Variational Calculus and its Applications in Control Theory and Nano mechanics
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Special cases / Invariance
Part-1

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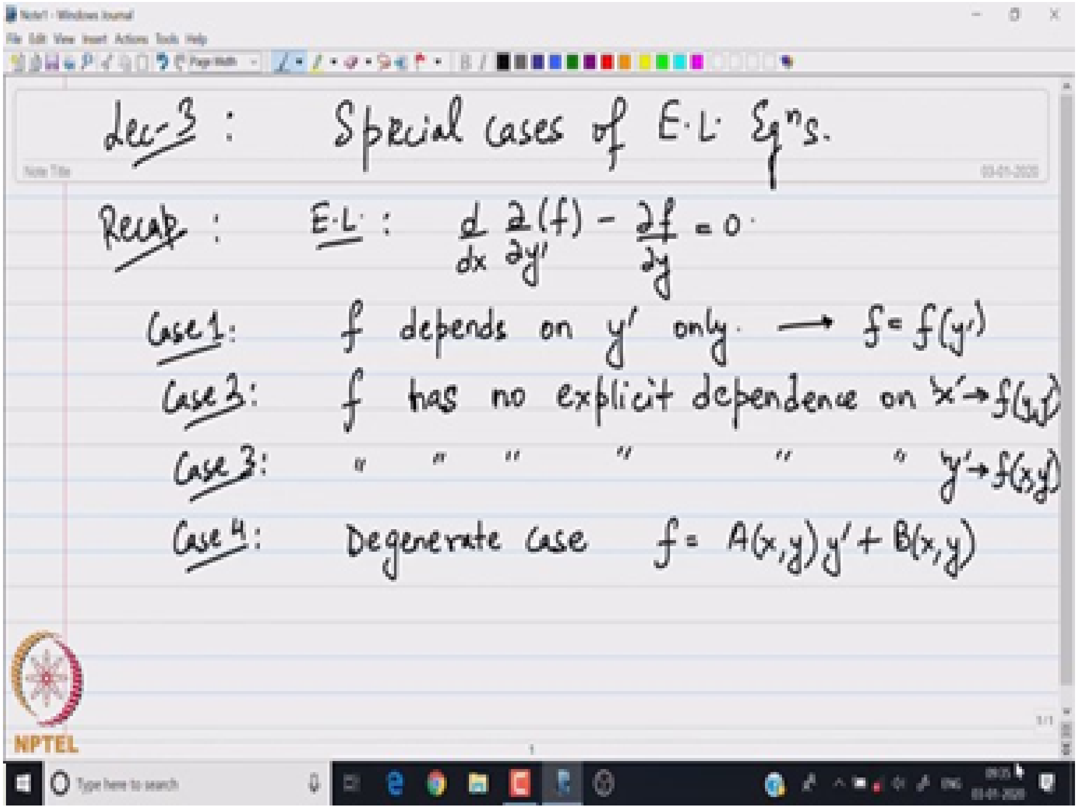

Good morning, everyone. So, in today's lecture, I am going to talk about the different, special cases of Euler-Lagrange equations. So, just a brief recap. In the last lecture we have found the major result in this course, namely the Euler-Lagrange equations $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$. We have found that the extremal to the functional satisfies Euler-Lagrange equation.

In today's lecture, I am going to look at some special cases, so let me just categorize the different special cases in which Euler-Lagrange equation simplifies. So, my first case would be where f depends, where my integrand to the functional depends only on $y^{'}$. So, in that case we will see how the Euler-Lagrange simplifies.

In the second case, we will look at the case when f has no explicit dependence on x. So, we will see that f is only a function of y and $y^{'}$.

And in the third case, we will look at the scenario where f has no explicit dependence on y and f depends on x and $y^{'}$.

And then there is a special fourth case where f has a linear dependence on $y^{'}$ or I call this as the degenerate case. And why it is called the degenerate case we will look it later. So, in this case my f has a linear dependence on $y^{'}$. So, let me now look at each of these cases one by one. So, let us look at the first case and how the Euler-Lagrange simplifies in this first case.

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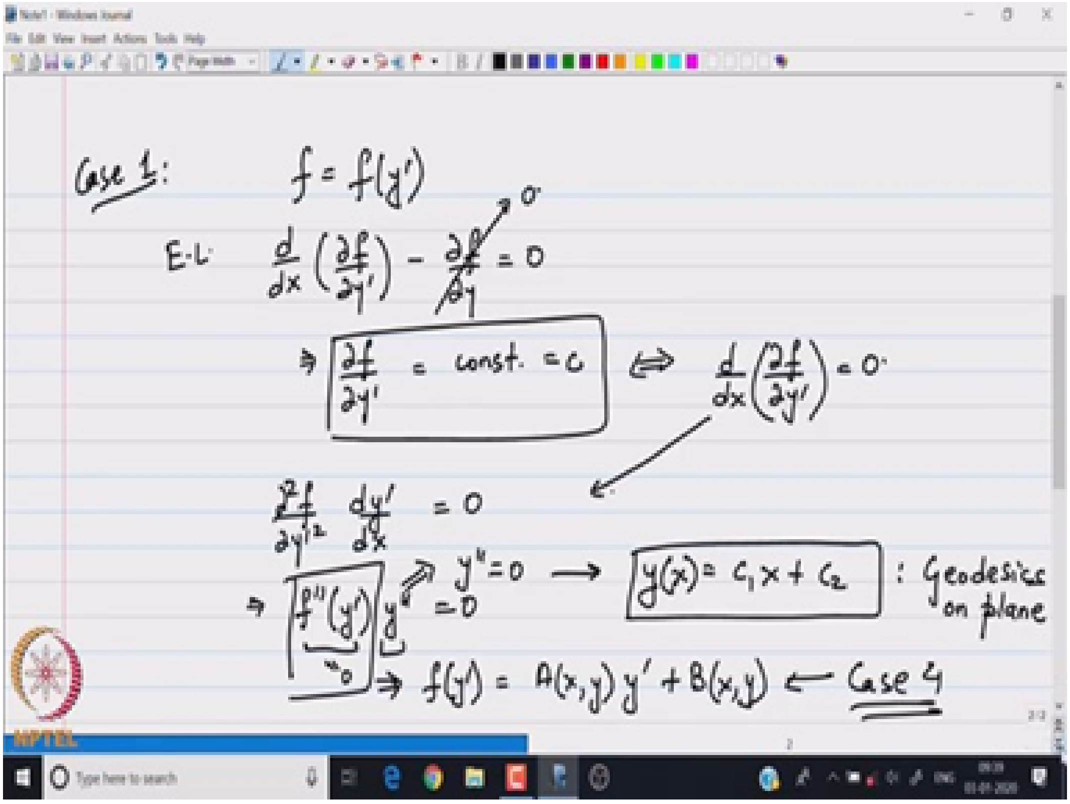

As I said, in the first case my f is only a function of $y^{'}$. So by Euler equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y^{'}}\right) - \frac{\partial f}{\partial y} = 0$, we see that since f is not a function of y explicitly so $\frac{\partial f}{\partial y} = 0$ and the equation shows that we can immediately integrate with respect to x

$$\Rightarrow \frac{\partial f}{\partial y^{'}} = constant = C \qquad \Leftrightarrow \qquad \frac{d}{dx}\left(\frac{\partial f}{\partial y^{'}}\right) = 0$$

So, in this case the Euler-Lagrange equation reduces to a first-order ODE or differential equation.

$$\Rightarrow f^{''}(y^{'})y^{''} = 0$$

So, we see that we could have two choices.
If $y^{''} = 0 \qquad \rightarrow \qquad y(x) = C_1 x + C_2$ (Geodesic on plane)
If $f^{''}(y^{'}) = 0 \qquad \rightarrow \qquad f(y^{'}) = A(x, y)y^{'} + B(x, y)$ (case 4)

So, I am not going to consider the case where $f^{''}(y^{'}) = 0$ is 0. So, which means we are only left with the fact, $y^{''} = 0$ and in this case we get the extremal y satisfies an equation of a straight line, the extremals are always straight lines.

And we have seen some examples in this case. Let me just recap some of the examples. Well, the example that we have seen earlier in our lecture was that of the geodesics on planes. And we saw that it naturally boils down to a straight line. Now let us continue our discussion on the second case.

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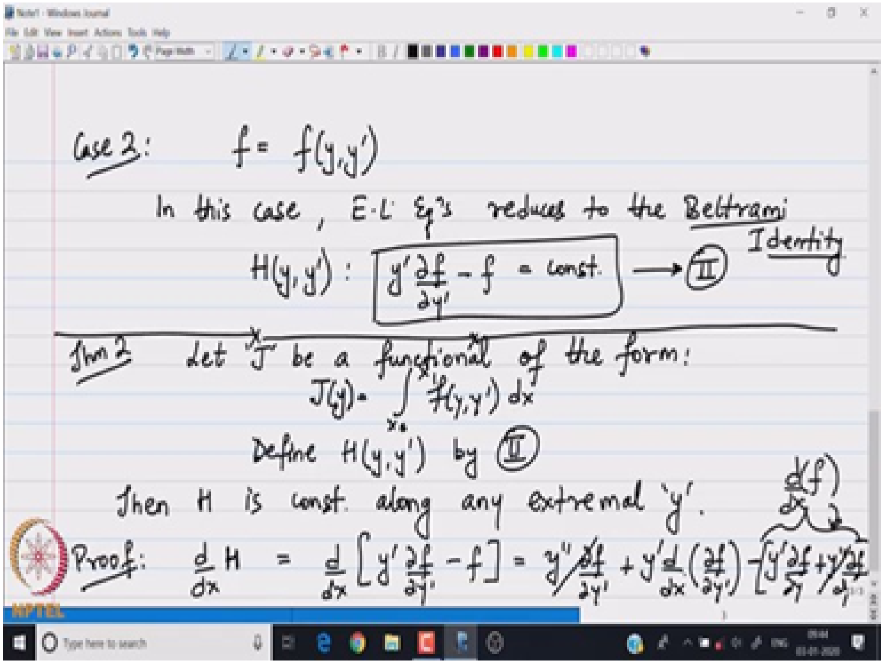

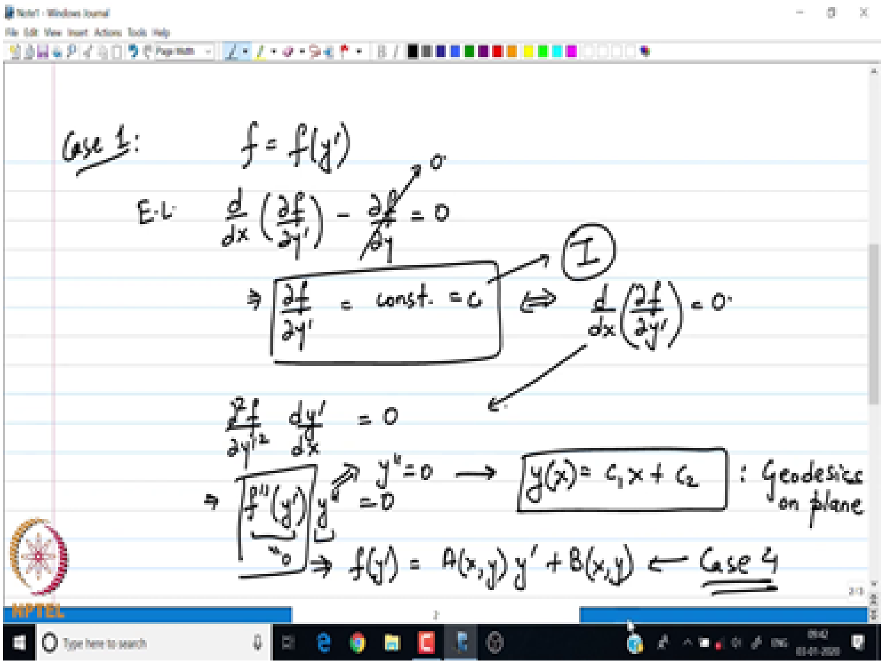

So, our second case is when we have f is only a function of y and $y^{'}$ ie f= f(y, $y^{'}$), so there is no explicit dependence of x on the integrand f and we will see that in this case, the Euler-Lagrange equation again reduces to first order differential equation, but it satisfies a particular identity. so what I just said is the following, in this case the Euler-Lagrange equation reduces to the Beltrami identity

So, it reduces to the Beltrami identity. And the identity is this particular function of y and $y^{'}$.

$$H(y, y^{'}) = y^{'} \frac{\partial f}{\partial y'} - f = constant \qquad \textbf{II}$$

Notice that the Beltrami identity is only a first-order differential equation. So, it is a reduced order

Euler-Lagrange equation and it is much simpler to solve that the original Euler-Lagrange equation.

So, let me just state this result in this case in the form of a theorem. So, I am going to continue to number the theorem, starting from the first lecture onwards we have so far shown one result in the form of one theorem and so this is my second theorem.

let J be a functional of the form

$$J(y) = \int_{x_o}^{x_1} f(y, y^{'})dx$$

and define H(y, $y^{'}$) by **II**

Then H is constant along any extremal 'y'

So, all we have to show is that the derivative of H with respect to x, which is the independent variable is 0 and the result follows right away that the function will be a constant.

proof:

$$\frac{d}{dx}H = \frac{d}{dx}\left[y^{'}\frac{\partial f}{\partial y'} - f\right] = y^{'}\left[\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right] = 0$$

Since y is extremal, therefore it satisfies E.L equation and that completes our proof that H is a constant

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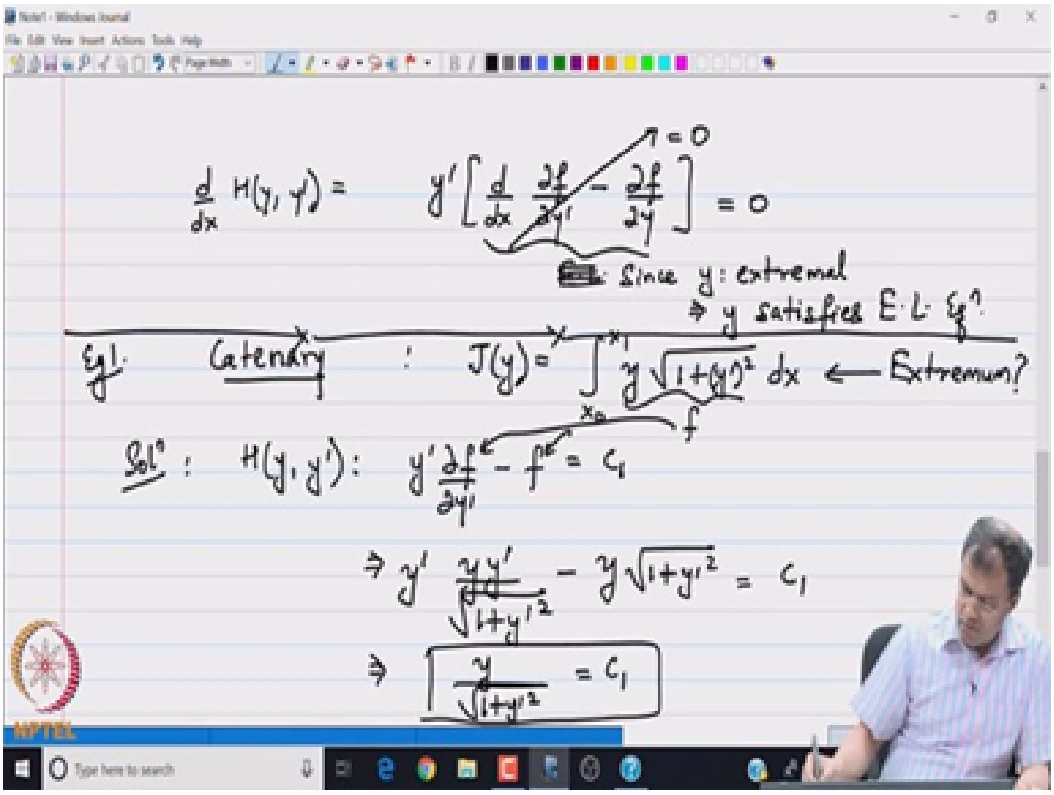

So, we can see the power of the result in this case. The Euler-Lagrange equation has reduced to H which is a first order differential equation as opposed to a second-order differential equation governed by Euler-Lagrange equation.

So let us look at, or revisit some of the cases, some of the examples in this second case. My first example is on the case of catenary. So, we have looked at the problem of catenary and recall that the functional involved in this case is as follows

$$J(y) = \int_{x_o}^{x_1} y\sqrt{1+(y')^2}dx \qquad Extremal?$$

Solution we can see that the integrand in this case is purely a function of y and y prime, so we can, this very nicely fits with the second case so we can directly apply the Beltrami identity and set it equal to a constant.

So, in this case, we have to find the extremum and it will satisfy the Beltrami identity and we see that the Beltrami identity is given by

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f = c_1$$

after plugging in the value of f and after simplification we get the following expression

$$\Rightarrow y'\frac{yy'}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = C_1$$

$$\frac{y}{\sqrt{1+(y')^2}} = C_1$$

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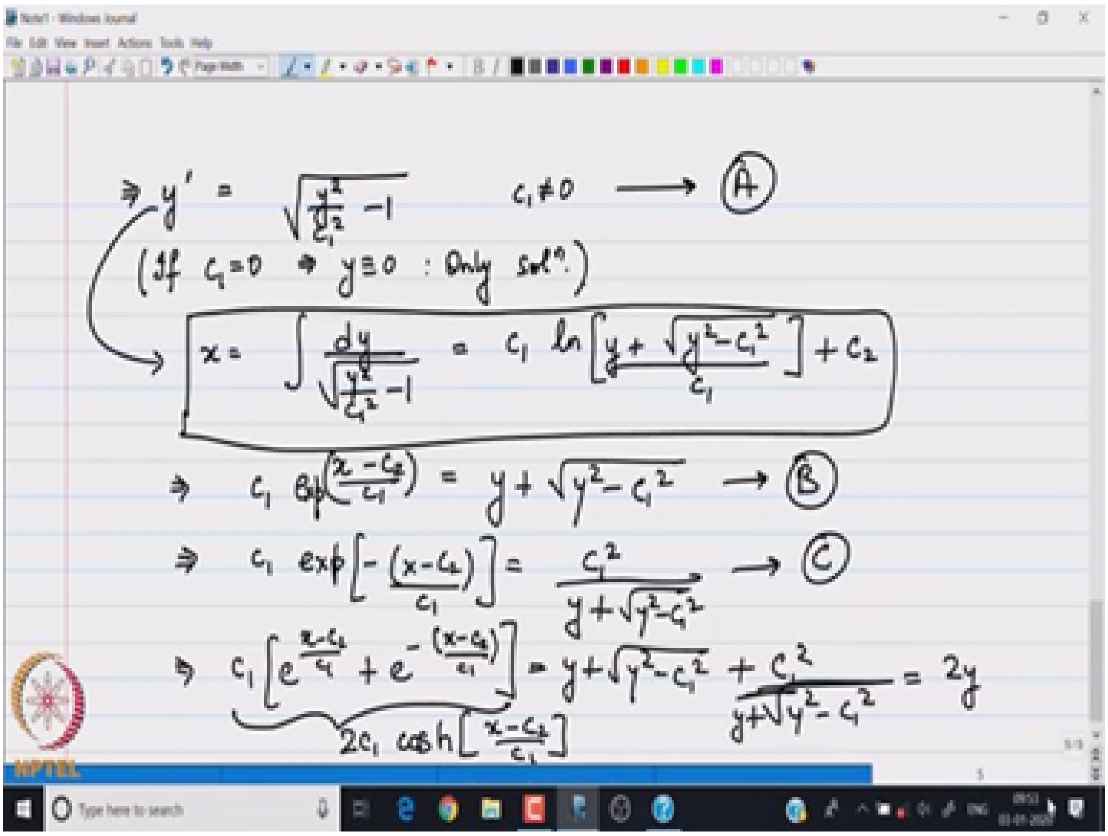

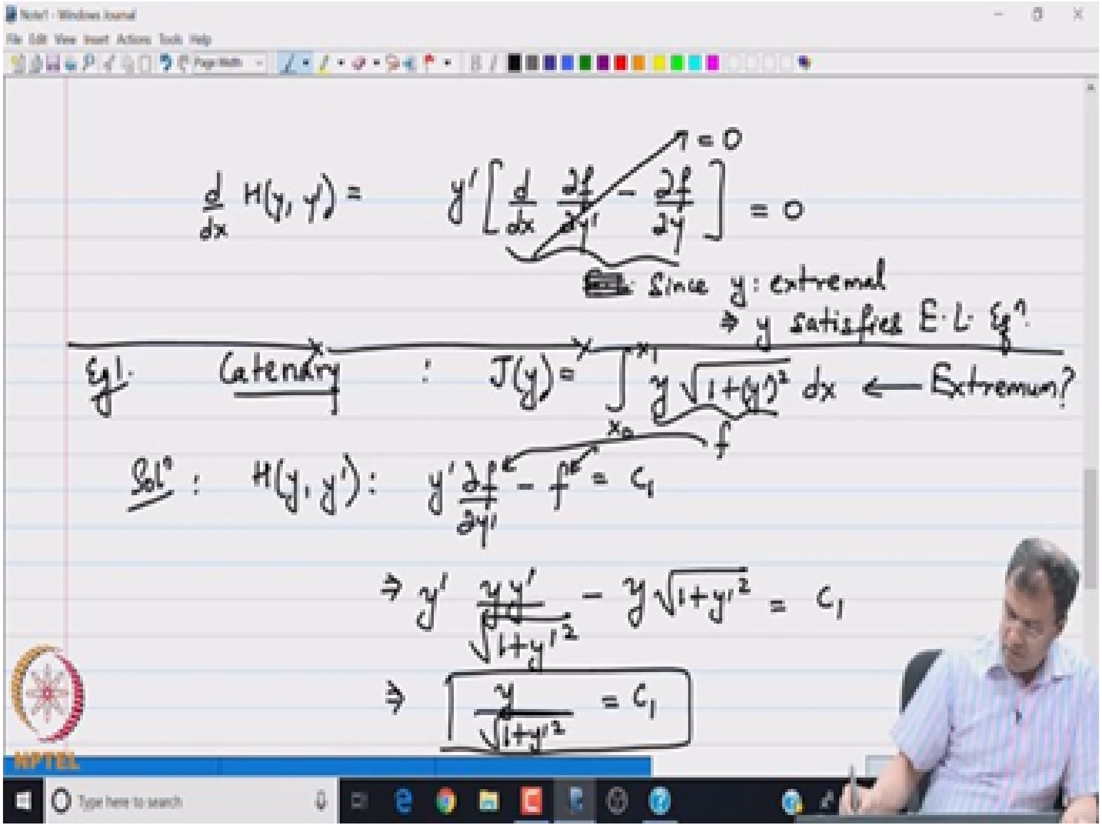

Let us integrate this equation and so when we do that we see that, well, so the equation here is the following

$$y^{'} = \sqrt{\frac{y^2}{{C_1}^2} - 1} \qquad C_1 \neq 0 \qquad \textbf{A}$$

If $C_1 = 0 \Rightarrow y = 0$ (only solution)

$$x = \int \frac{dy}{\sqrt{\frac{y^2}{{C_1}^2} - 1}} = C_1 \ln\left[\frac{y + \sqrt{y^2 - C_1^2}}{C_1}\right] + C_2$$

$$\Rightarrow C_1 exp\Big(\frac{x - c_2}{C_1}\Big) = y + \sqrt{y^2 - C_1^2} \qquad \textbf{B}$$

$$\Rightarrow C_1 exp\Big(-\frac{x - c_2}{C_1}\Big) = \frac{{C_1}^2}{y + \sqrt{y^2 - {C_1}^2}} \qquad \textbf{C}$$

$$\Rightarrow C_1\left[e^{\frac{x-C_2}{C_1}} + e^{-\frac{x-C_2}{C_1}}\right] = y + \sqrt{y^2 - C_1^2} + \frac{{C_1}^2}{y + \sqrt{y^2 - {C_1}^2}} = 2y$$

$$\Rightarrow y(x) = C_1 \cosh\left[\frac{x - c_2}{C_1}\right]$$

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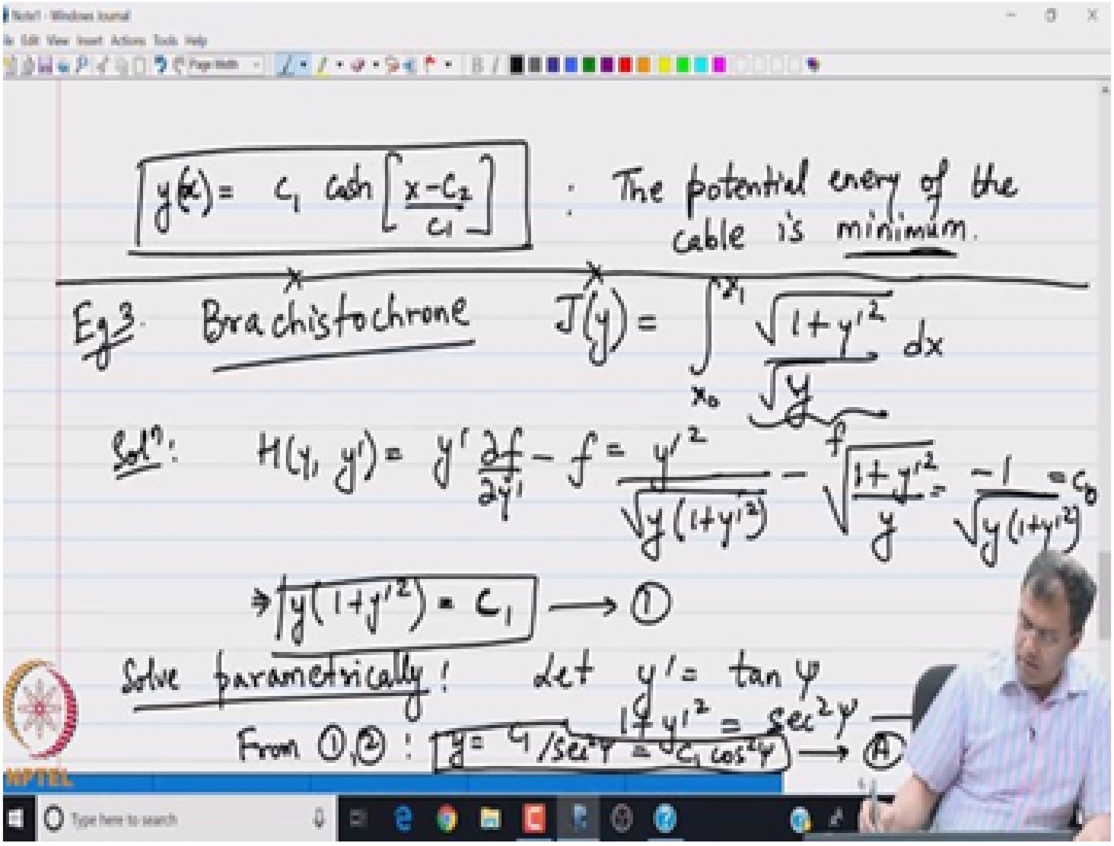

So, this is a much more convenient form of expressing the extremal in this particular case. And if we recall in this catenary problem, we are trying to minimize the potential energy of the cable.

So, it turns out that if a potential and if the cable has this shape of a cos hyperbolic function, then the potential energy which was represented by the functional, the potential energy of the cable is minimum. Well, I am just saying minimum, we have not figured out whether it is minimum or maximum, but at least the potential energy has reached its extremum.

Later on when we will talk about sufficient condition, we will show that this extremum is indeed the minimum, so right now let us just assume that the extremum that we have found is a minimum. So, that completes the discussion of this example. Let me look at another case study or **example** that I have described in my first lecture that is the example of Brachistochrone. So, let me just revisit that problem.

So, the problem involved a functional, so I am not going to write the entire detail of this problem but just the functional.

$$J(y) = \int_{x_o}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx$$

Notice that the quantity on the numerator is nothing but the arc length and the quantity in the, well we had a square root y and the quantity in the denominator represented a form of velocity. So, this is the total length divided by the total velocity so we are extremizing the time functional in this case.

Again, notice that in this case also the integrand in this integral is purely a function of y and y'. So, we can very safely use our Beltrami identity. And in this case my Beltrami identity will reduce the

Euler-Lagrange as follows

$$H(y, y^{'}) = y^{'} \frac{\partial f}{\partial y^{'}} - f = \frac{y^{'2}}{\sqrt{y(1+y^{'2})}} - \sqrt{\frac{1+y^{'2}}{y}} = -\frac{-1}{\sqrt{y(1+y^{'2})}} = C_o$$

So, from here we can solve for y as a function of x. The first step is to just invert this expression and take the square. So, just for bookkeeping purpose let me call this as C_o, a constant C_o and now we invert this expression and take the square on both sides, we see that now the new expression becomes $y(1+y^{'2})$, this is equal to another constant C_1. I do not care, I do not want to simplify in terms of C_o, let me call this as another constant .

$$\Rightarrow y(1+y^{'2}) = C_1 \qquad \mathbf{1}$$

So, from here we see that the direct solution is not possible. So, this is our extremal, the extremal curve satisfies the ODE but the direct solution will not be possible so we use an alternative approach. We solve this equation parametrically.

So, what I mean by solving parametrically is as follows, I am going to introduce a new parameter on which both y and x will depend. So, I am going to explicitly find y as a function of that parameter and x as a function of that parameter and that will describe the extremal.

let $y^{'} = \tan\psi$, $y^{'}$ is a slope of a curve so it is $\tan\psi$.
So, ψ is our new parameter. So, $1 + y^{'2} = \sec^2\psi$ **2.**

From **1** and **2** $\quad y = \frac{C_1}{\sec\psi^2} = C_1 \cos^2\psi$ **A**

So, I have already found the parametric representation of y with respect to the parameter ψ and now my next task is to find the expression for the x component in term of the parameter ψ.

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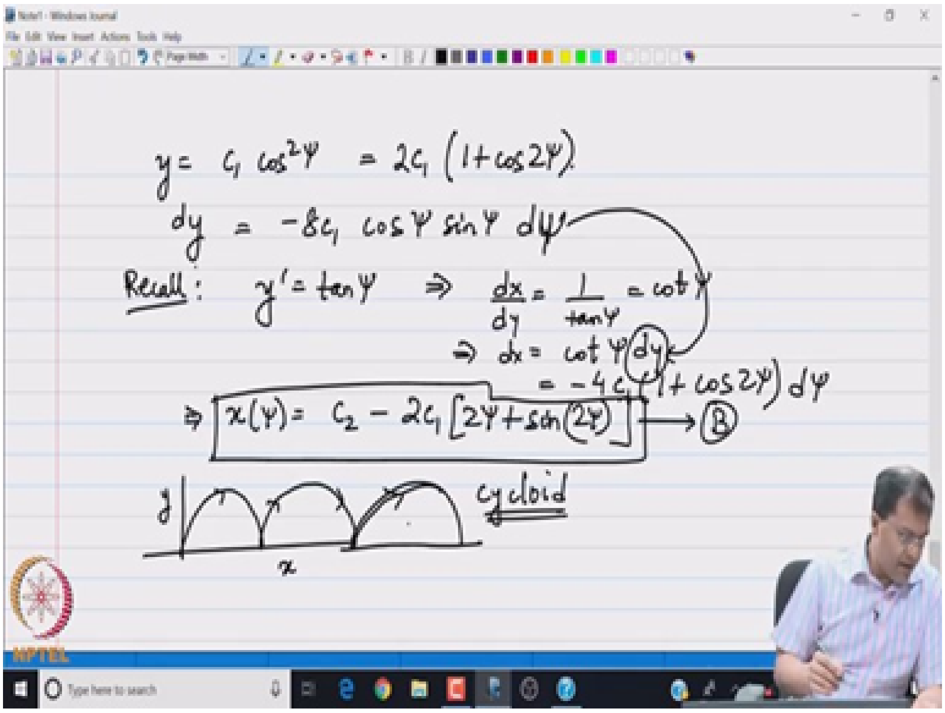

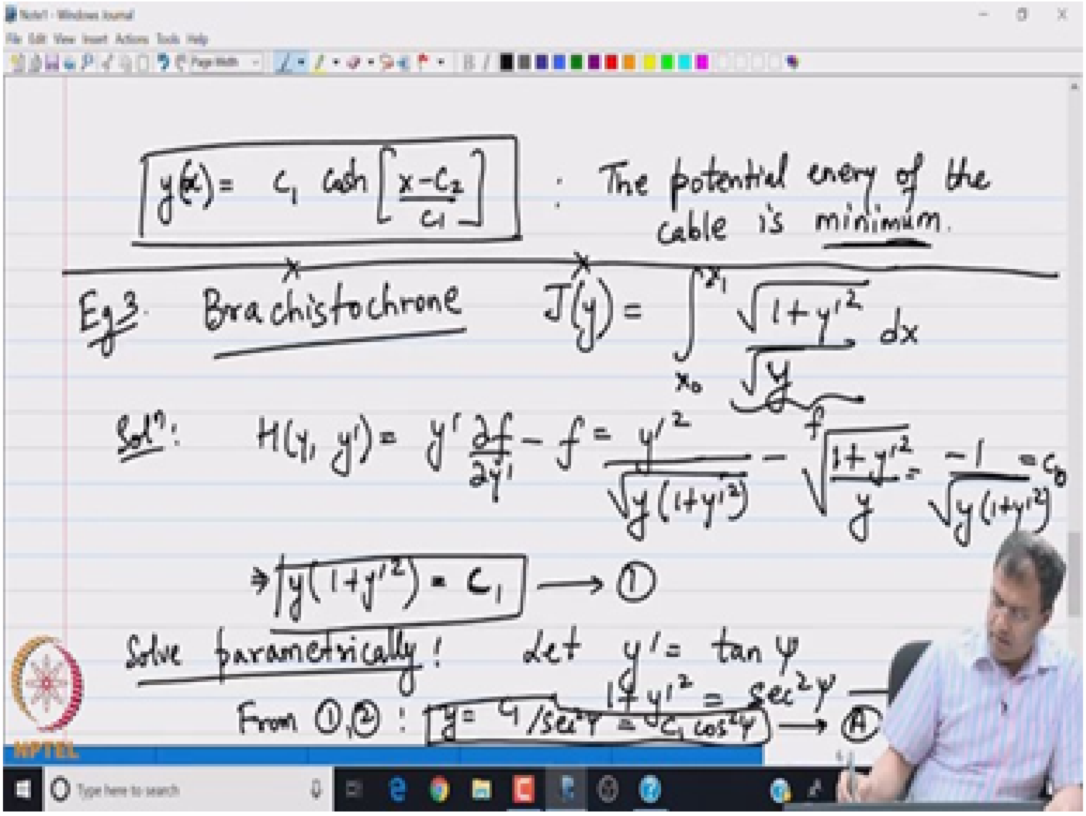

So, to do that let us notice the fact that, $y = C_1 \cos^2 \psi$ and let further simplify this using the double angle formula

$$C_1 \cos^2 \psi = 2C_1(1 + \cos 2\psi)$$

$$dy = -8C_1 \cos \psi \sin \psi d\psi$$

$$Recall \quad y^{'} = \tan \psi \Rightarrow \frac{dx}{dy} = \frac{1}{\tan \psi} = \cot \psi$$

$$\Rightarrow dx = \cot \psi dy = -4C_1(1 + \cos 2\psi) d\psi$$

$$\Rightarrow x(\psi) = C_2 - 2C_1[2\psi + \sin 2\psi] \qquad \mathbf{B}$$

So, now we have completely described our extremal motion x and y in the form of the parameter ψ.

Now, students should recall that we had already revealed the solution to this problem the Brachistochrone problem and we had a similar, I had written a similar solution except that instead of 2 ψ we had ψ. So, you can just replace 2 ψ by ψ everywhere. And if people recall, if we were to plot x versus y, if we were to plot x versus y, the plots will follow a locus which is a point on the rim of the bicycle wheel.

So, these are nothing but, these locus of points are nothing but following the curve which is known as the cycloid or a point which lies on the rim of a bicycle wheel. So, that completes the discussion on this example.