Basic Calculus - 1

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Lecture 5 - Part 2

Limits of functions - Part 2

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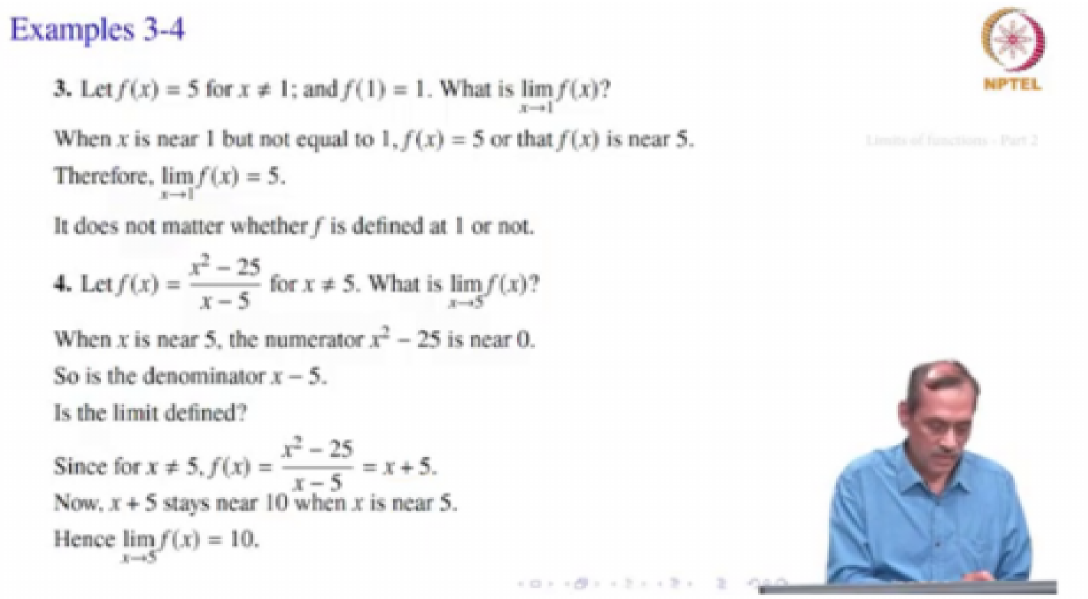

Let us take another example, say, $f(x) = 5$ for $x \neq 1$, and $f(1) = 1$. So, f is defined over the whole of real numbers; at 1 it is taking the value 1, everywhere else it is 5. Now, what should be a limit of $f(x)$ at $x = 1$? If you see your nearness argument, whatever you choose near 1, but not equal to 1, its value at that point is 5. That is, $f(x)$ is near 5; it is equal to 5 really. That is what we want to show formally, and this is what the limit would become.

Now, how do you do that formally? Well, you choose epsilon equal to delta there. Suppose an $\epsilon > 0$ is given. Choosing $\delta = \epsilon$, suppose we take $|x - 1| < \delta$. Now, when you take $f(x)$, it is 5 anyway. Then, $|f(x) - 5| = 0 < \epsilon$ as $x \neq 1$ here. Anyway, this is less than your epsilon. So, here if you choose any δ, not even this epsilon, that also does not matter. Because your value remains constant there, $f(x) = 5$ throughout. That is how we will be very verifying it formally.

Let us take the next example. Here, $f(x) = (x^2 - 25)/(x - 5)$ for $x \neq 5$; and $f(x)$ is not defined at $x = 5$. And we know it does not matter for computing the limit of $f(x)$ as x goes to 5; at different points it might. So, what happens for this? How do you get this limit? Well, when x is near 5, this x is equal to 5, this becomes 0, but x is near 5, the denominator is never 0. That means you can really simplify this. It says that you take $(x^2 - 25)/(x - 5)$ and write this as $x + 5$. This is what you can see because $x \neq 5$. That is what our definition requires. So, x remains near 5 but not equal to 5. In that case, the function $(x^2 - 25)/(x - 5)$ is really $x + 5$. Is it that it is given this way just to confuse? Well, because x is never equal to 5, you can always cancel and this turns out to be $x + 5$

really. And as you see, when x goes to 5, +5 should go to 10. That is what we feel, 10 should be the limit. And formally again, you can try to verify this by taking $\delta = \epsilon$. As earlier in our first example of $x + 2$, it goes the same way here. Fine. So, limit is equal to 10.

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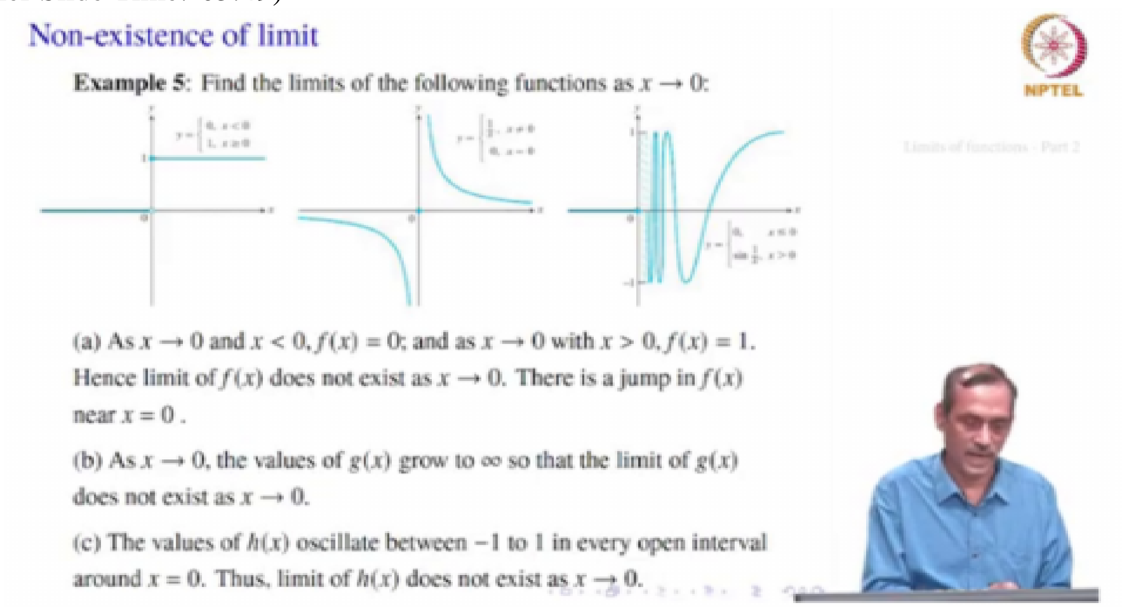

Sometimes, a limit may not exist. We want to see how it happens. So, find the limit of the following function as x goes to 0. In the first case $x < 0$, our function $y = f(x)$ is defined to be 0; and when $x \geq 0$, our function is defined to be 1. This is the graph. Here, 0 is excluded, so you see a hollow dot, a hollow small circle, and then the top is a solid one telling that 1 is included there. At that point $x = 1$ is included. That is how the graph looks.

When you take the limit of the function as x goes to 0, we have x remains near 0. That means x can be somewhere here or x can be somewhere here. On the left side, $f(x)$ becomes equal to 0, which means it remains near 0. And on the right side $f(x)$ becomes 1, that means, it remains in near 1. So, for any point x here, $f(x)$ does not remain close to anyone.

Of course, whether this side or that side, whichever side you take, it does not remain close to anyone, neither close to 0 nor to 1. If you say it is close to 0, then I would give a point on the right side and ask "what about this"? No, it does not. If you say it remains close to 1, I give one point on the left. And you see that it does not remain near that. So, it neither remains close to 0, nor it remains close to 1. And it remains close to no other number because it is either 0 or 1. Therefore, this does not have a limit as x goes to 0.

Let us look at another example; the second picture. There, we have the function as $y = f(x)$, which is 0 at $x = 0$ and it is $1/x$ when $x \neq 0$. So, at 0, it is blue small circle, and you can see that it is included. On the other side, it is $1/x$ when x is positive, also when x is negative, it is $1/x$; so it comes this way. Now, what happens when x is near 0?

When x is near 0 on the right side, you see that I can make $1/x$ as large as possible. That means if I want that my $f(x)$ should be bigger than 100, I will choose x to be smaller than $1/100$. So, I can make it large and large, as large as possible. And on the left side, I can make it as small as possible. For, suppose that you want to say that it is less than -10; then I will take something between $-1/10$ to 0 so that it becomes bigger. That is how it does not have a limit.

But it is slightly different from the earlier case. Here, on the right side we are getting very large quantities as the functional values, when x is remaining near 0. On the left side, it is taking very small values. It is not like 0 or 1. These values are not specific number here, near which it stays when it is near 0. There, of course on the left side, when x is near 0, $f(x)$ is staying near 0; and on the right side, $f(x)$ was staying near 1. Here, on the left side, it does not stay near any real number; no negatives even; and on the right side also it does not stay near any real number.

Let us look at the third one. Here it says that $f(0) = 0$. Not only that but it is also 0 for all the negatives. That means this one. This is its graph. And it is $\sin(1/x)$ when $x > 0$. You have to really plot it to see why the graph of $\sin(1/x)$ looks like this. Now, when x is remaining near 0, $\sin(1/x)$ is oscillating taking values as any number between -1 to 1. It goes on taking values between -1 to 1, it is not staying near any number between -1 to 1; but almost going near every number between -1 to 1. This is an oscillating behavior near x equal to 0.

These three cases are different. These are various ways the limit may not exist. It can vary between two numbers, jump between 2 numbers, or may not even remain close to any number. But one becomes very large one becomes very small. And, here it is really bounded between -1 to 1, but it does not remain near any number; it is oscillating between -1 to 1. That is how a limit may not exist. That is what we have explained. As $x < 0$, we see that $f(x) = 0$, and if $x > 0$, $f(x) = 1$. So, the limit really jumps between 0 and 1 at $x = 0$. In the second one, it grows to become very large or very small. And in the third one, it really oscillates between -1 to 1.

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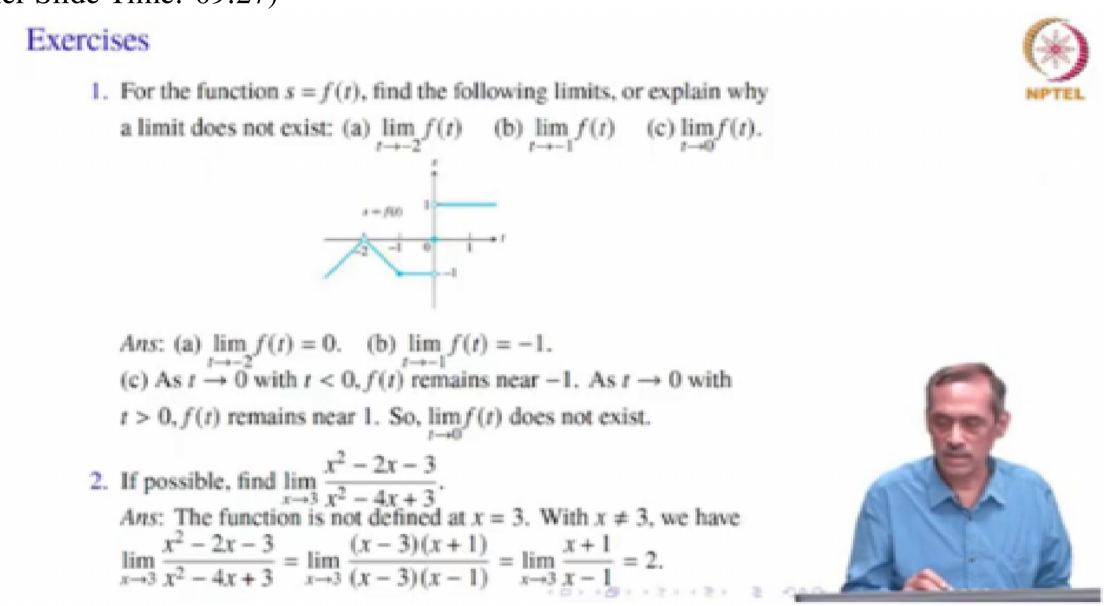

Let us see how to apply these ideas, whatever we have learned till now, on some problems. Let us take the function $s = f(t)$ instead of $y = f(x)$. Sometimes we write $s = f(t)$ also; that should not worry you. So, find the following limits or explain why limit does not exist.

In the first case, limit of $f(t)$ as t goes to -2. This is the t-axis, which was the earlier x-axis;

this is s, which was y-axis earlier. Now $s = f(t)$ is given by this picture. What does that mean? Let us read the picture. At 0, its value is 0; when $t < 0$, it has three pieces. Between -1 to 0, its value is -1; between -2 to -1, it goes as a straight line: here joining the two points $(-2, 0)$ to $(-1, -1)$; beyond -2, it is the other straight line; at -2, it is not defined; and on the other side, if $t > 0$, it is behaving like the straight line $s = 1$.

We are trying to find out the limit as t goes to -2. So, -2 is here. Other things are not relevant; we are not concerned now, because we are concerned about numbers in a neighborhood of -2. We can always limit it to a smaller window, whichever is convenient for us; because that is the way we can choose your δ and make it smaller and smaller. So, near -2 what happens? From this side it goes to, rather, it remains near 0; its value remains close to 0. On the right also, it remains close to 0. It looks like the limit should be equal to 0. That is what we see; the limit should be equal to 0. And you can justify it formally, of course.

Now, what about Part (b)? For the same function $f(t)$, we are asking the limit of $f(t)$ as t goes to -1. As t goes to -1, if you take a point here, take a neighborhood, and limit it conveniently. On the one side, it is a straight line joining these two points $(-2, 0)$ here and $(-1, -1)$ on the left side. When t is near -1, s is remaining also near -1. On the other side, $s = -1$. So, the limit should be -1. That solves Part (b).

Next, we are asked to find in (c), the limit of $f(t)$ as t goes to 0. So, at 0, we will take a small neighborhood around 0. On the left side the value of the function is remaining near -1, it is exactly equal to -1. On the right side, it is 1. So, it should not have any limit there. It does not matter whatever way $f(0)$ is defined. So, it says that as t goes to 0 with $t < 0$, $f(t)$ remains near -1, and as t goes to 0 with $t > 0$, $f(t)$ remains near 1. So, limit does not exist. Fine, I think this is clear.

Now, let us find out the limit of this function which is given as $(x^2 - 2x - 3)/(x^2 - 4x + 3)$ as x goes to 3. First thing we should check what happens for the denominator if it is 3. So, it is really $9 - 12 = 0$ if $x = 3$. But this is all right because the function may not be defined at $x = 3$ for getting a limit. Now, you have to see what happens for this. If there is a simplification, it will be easier. Yes, there is a simplification. What do we do?

With $x \neq 3$, we can factorize. The top one is $(x-3)(x+1)$ and the bottom one is $(x-3)(x-1)$. So, that can cancel and we get $(x + 1)/(x - 1)$. Now, we can take the limit of this. The top one remains near 1. This is 3, so this is $3 + 1 = 4$. And, the down one remains near $3 - 1 = 2$. We expect that this ratio should be near 2. Of course, to justify it formally will take some time. Here, how to choose your δ? But later we will find a shortcut; we do not have to formally do always everything. Our feeling is that it should be remaining near 2, and we write limit equal to 2. You can try justifying it. So, let us stop here today.