Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 1
The Real Line - Part 1

Hello, this is a course on Basic Calculus. It is indeed the first part of the Basic Calculus course. I will be here to help you learn this topic. And this is Lecture 1. We will be talking about the real line on which we have the functions. We will be doing something with the functions in this course, which are defined on the real numbers to real numbers. So, this real line will be the topic of the first lecture.

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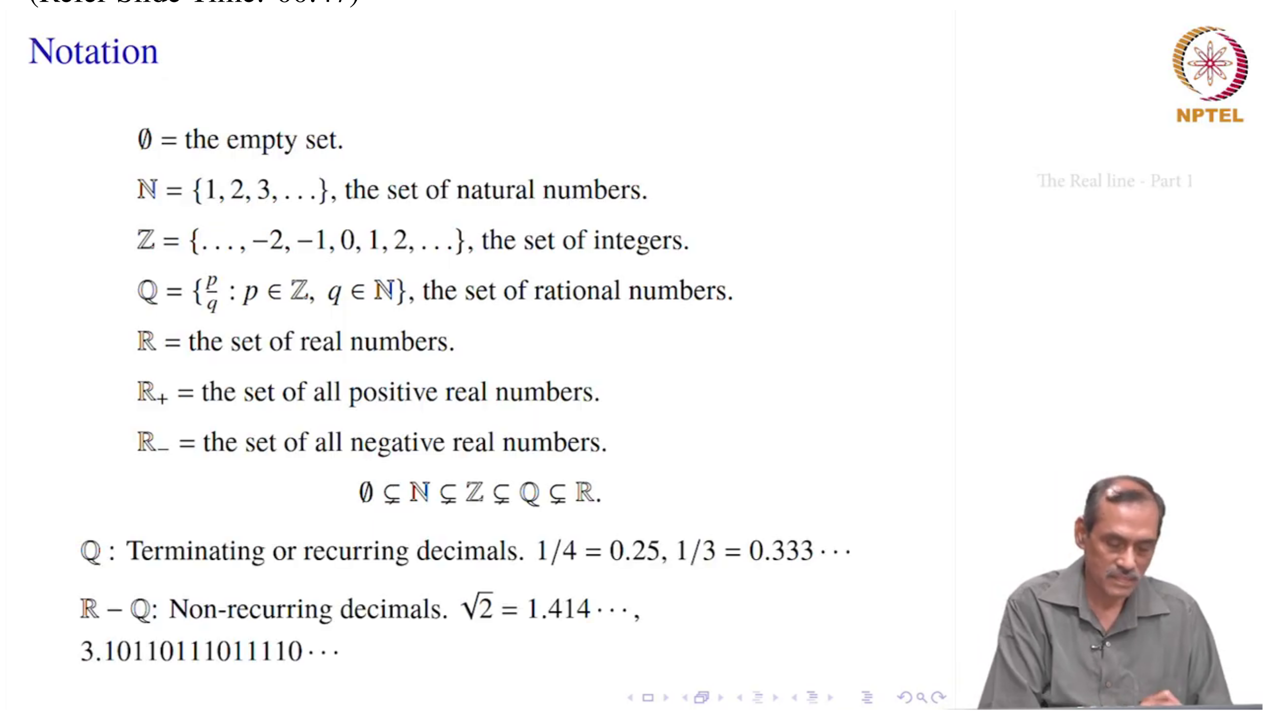

We will be talking about many things about real numbers. First, let us fix some notation. It will be helpful to read the written text. The first is the empty set, which we write like $\varnothing$; it looks like the Greek letter phi, but it is not exactly phi. We will call it the empty set. And then, the set of natural numbers which we will be writing as $\mathbb{N}$, the blackboard font N, which consists of numbers 1, 2, 3 and so on. Then we have the set of integers, which is $\mathbb{Z}$; it includes the set of all natural numbers, and along with that we have some more numbers which are the minus of all those natural numbers, and 0. We will be writing like 0, 1, 2, 3 and so on; on the other side -1, -2, and so on. In fact this 'other side' means they are ordered, but we will come to it later. And, the set of rational numbers will be denoted by $\mathbb{Q}$. A rational number looks like p/q, where p is an integer and q is a natural number, so that 0 and negative numbers are avoided in the denominator. When we write $-2/-3$, we cancel this minus sign and say that it is $2/3$. So, there is no loss in telling that all rational numbers are in the form p/q, where p is an integer and q is a natural number.

Then we have the set $\mathbb{R}$ which is the set of real numbers, our main concern. We will be defining it slowly; we will wait for some time to say what it is exactly. Then this $\mathbb{R}_+$, this will denote the set of all positive real numbers. Among the real numbers there are some which are positive, there are some which are negative and there is also 0. The set of all positive real numbers is written as $\mathbb{R}_+$. Similarly, $\mathbb{R}_-$ is the set of all negative real numbers.

Now you can see that it is in the increasing order, in the sense that the empty set is a subset of every set. So, $\varnothing \subsetneq \mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$. But it is not only the subset; the sign here says that it is a subset and it is not equal to the next one. It means a proper subset. So all that we see here is the empty set is a proper subset of $\mathbb{N}$, which is a proper subset of $\mathbb{Z}$, which is a proper subset of $\mathbb{Q}$, and that is a proper subset of $\mathbb{R}$.

And for the decimal representation, we know that the set of rational numbers consists of all terminating or recurring decimals. For example, $1/4$, which is 0.25 terminates there. After that the digit 0 is repeated, which we do not usually write. But, if you take $1/3$, which is $0.333\cdots$, the digit 3 is repeated infinitely often. So, it is called a recurring decimal where one or some finite number of digits get repeated. That is the set of rational numbers. Each rational number can be written as a terminating decimal or as a recurring decimal.

The set of real numbers includes all rationals, but it is a proper superset of rationals. Sp, there are some real numbers which are not rationals, they are irrational numbers. The set of irrational numbers is written as $\mathbb{R} - \mathbb{Q}$, over-using that minus symbol. These will have the decimal representations which are non-recurring and non-terminating. For example, $\sqrt{2}$, it is $1.414\cdots$. No digit will be recurring there. Take another, for example, $3.1011011101111\cdots$. Here, there is a pattern. We have written the pattern here to show that it is not repeating. It is an irrational number. These are the decimal representation of real numbers.

There is another alternative way of defining the real numbers. It starts with what we do with the real numbers. That question gives to some operations on the real numbers.

These operations satisfy certain properties. That gives structure to this set of real numbers. The first is the commutativity property, which says that if you take $a+b$ or $b+a$, no matter, they are the same. Similarly ab and ba are same. And there is further nicer thing. We can curtail many more brackets through the associativity property. We say that $a + (b + c)$, which means $b + c$ obtained first, then a is added to it; it is the same thing as $a + b$ and then plus c. On the right hand you see that first one we obtain is $a + b$, then add c to it. They are same. Similarly when you multiply, you take bc, multiply it with a on the left; and take ab multiply with c on the right; they are same. Of course it is commutative; so left and right will go away.

We assume that there are two special numbers in the set of real numbers, which are 0 and 1. What do they do? If you take any real number a and add 0 to it you will get back a. So you say that 0 is the identity of addition. Similarly, 1 is the identity of multiplication. We say, any real number a times 1 is equal to a itself.

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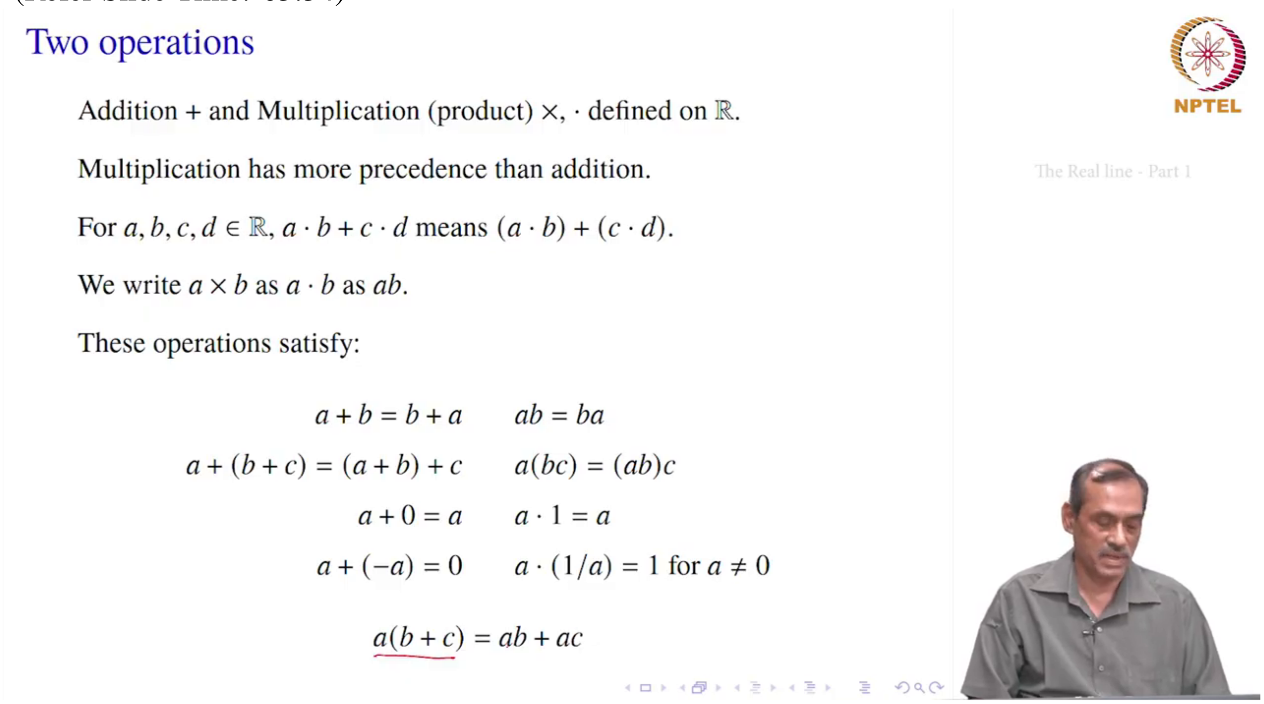

Next one is that corresponding to every a we have another number called $-a$, and what will happen to that? If you add them together you will get 0, the identity element of plus. And similarly for multiplication if you take a times $1/a$, you will get 1. But here we have a condition that a should not be 0. Because we want to avoid $1/0$. So, $a \times (1/a) = 1$ whenever $a \neq 0$. That is, corresponding to any non-zero number a, we have another such number, say $1/a$ which when multiplied give 1.

In fact, from all thes properties we will be able to find that corresponding to this a this $-a$ is unique; corresponding to this a which is non-zero, this $1/a$ is also unique. Then we will assume that uniqueness here. Now, how do they interplay? That is the distributive property. It says that multiplication distributes over addition. So, $a(b + c)$ is equal to ab, ac and both of them added together. So, these are now the properties which happen for the operation of addition and multiplication.

We assume that there are two operations defined on the set of real numbers; one is written as +, which is addition, and other is product or multiplication which we write sometimes with $\times$ symbol, cross or with just a $\cdot$. Here we assume that multiplication has more precedence than addition, which will just curtail some brackets while writing. When you write $a \times b + c \times d$, this will mean $a \times b$ first and then $c \times d$ next, and both of them are added. We will not interpret it otherwise. So, that is the meaning of telling multiplication has more precedence over addition. Also, this $a \times b$ will be written as $a \cdot b$; sometimes $\cdot$ is also omitted and we just right ab.

And over and above we have something else, a relation defined on the real numbers. It is a binary relation. It talks about two elements and what relation they have in-between. So, suppose a and b are any real numbers and we have this relation: we will write it as $<$, less than. This property holds: which is, either $a < b$ or $b < a$ or $a = b$. This is called the Law of Trichotomy; it holds for this order relation $<$.

Then we have the associativity or sometimes called transitivity. It says that if $a < b$ and $b < c$, then $a < c$. And, we have another property, it says how this less than behaves with respect to addition. It is really preservinng addition. That is, if $a < b$, then $a + c < b + c$ for whatever a, b, c are. And with respect to multiplication we have a restriction there. That is, if c is positive, $< c$, $a < b$, then you get $ac < bc$. What happens when c is negative? We will come to it. The result follows from these properties; we will see later.

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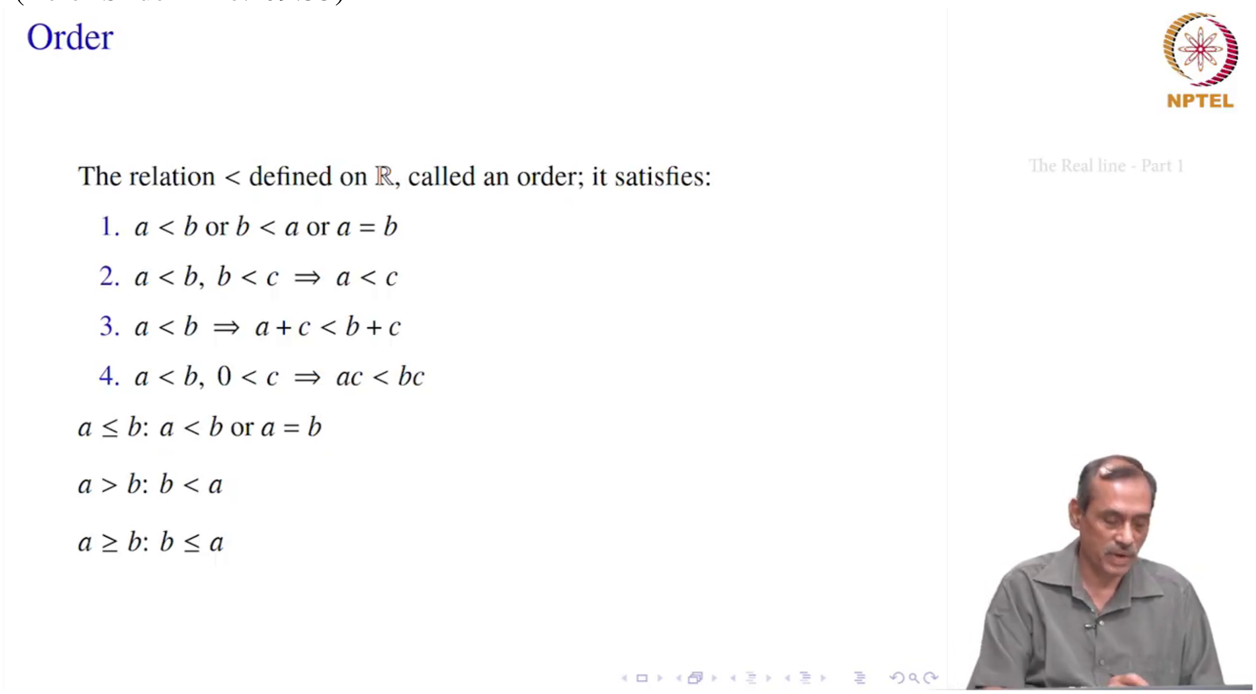

There is a notation here. For $a < b$ or $a = b$, we will have an abbreviation. We will write $a \leq b$. And, there is another abbreviation. We write $a > b$ for $b < a$. This is another symbol which is really coming from less than. Similarly, we will say $a \geq b$ whenever $b < a$ or $b = a$.

The next property is the most important one. It helps really a lot to prove many theorems; it is called the Completeness Property of the set of real numbers. We will come to that slowly but first let us see how these order relations are behaving. From the above properties it will follow that if $a < b$, then $a - c < b - c$. What is the meaning of $a - c$? It is again an abbreviation for $a + (-c)$. Similarly on the right side, it is $b + (-c)$. This property says that if $a < b$, then then you can take away something, say c from both the sides and still the less than relation is maintained.

Similarly, if $a < b$ and $c < 0$, that is, c is negative, then the inequality is altered. It is no more $ac < bc$, but it is $ac > bc$. And, if c is positive, remember that ac will be less than bc. If it is negative then it will change the inequality. Now if $a < b$, then $-b < -a$. This is how you can now think how the other property is related. Now, take the sixth one. Due to the earlier fact that c is positive gives $ac < bc$, and c is negative we gives $ac > bc$. This is now reflected here. If $a < b$, then $-b < -a$. And if a is positive, then $1/a$ is also positive. This is another property which follows from the earlier properties.

If we take any two numbers, say, a and b such that a is in between 0 and b, then both a and b are positive with $a < b$. When you take the reciprocal that is 1 by, then the inequality will be reversed. You will say that $0 < 1/b < 1/a$. These are all the properties we know and we are familiar with. So we are just recalling these; and it also fixes the notation.

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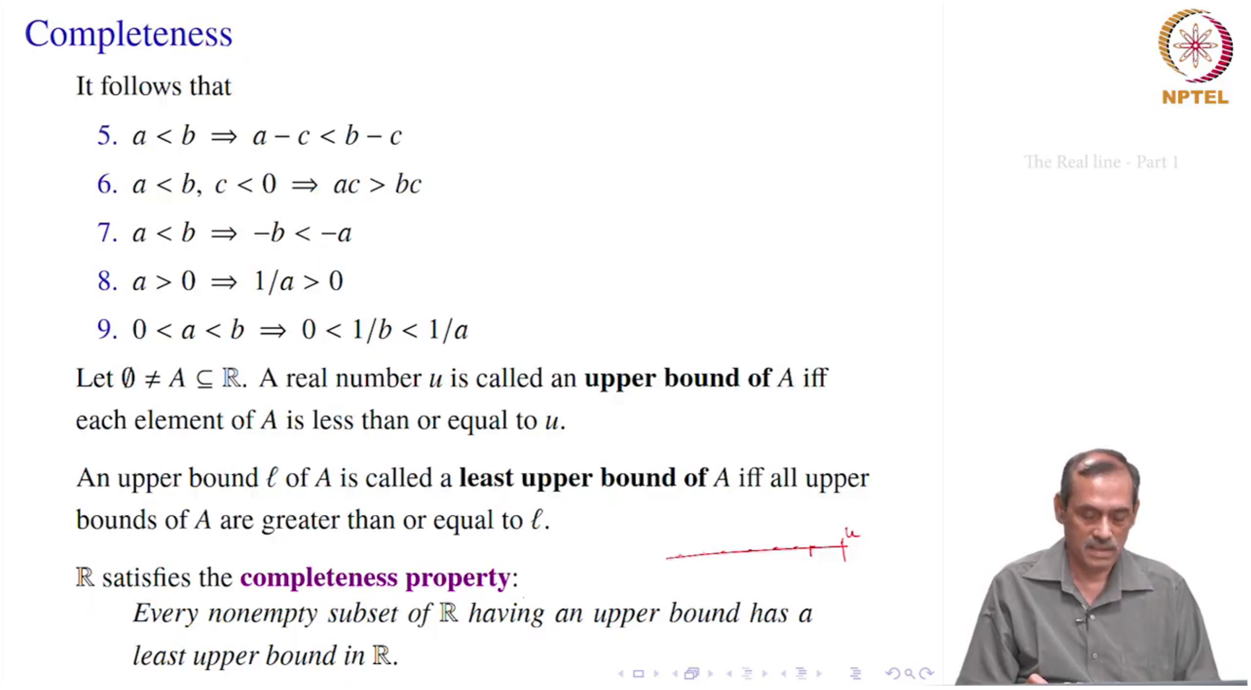

Now we will be talking about the Completeness Property, for which we need a small definition. It will be useful and helpful for writing this completeness property in a compact way. Suppose we start with a set of real numbers A and it is a not an empty set. Which means there are some real numbers in A. Then take another number u which is from $\mathbb{R}$; it may be inside A, maybe outside A, it does not matter. So, a real number u is called an upper bound of this set A, if each element of A is less than or equal to u. That means something like if your A is somewhere here, then u can be to the right of A; so that u is greater than or equal to every element of A. That is why we say that u is an upper bound of A.

Suppose we take one upper bound of A which is ℓ. We call this ℓ as a least upper bound, if all upper bounds of A are greater than or equal to l. That is, it is the least of all the upper bounds of A. That is why we will call it the least upper bound. Sometimes we just write lub for this, lub instead of least upper bound.

And now we can state the Completeness Property. It says that every non-empty subset of $\mathbb{R}$ having an upper bound has a least upper bound in $\mathbb{R}$. Suppose you have some set A here and then you have some upper bound u; then you can always find something which is the least of all those upper bounds. That looks very obvious but not satisfied everywhere. For example in $\mathbb{Q}$, it is not satisfied. $\mathbb{Q}$ does not satisfy the Completeness Property; what is the reason? Suppose you take the set of all numbers which are less than $\sqrt{2}$. Then the least upper bound of this set will be $\sqrt{2}$. We know that $\sqrt{2}$ is not a rational number; it is irrational, which is not in the set. So, $\mathbb{R}$ really satisfies completeness but $\mathbb{Q}$ does not. However, $\mathbb{N}$ satisfies this property also. But $\mathbb{N} \neq \mathbb{R}$ because there are some other properties which is not satisfied by $\mathbb{N}$.

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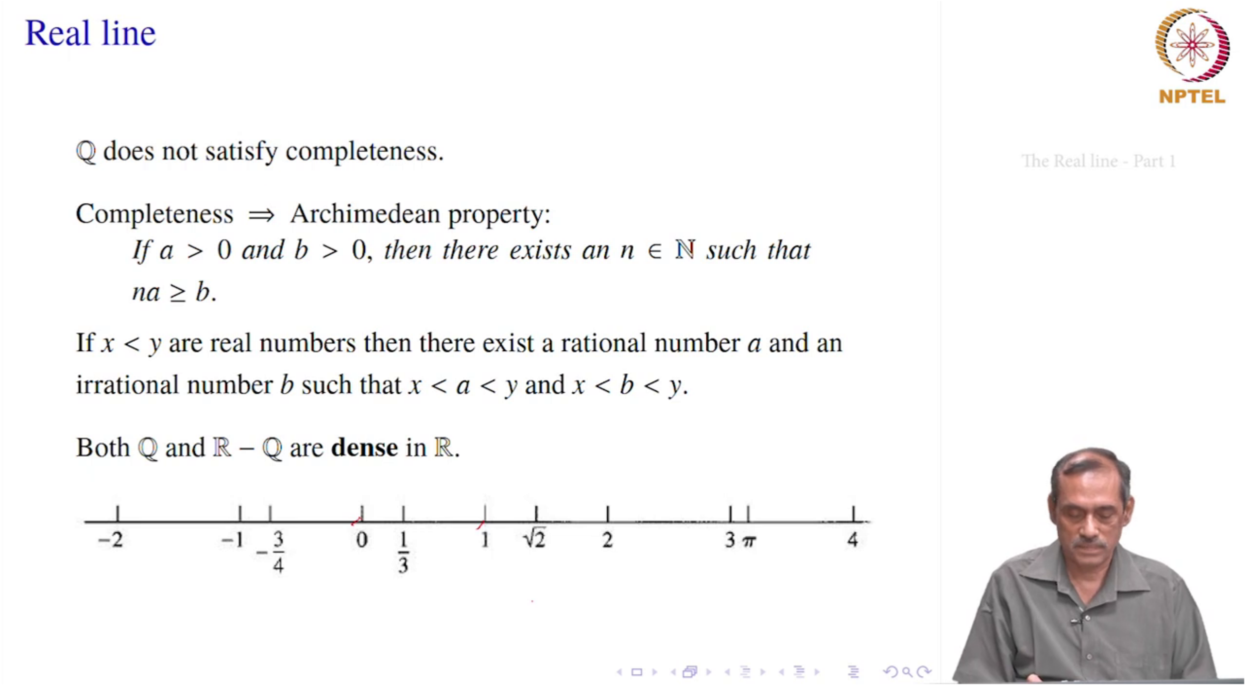

The most important thing inference from this Completeness Property is the so-called Archimedean property. It says that if you have two positive numbers a and b, then you will always find one natural number n such that $na \geq b$. This really strikes in; it is told in a funnier way because of Archimedes.

Suppose you have a very big bath tub and you have a very small spoon and you try to empty the bath tub with the spoon. Then it says that after sometime you will be able to do it. The bath tub will be empty by the spoon. It may take more time because this n can be large. There is no problem. But there exists one n such that na will be greater than or equal to b. This is called the Archimedean Property. We may not really use it very explicitly. But there are some theorems where exactly this Archimedean Property and this Completeness Property with its full strength will be used.

Let us look at one property which follows from this Archimedean Property. We are not going to prove it of course; it is called the denseness of rationals. Suppose you choose any two real numbers x and y where $x < y$. Then, it says that there is always a rational number between these two numbers x and y. That is, if x and y are real numbers, then there exists a rational number a such that $x < a < y$. Similarly, the irrational numbers are also dense in $\mathbb{R}$. That means given any two real numbers x and y, where $x < y$, you will always find one irrational number b such that $x < b < y$, in between this two numbers. So, that is what we write: both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are dense in $\mathbb{R}$.

These properties help us to visualize real numbers as the real line. The set of real numbers can be thought of as a line. If we have marked 0 somewhere and 1 is another, say this is our unit length.

So 0 to 1, then 2 will be here, these are the natural numbers 3, 4. And here on the left side, we mark the negatives of that. So, we have the integers inside $\mathbb{R}$. Then we have rational numbers inside. You take any fraction, say, $1/3$; it is here, and that is marked on the real line itself. And there are also irrational numbers like $\sqrt{2}$, which is between 1 and 2, also π, an irrational number; we will meet also e and so many others. We can think of the set of real numbers as a real line having this order property. The numbers which are on the left are less; those on the right are bigger; like 1 is bigger than 0, and minus 1 is smaller than 0, and so on. This is how we can visualize the real line.

On this real line we are going to fix some more terminology. So, today we will be simply devoting most of our time to terminology. These are called the intervals. These are very nice subsets of real numbers; they look like line segment on the real line.

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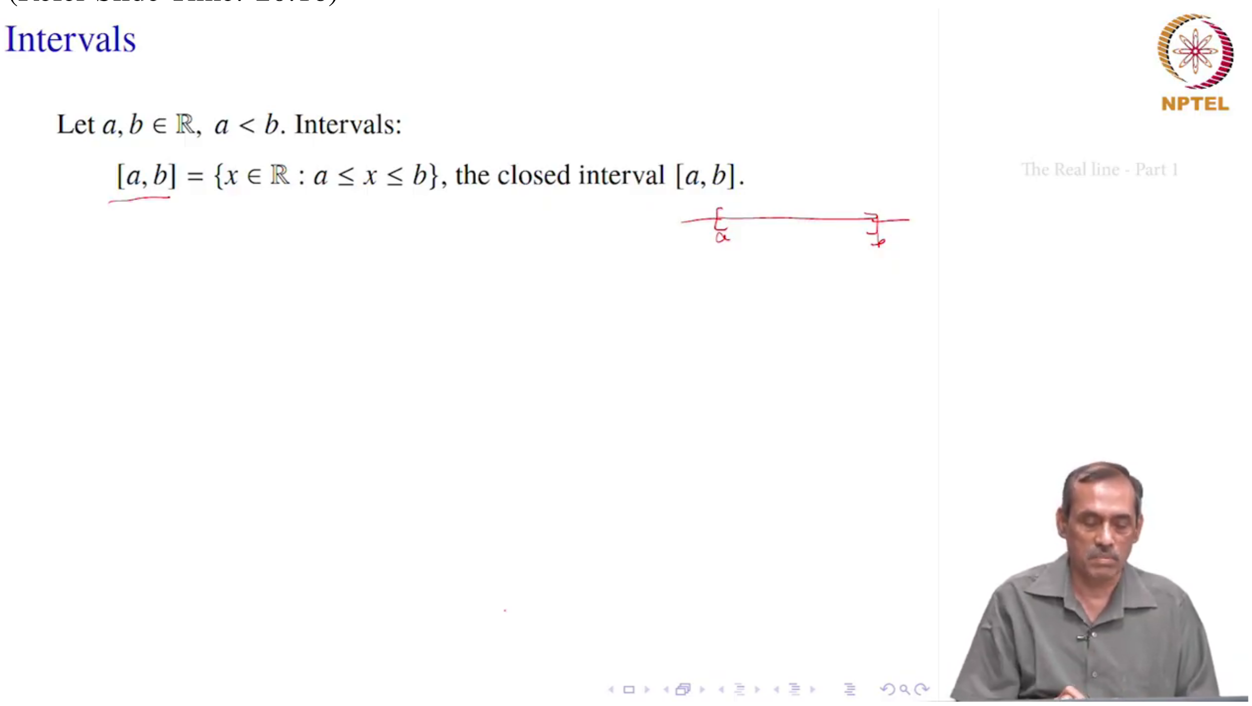

Suppose we start with two real numbers a and b where $a < b$. This notation which reads $[a, b]$, with closed left bracket and then closed right bracket, the square brackets, will denote the set of all numbers between a and b and including those two also. So, $[a, b]$ consists of all numbers inbetween and along with a and b. The numbers less than a and bigger than b are not inside that interval; they are in $\mathbb{R}$, of course. This is the closed interval. We will say closed interval a b.

Then we have similarly left open interval. Look at the parenthesis in $(a, b]$; it is not the bracket, it is a parenthesis. We read it as open a, closed b. It is a semi-open interval $(a, b]$. It consists of all numbers in the closed interval $[a, b]$ except a. Since a is omitted, we write an open bracket. Similarly, we have the other interval: on the left side it is closed, and on the right side it is open: $[a, b)$. That means a is included but b is not. It consists of all real numbers between a and b including a but not b.

Similarly, both can be open so that both are excluded: (a, b). This is called an open interval.

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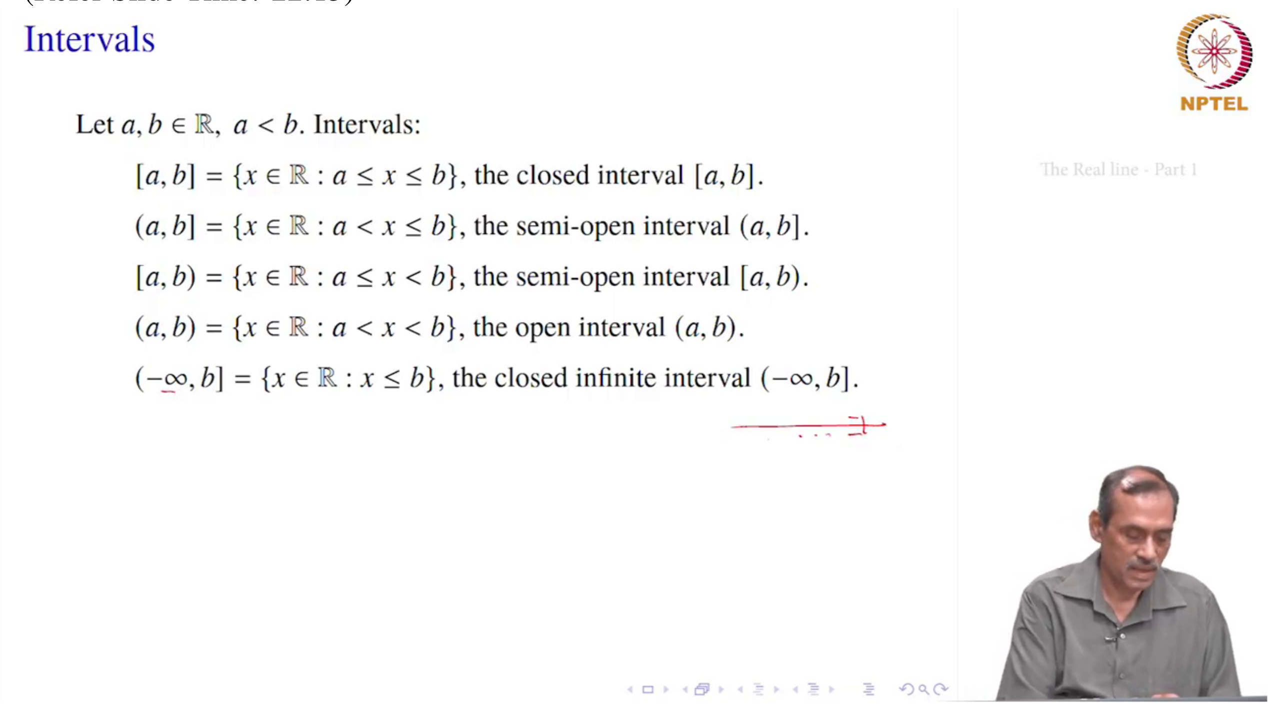

Now we introduce this symbol ∞; this is the infinity symbol. This is not a real number. It is just a notation. Writing some set in this form: $(-\infty, b]$; it has to be open because $-\infty$ is not in $\mathbb{R}$; it means it is the set of all numbers which are less than or equal to b. Here, nothing to the right of b is includes, but everything to the left is included. So minus infinity is omitted, that is why this open paranthesis. In fact, it is not a real number, it is just a notation. It is the set of all real numbers less than or equal to b.

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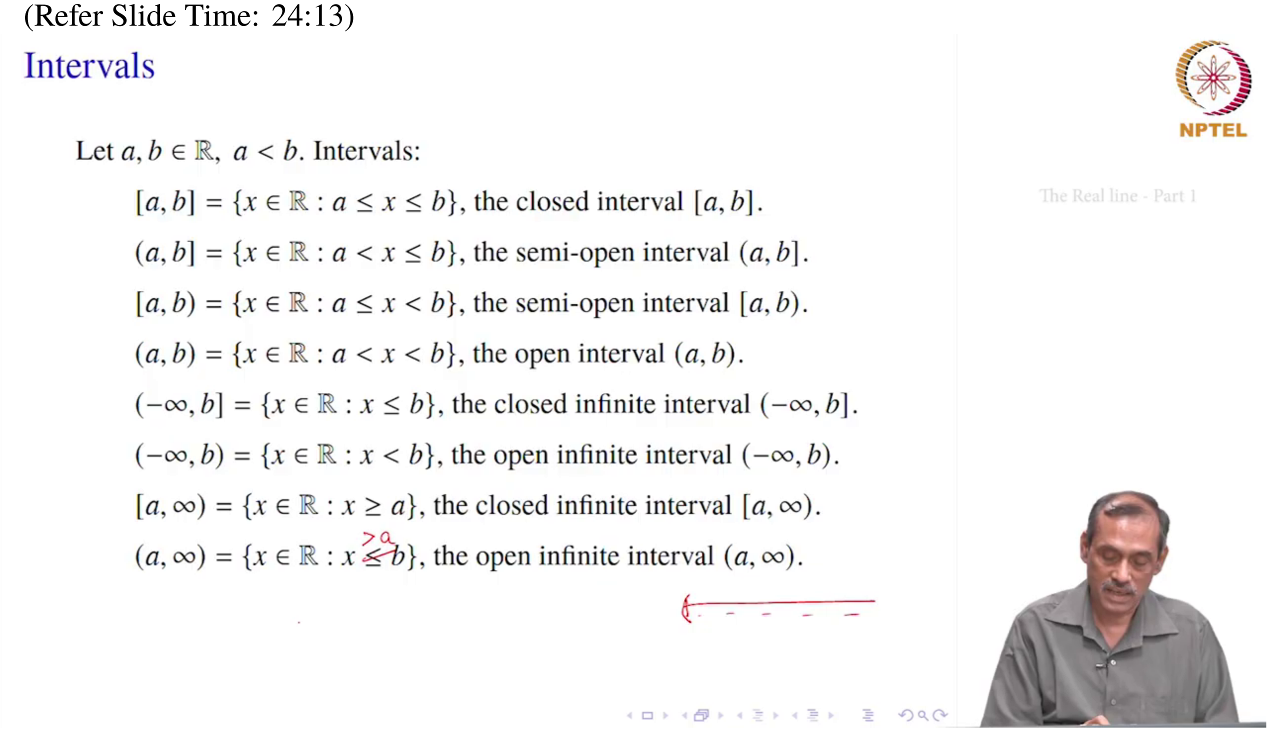

Similarly, we have $(-\infty, b)$, where at b it is also open. It includes all real numbers which are less than b. And then we have $[a, \infty)$; it is the set of all real numbers which are greater than or equal to a. Here, a is included but nothing to its left, but everything to its right is included.

Then we have (a, ∞). It includes all numbers which are greater than a. All numbers to the right of a are included, but a itself is not.

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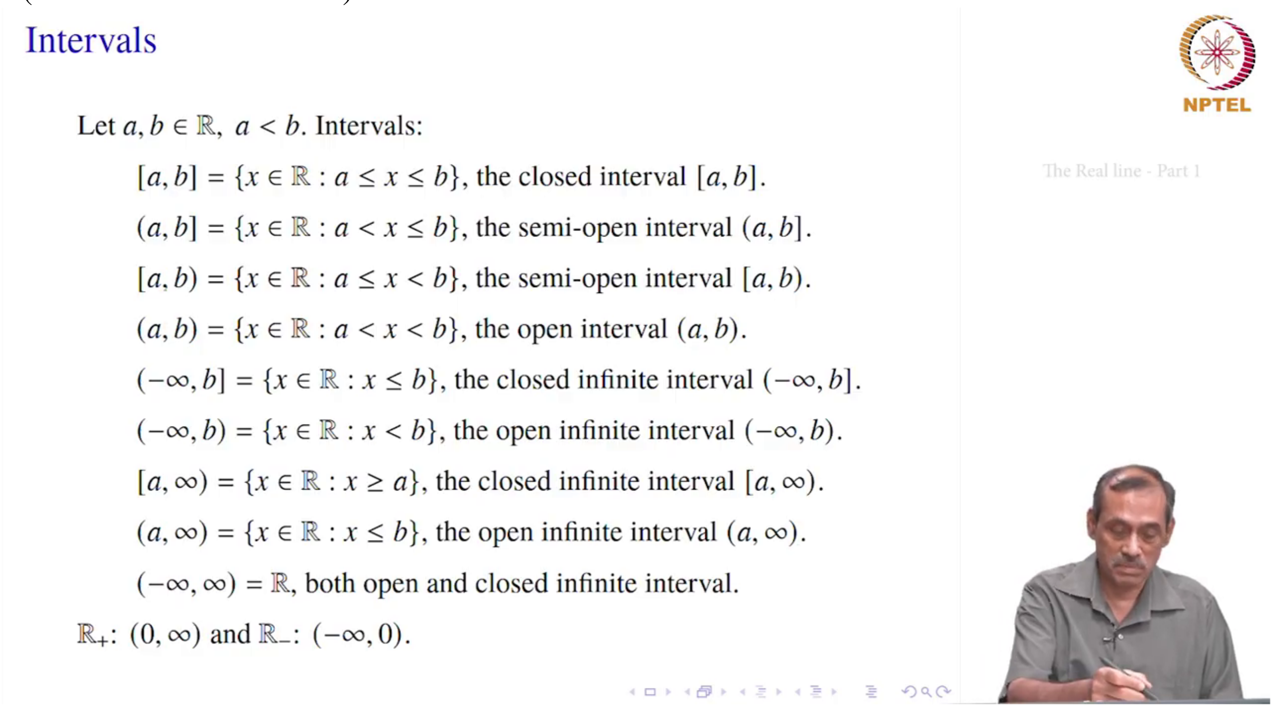

Similarly, we can have $(-\infty, \infty)$; that means all real numbers. There is no a specified here. So, this is another notation for $\mathbb{R}$ itself. It is considered to be both open and closed infinite interval.

Then we have the positive real numbers which we can now write in this way: $(0, \infty)$, because both open means all the real numbers bigger than 0. That is exactly $\mathbb{R}_+$, all positive numbers.

Similarly, $\mathbb{R}_-$ will have all real numbers less than 0, the negative real numbers.

All these sets are called intervals. Whenever we write one of these, that is called an interval. When we later say that "let I be an interval", then it will mean that I can be any of these.