Variational Calculus and its Applications in Control Theory and Nano mechanics
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Special cases / Invariance
Part-2

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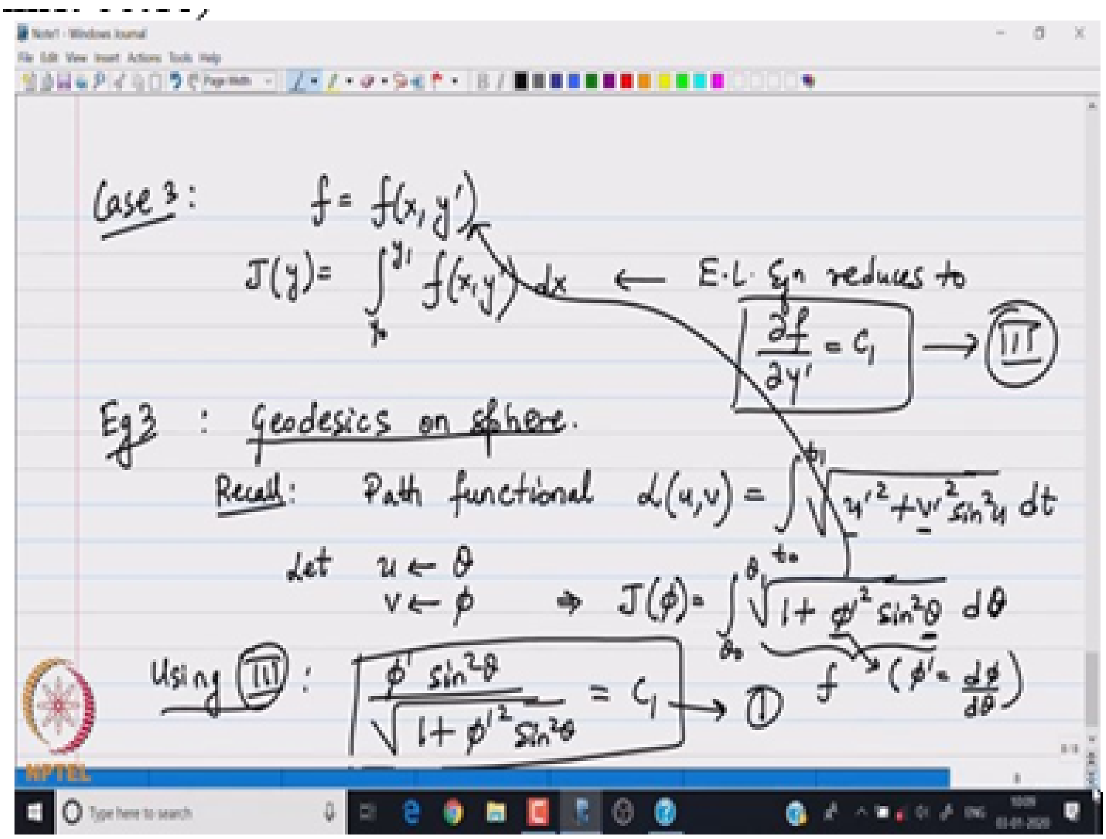

Our next case that is the third case . So, the case where f is explicitly a function of x and y', i.e f = f(x, y') so there is no explicit dependence of y on the integrand f. So, in this case functional will have the following form,

$$J(y) = \int_{y_o}^{y_1} f(x, y')dx$$

And then if we recall my Euler-Lagrange equation involved partial derivatives with respect to y and partial derivatives with respect to y'. So, the partial derivatives with respect to y is going to vanish because there is no explicit dependence on y in this integrand.

So in this case my Euler-Lagrange equation reduces to the following simple form $\frac{\partial f}{\partial y'} = C_1$ **III**
So, that can be very quickly derived from the Euler-Lagrange equation.

Now, one example that comes right away is the example of geodesics on the surface of a sphere. Well, the geodesics on the plane is nothing but straight line and that falls in the case 1 discussion. In case 3 discussion is the geodesics on a sphere. This is one very nice example.

So, let me just recall. In our first lecture I had introduced the functional for the geodesics on a sphere. So, let us recall from our first lecture that my geodesic or my path functional we are trying to minimize, the path of a curve lying on the surface of a sphere. So, my path function

Recall: Path functional L(u, v) = $\int_{t_o}^{t_1} \sqrt{u'^2 + v'^2 \sin^2 u}dt$.
where these derivatives $u^{'}$ and $v^{'}$ these are the derivatives taken with respect to the parameter t, Let replace variable u by θ and v by ϕ

$$J(\phi) = \int_{\theta_o}^{\theta_1} \sqrt{1 + \phi'^2 \sin^2 \theta} d\theta$$

Using **III** $\frac{\phi^2 \sin^2 \theta}{\sqrt{1+\phi'^2 \sin^2 \theta}} = C_1$ **1**

So, what we have assumed is θ is our independent variable and ϕ depends on theta. So, we have completely removed the parameter t. So, instead of working in the parametric form we are now directly working in the form of the two variables θ and ϕ.

notice that the quantity in the denominator is certainly bigger than the quantity in the numerator, so the quantity in the numerator cannot exceed the quantity in the denominator because the sin function is bounded above by 1. So, we are guaranteed that the quantity in the denominator is bigger, so what this says implicitly is the following.

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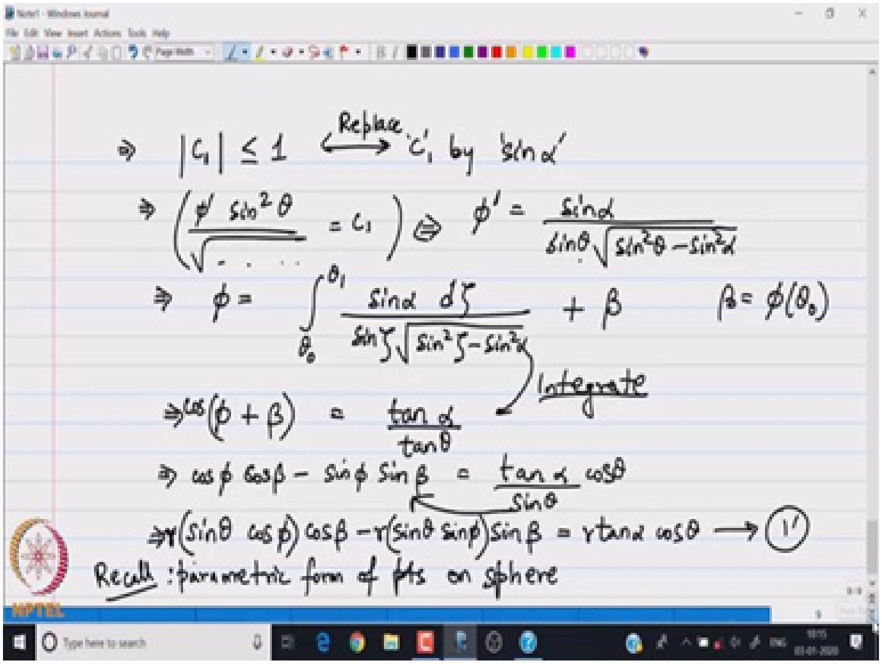

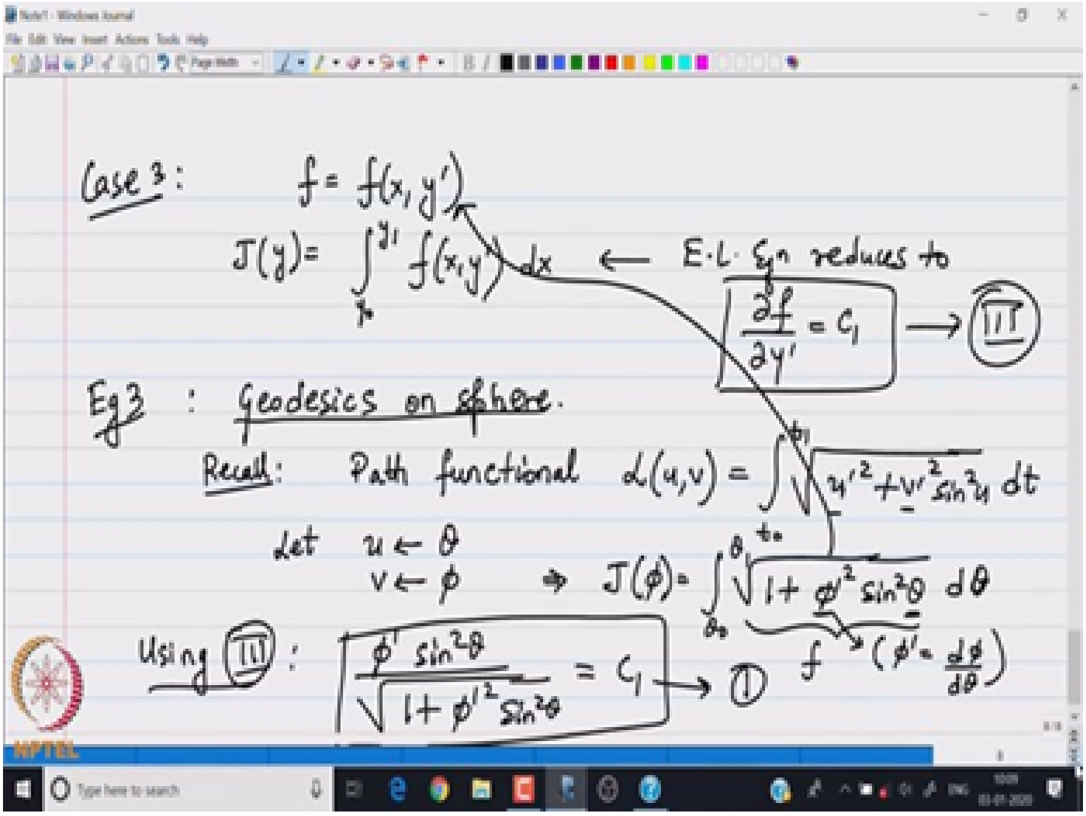

if the particular quantity is less than 1 which means that the absolute value of the quantity on the right hand side is also less than 1. So, $C_1 \leqslant 1$. So, which means I can replace C_1 let us say a quantity which is bounded above by 1, replace C_1 by $\sin\alpha$.

We will soon see why we are doing that. and let me rewrite my equation for the extremal.

$$\Rightarrow \frac{\phi^{'} \sin^2\theta}{\sqrt{1+\phi'^2\sin^2\theta}} = C_1 \qquad \Leftrightarrow \qquad \phi^{'} = \frac{\sin\alpha}{\sin\theta\sqrt{\sin^2\theta - \sin^2\alpha}}$$

So, then the next step will involve integrating ϕ with respect to θ. So, ϕ is integral from θ_o to θ_1 , so notice that since we are using the limits θ_o and θ_1 so let me just change the variable θ with another dummy variable ζ.

$$\Rightarrow \phi = \int_{\theta_o}^{\theta_1} \frac{\sin\alpha d\zeta}{\sin\zeta\sqrt{\sin^2\zeta - \sin^2\alpha}} + \beta$$

Where β is constant of integration, where β is nothing but ϕ at the original point, the starting point θ_0

So, then once we integrate, I am not going to go through all the steps of the integration because that is part of the students who have done multivariate calculus, but let me write down the final answer after integrating.
$\cos(\alpha+\beta) = \frac{tan\alpha}{\tan\theta}$

$$\Rightarrow \cos\phi\cos\beta - \sin\phi\sin\beta = \frac{tan\alpha\cos\theta}{\sin\theta}$$

$$\Rightarrow r(\sin\theta\cos\phi)\cos\beta - (\sin\theta\sin\phi)\sin\beta = rtan\alpha\cos\theta \qquad \mathbf{1}^{'}$$

Notice that recall the parametric form of points on sphere. So, students should recall the parametric form of the points on sphere whose x component is $r\sin\theta\cos\phi$, $y = r\sin\phi\cos\theta$ and $z = r\cos\theta$ where r is the radius of the sphere

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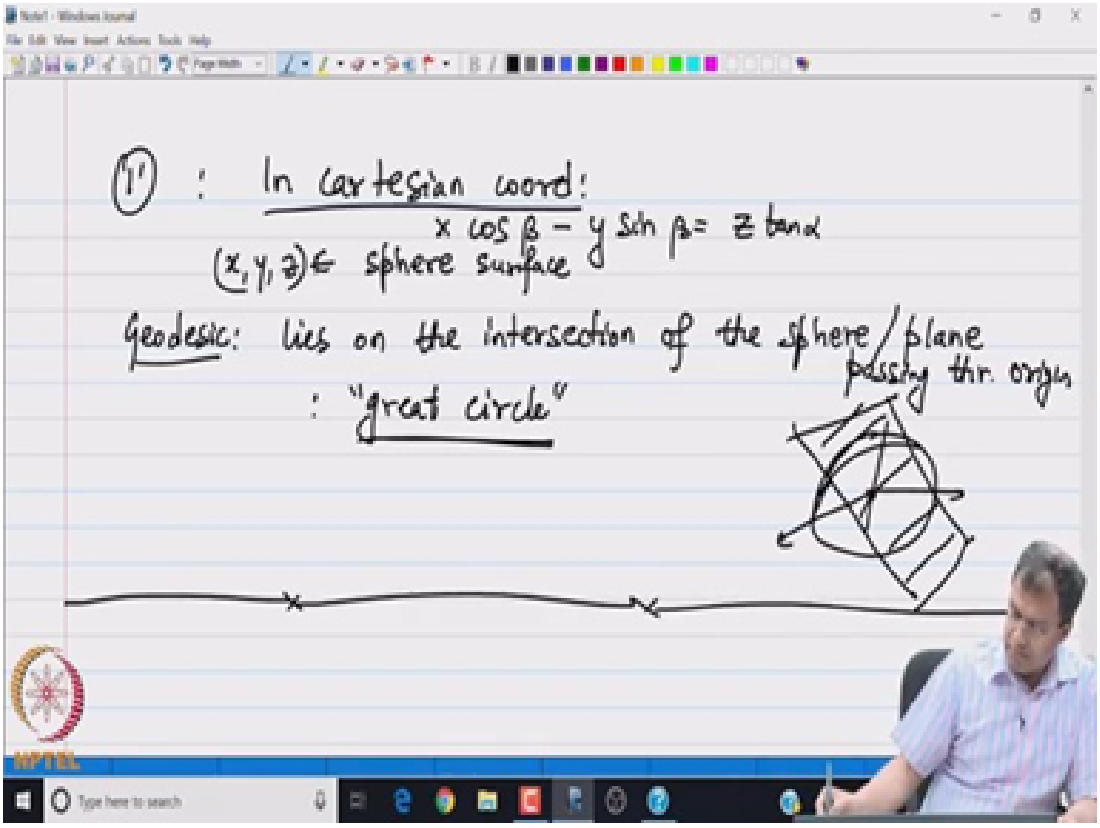

we can rewrite in cartesian coordinates $x\cos\beta - y\sin\beta = z\tan\alpha$, where (x, y, z) $\in$ sphere surface.

And further (x, y, z) satisfy equation of a plane which passes through the origin because (x, y, z) = (0, 0, 0) certainly satisfies this equation. So, my extremal, so the moral of the story is that the geodesic or the extremal solution lies at the intersection of the sphere and the plane which passes through the origin.

So, geodesic lies on the intersection of the sphere and the plane passing through the origin.

So,let me just show the diagram quickly, here we have a sphere and we have a plane. We have a plane which cuts the sphere and it passes through the origin and then the intersection of this plane and the sphere can only be an arc which is the equator of the sphere, which is the great circle. So, the great circle is one of the equator circle with the largest diameter, so that is the geodesic on the sphere.

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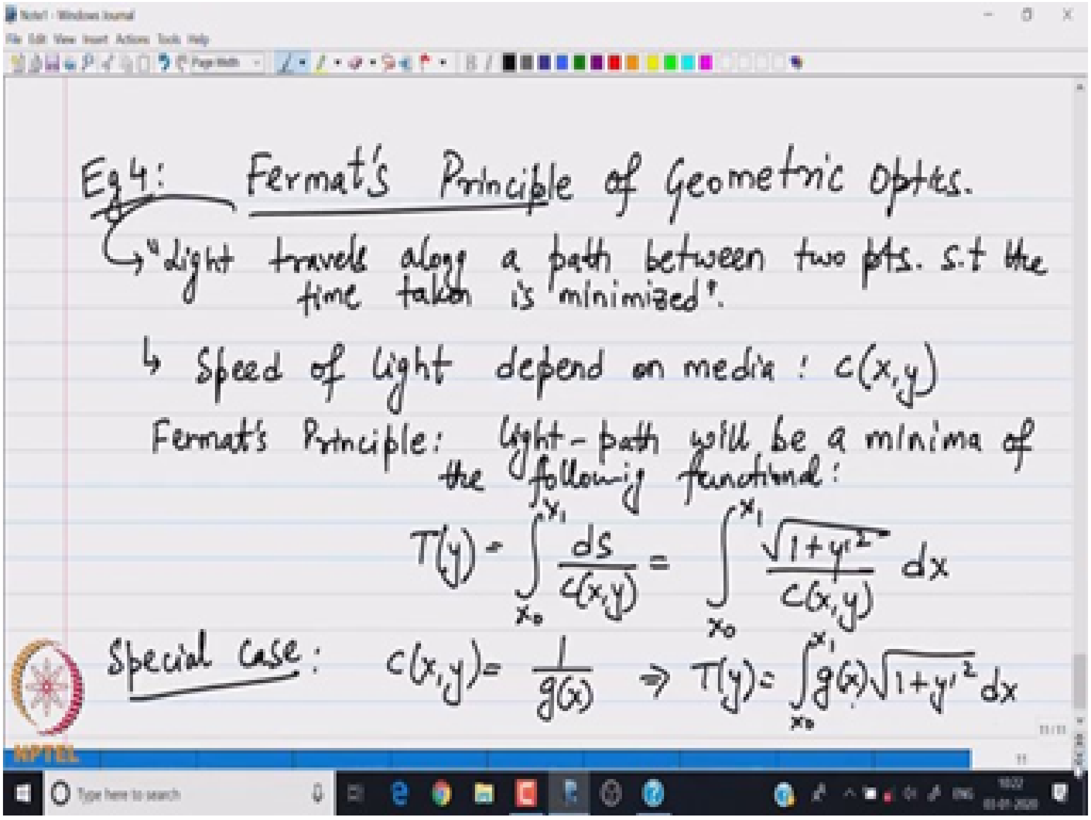

So, then let us look at another example which falls under this case, very interesting example that is how light travels through different medium. So, students who have done science is class 12th, they must have recalled the Snell's law and the Fermat's principle of geometric optics namely if we have different media we have that the light follows a path such that certain relation is satisfied. So, students in class 12th are taught certain relations on how the light travels through different medium.

In today's lecture I am going to show it using calculus of variations as to how to get that result. So, the result is as follows: so the Fermat principle says that light travels along a path, see nature always follows a principle such that it obeys the minimum energy, minimum time, minimum path and so on. So, in this case the light is going to travel along a path between two points such that the time between the travel of these two points is minimized, So, to set up this problem let us say that the speed of light, in general the speed of light depends on the medium in which it travels, so it dependent, it depends on the media.

Let me denote the speed of light by C(x, y), so I am doing a problem in two dimensions in cartesian coordinate x and y. So, let me say that in general the speed of light depends on both the coordinates x and y. So, the Fermat's principle says the following.

The Fermat's principle says that the path of the light will be a minima of the following functional. So, the functional that we have is the time functional, so the time functional is the integral of the arc length or the length of the path traveled by the light divided by the speed of the light which is the time.

$$T(y) = \int_{x_o}^{x_1} \frac{ds}{C(x,y)} = \int_{x_o}^{x_1} \frac{\sqrt{1+{y'}^2}}{C(x,y)} dx$$

So, then in order to further solve this problem let us use a simplifying assumption.so let me call this as a special case. So, the special cases we have C(x, y) = $\frac{1}{g(x)} \Rightarrow T(y) = \int_{x_o}^{x_1} g(x)\sqrt{1+{y'}^2}dx$

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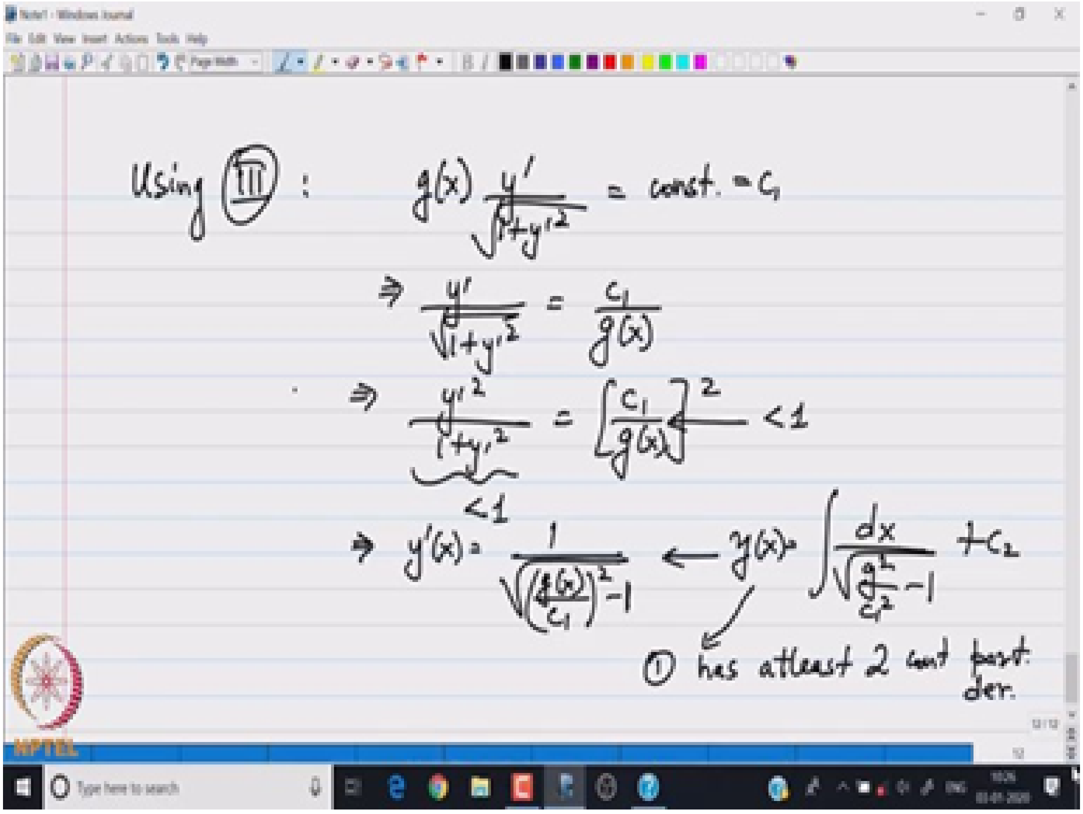

So, let me use the boxed result **III** to see that my extremal is going to satisfy the following expression

$$g(x)\frac{y^{'}}{\sqrt{1+y^{'2}}} = constant = C_1$$

$$\Rightarrow \frac{y^{'}}{\sqrt{1+y^{'2}}} = \frac{C_1}{g(x)}$$

$$\Rightarrow \frac{y^{'2}}{1+y^{'2}} = \left[\frac{C_1}{g(x)}\right]^2 < 1$$

Notice now that the quantity on the left hand side is less than 1. So, which means that the quantity on the right hand side is also strictly less than 1.

$$\Rightarrow \qquad y^{'}(x) = \frac{1}{\sqrt{\left(\frac{g(x)}{C_1}\right)^2 - 1}}$$

$$\Rightarrow \qquad y(x) = \int \frac{1}{\sqrt{\left(\frac{g(x)}{C_1}\right)^2 - 1}} dx + C_2$$

C_2 is constant of integration. So, further integration of this expression is not possible unless and until we know the exact form of this function g. But we can go ahead and see some intuitive results. We can still extract the physics out of this problem.

So, well, before we do that we must mention the following. y is the extremal, so y has at least two continuous partial derivatives, that is the underlying assumption of the Euler-Lagrange equation and that is fine as long as the velocity here defined by this function g is continuous. The moment we have discontinuities we have some problems. So, let me look at the case.

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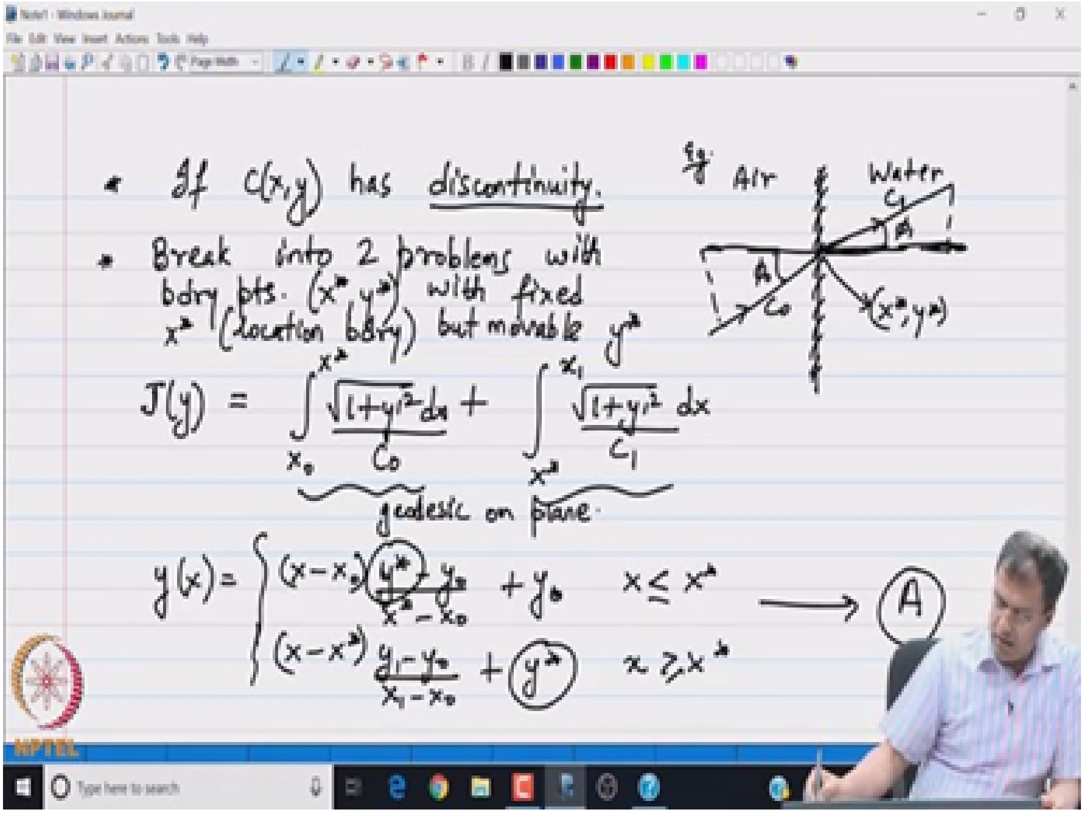

So, what I have just described is as follows. let me draw the figure. So, suppose we have two media, let us say we have air and we have water so just an example here. So, we have two media and we have light which passes through the first media and the light which passes through the second media and of course the speed of light in the first media will be very different from the speed of light from the second media which means that in this case C(x, y) will have discontinuities, either discontinuity of first order or second order and so on. So, the speed of light is different from in air than in water and we will see that light travels very differently in the different media.

So, in this case now I cannot use my Euler-Lagrange equation to solve the problem, because the moment the velocity has discontinuity the speed has discontinuity, we cannot use our underlying assumption of having two continuous derivatives of the integral will fail.

So, then we can still move ahead, so let me just draw this problem a little bit more carefully, let me say that the speed of light, this path taken by the light makes an angle ϕ_o and the path taken by the light in water in the second medium makes an angle ϕ_1 with the horizontal.

So, in this case what we are going to do is, well of course there are discontinuities when we consider the problem with two media but in each of the single medium problems the path taken by the light has continuous solution. So, we can definitely apply our Euler-Lagrange equation in each of the media and try to combine the solution somehow.

So, the idea is as follows, what we do is we break the problem into two problems with boundary points x^*, y^* such that we do a little bit more simplification such that the x component of the boundary point is fixed and I have movable y^* .

So, this point, so moving the y^* is going to change my angles ϕ_o and ϕ_1 but x component of the location of the boundary is completely fixed. So,

$$J(y) = \int_{x_o}^{x^*} \frac{\sqrt{1+{y'}^2}}{C_o} dx + \int_{x^*}^{x_1} \frac{\sqrt{1+{y'}^2}}{C_1} dx$$

So let me call this speed of light is C_o in the first medium and in the second medium the speed of light will be C_1 . So, now we see that each of these integrals are purely a function of $y^{'}$. So each of individual functionals falls under case 1 or the extremal solution to these functionals are nothing but straight lines. So, this is all geodesic on a plane. So, this is the solution, the extremal solution will be all geodesic on plane which are straight lines.

So, let me right away write the solution, the solution y of x will be straight lines such that the first line passes through (x_o, y_o) and (x^*, y^*)

$$y(x) = \begin{cases} (x - x_o)\frac{y^* - y_o}{x^* - x_o} + y_o & x \leq x^* \\ (x - x_o \frac{y_1 - y_o}{x_1 - x_o} + y^* & x \geq x^* \end{cases} \qquad \textbf{A}$$

So x_o, y_o, x_1, y_1, x^* these are all fixed quantities except y^* which is the variable of the problem to begin with and now the solution to this combined problem will involve minimization of this function with respect to this variable y star.

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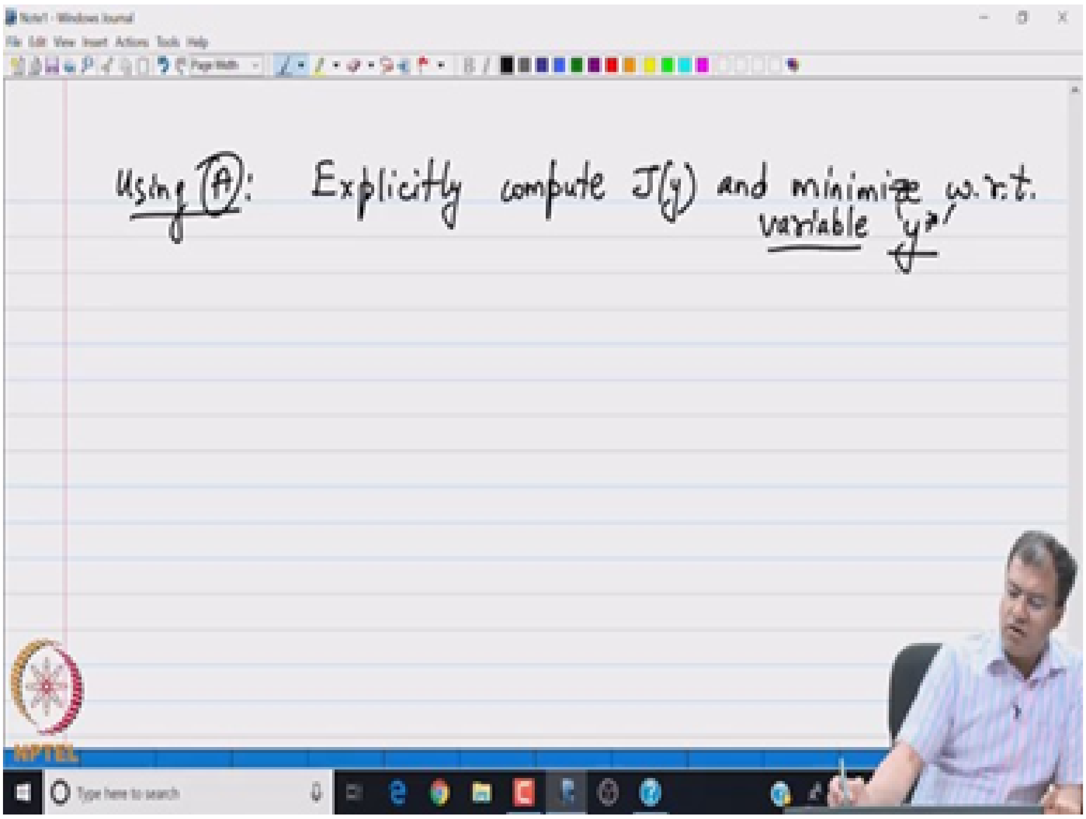

So, to get my final solution we have to do as ordinary minimization or the multivariate minimization with respect to the parameter y^*. So, explicitly compute J(y) and minimize with respect to the variable y^*. Notice now we are not doing a functional optimization, but we are doing an ordinary function optimization. So, we use the standard first derivative, second derivative tests to find the quantity y^* which minimizes our function y.