsewhere we have this inequality. Then we have to find the limit of $f(x)$ as x goes to c for all possible points c.

Suppose, it is at 0. Suppose $c = 0$. Then only one inequality is satisfied; we are looking at this only. This will not be relevant for x near 0. What happens is, we see that x^4 goes to 0, x^2 goes to 0, so $f(x)$ will go to 0. The limit of $f(x)$ must be 0 by sandwich theorem, when x goes to 0.

Similarly, suppose it is $c < -1$. Let us say c is something which is less than minus 1. Then this inequality will not be applicable. This is only applicable. Then you can think of what is going on here. Similarly, if $x > 1$ only this one is applicable, the second one. There, the first inequality will not be satisfied. And x at -1 and 1, this is really satisfied; this inequality. But, in the limiting process, we are thinking of the neighborhood, which can be near but not equal to. So, the other inequality also becomes relevant. As per our procedure, we may use only the sandwich theorem. At which points you can really find the limit by using the sandwich theorem? And, this information can only be used that way as of now. Let us ask, using sandwich theorem, what limit can we get? For which c, we can get the limit?

If we want to use sandwich theorem, then in either case, it is x^4 and we have x^2; this is x^2 and x^4; their limits should be same, then only you can use the sandwich theorem. When the limits of x^2 and the limit of x^4 are same, that means, c^2 must be equal to c^4. There, the limit, as x goes to c, of x^2 is c^2, and x^4 goes to c^4. So, it says that $c^2 = c^4$, and that would give c to be one of these points $-1,\ 0$ or 1. So, if you use only sandwich theorem, no other tools are there to be used, then we can possibly find out its limit at only those points: $-1,\ 0,\ 1$. Of course, by inspection you may get it, but this really makes it solid. Well, at these points what will be the limit?

The limit of $f(x)$ as x goes to -1, will be equal to what? If $x < -1$, this inequality is applicable. If $x > -1$, then this inequality is applicable. Using both the things that are applicable, we find that when x goes to -1, both the limit of x^4 and the limit of x^2 are 1. So, by sandwich theorem, the limit of $f(x)$ as x goes to -1 is equal to 1.

Similarly, at 0 only this one will be applicable. That gives us 0 on both the sides. Therefore, the limit is equal to 0. And, when $c = 1$, we need the limit of $f(x)$ as x goes to 1. Then, this is applicable for $x > 1$, and this is applicable for $x < 1$. But in both the cases, both sides functions go to 1. So, limit must be equal to 1. This is how we solve the third problem.

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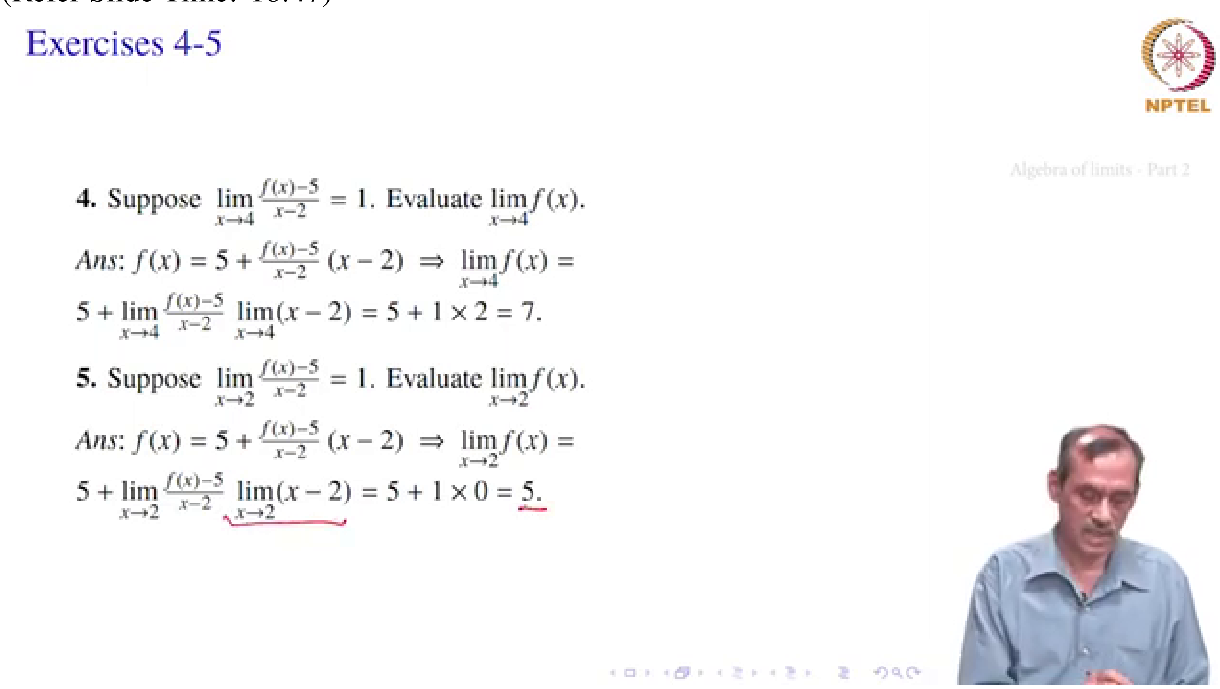

Let us take the fourth one. Suppose, the limit of $(f(x) - 5)/(x - 2)$ as x goes to 4 is equal to 1. You see here that the limit of the denominator is equal to 2. It is not 0. Then we are safe. We do not have to use any specific trick here; it should be quicker. We will just algebraically find out the limit of $f(x)$ as x goes to 4. We just rewrite $f(x)$. That is, $f(x) = 5 + [(f(x) - 5)/(x - 2)] \times (x - 2)$.

You can verify it easily. Now, using the given limit as 1, we see that this limit is equal to 1 times the limit of $(x-2)$; that gives 1 times 2. So, it is $5 + 1 \times 2 = 7$.

One more similar problem is there. It is given that the limit of $(f(x) - 5)/(x-2)$ as x goes to 2 is equal to 1. As x goes to 2, the denominator has the limit 0. And, if this limit is 0, then as we have learned earlier, the limit of the numerator must also be 0. Therefore, limit of $f(x)$ as x goes to 2 must be equal to 5.

But we can also use the earlier method of rewriting and see what happens. We can write $f(x) = 5 + [(f(x) - 5)/(x-2)] \times (x-2)$ as earlier. We then take limit as x goes to 2. That gives you 5 plus this limit, times the limit of $x - 2$. That is, $5 + 1 \times 0 = 5$. That is easy.