Variational Calculus and its applications in Control Theory and Nanomechanics

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Lecture 66 Introduction to Nanomechanics Part 6

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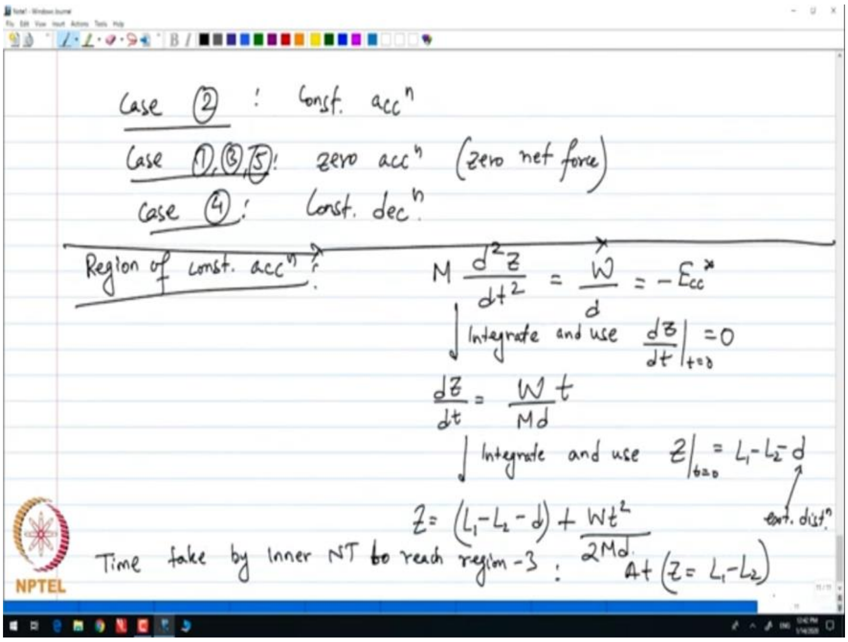

So, what I have is a following, we have region, it turns out that case 2 is a case of constant acceleration. We are going to measure this or quantify this acceleration purely via Newtonian arguments. Case, case 1, 3, 5 are my zero acceleration or zero net force and my case 4 is my constant deceleration. So, let us try to find all these regions of constant acceleration and constant deceleration. So, let us look at the region of constant acceleration and try to find how much time is taken by the inner nanotube to spend in this region, which is case 2, region of constant acceleration. So, in this case, we set up our Newtonian equation that is via Newton's second law:

$$M\frac{d^2z}{dt^2} = \frac{W}{d} = -E^*_{cc}$$

So, this is for region number 2. We need to integrate this equation twice to find out Z as a function of t and to do that, we also need the initial condition note, we assume that in case 2, the particle or the inner nanotube starts with a zero velocity and let us integrate once to see what happens. So integrate and use the fact that $\frac{dz}{dt}$ at t=0 is equal to 0. Integrate once and see that I get:

$$\frac{dz}{dt} = \frac{W}{Md}t$$

So this is a case of constant. So minus W is negative or W is positive. So, we can peacefully use W to be a positive function, this is our W being positive. So that is the integration once and further we integrate once more and also use the position coordinate, note that Z at t equal to 0 is at a position coordinate. So, we want to find out what is the value of the position between the boundary point 2 and

3. So, that will be given by this particular value L1 minus L2 or I see that Z at t equal to 0 is L1 minus L2 and suppose the extrusion distance was d, so, we will get minus d. So,

$$Z = (L_1 - L_2 - d) + \frac{Wt^2}{2Md}$$

We want to see what is the value time taken by this oscillator, this inner nanotube to reach to region 3 is when it has travelled a distance, Z is equal to L1 minus L2. So, we have to, to reach region 3 from region 2, then the centre of the nanotube must have travelled a distance of L1 minus L2. Again this is directly coming from our description of the cases. So, the cut-off point is L1 minus L2 going from case 2 to case 3. So, we plug that value of Z and find out how much time is taken by the oscillator to stay in region 2.

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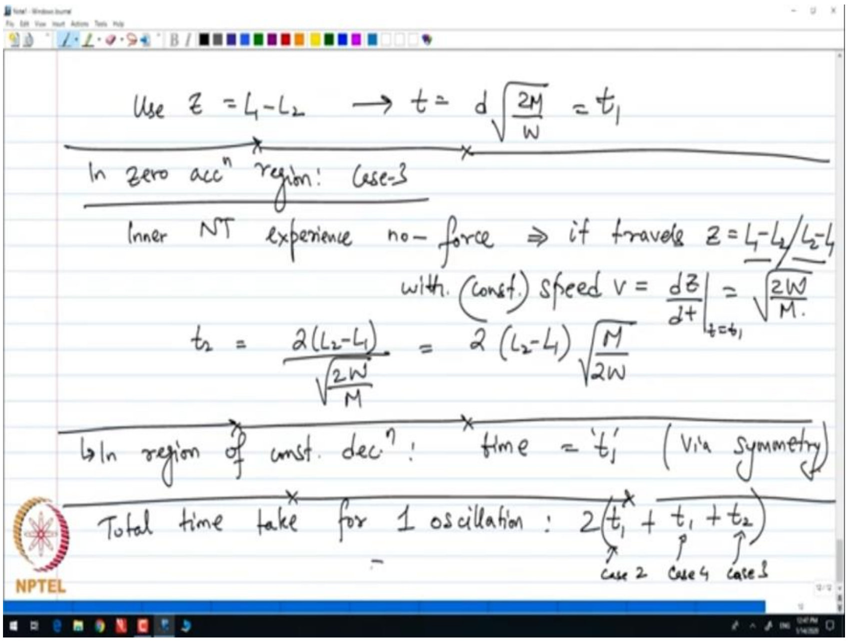

So, so when we use Z equal to L1 minus L2 and from here I see that my time taken by the oscillator to stay in region 2 is coming out to be

$$t = d\sqrt{(2M/W)} = t_1$$

Similarly, I can find out in the region of no acceleration, I see that my inner nanotube experiences no force because then it is going to travel with a constant velocity and the constant velocity is given by plugging the value of t_1 in our previous results. So the nanotube travels from one cut-off point from Z is equal to L1 minus L2 to the other cut-off point which is L2 minus L1, from one cut-off to the other cut-off is marking the region which is denoted by 3 with constant speed V given by the value of

$$V = \left.\frac{dz}{dt}\right|_{t=t_1} = \sqrt{(2W/M)},$$

$$\text{Then}, t_2 = \frac{2(L_2 - L_1)}{\sqrt{(2W/M)}} = 2(L_2 - L_1)\sqrt{(M/2W)}$$

So finally, we also have the region of constant deceleration. In region of constant deceleration we are completely modelling a system with zero friction. So, the region of constant deceleration will be symmetric to the region of constant acceleration or students who do not believe this should directly find the time taken in this constant deceleration region, similar to our region for constant acceleration case. So, in case of constant deceleration the time taken to spend in this region will be again 't_1', this is via symmetry arguments of the system. So, that is the result in short, so now let us add up the total time taken by the system. So, what is a total time taken? The total, the total time taken for 1 oscillation, it will be the time taken in the region of constant deceleration. So, that is in case 2 plus the time taken in case 4, which is via symmetry the same time plus the time taken in case 3 which is the region of constant speed times 2 because the motion gets repeated after going from one end to the other.

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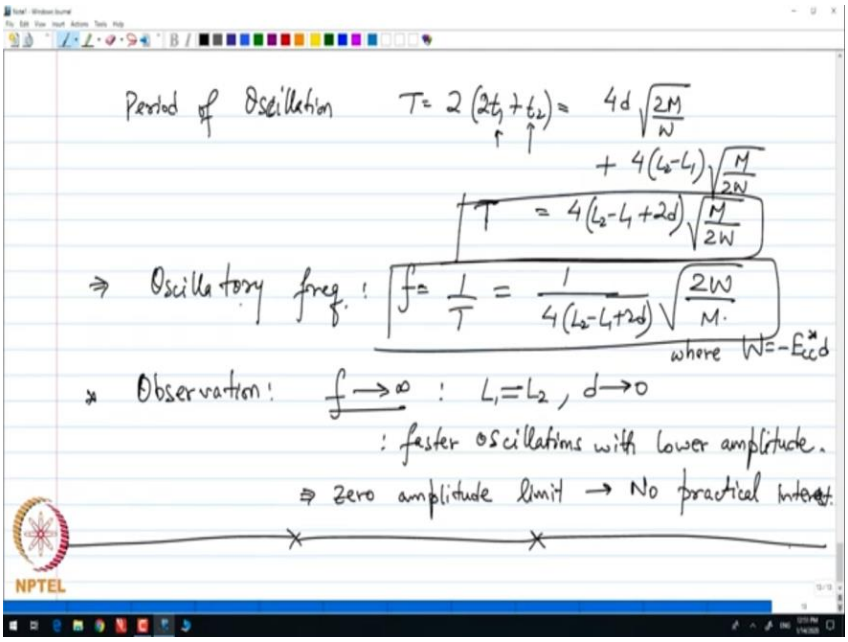

So now, we have that the total period of oscillation or it implies the period of oscillation becomes

$$T = 2(2t_1 + t_2) = 4d\sqrt{(2M/W)} + 4(L_2 - L_1)\sqrt{(M/2W)}$$
$$T = 4(L_2 - L_1 + 2d)\sqrt{(M/2W)}$$

So, that is my period t of oscillations and in this case let me finally write down what is the oscillatory frequency.

$$f = \frac{1}{T} = \frac{1}{4(L_2 - L_1 + 2d)}\sqrt{(2W/M)}$$

where my work done is $W = -E^*_{cc}d$. So, this is my period of oscillation, which has now been found completely through the Newtonian mechanics. Now, notice that if my nanorods are almost similar in length, if my L1 is equal to L2 and extrusion distance is minimal or d goes to 0, I see that my frequency goes to infinity, which means that small extrusion equal length nanorods. Frequency is almost immeasurable because it goes to infinity. So, this is just an informal observation. So, notice this observation, if we see that F goes infinity, when L1 is equal to L2 and d goes to 0. Or this is a case of faster oscillations with lower amplitude. So, in the limit of 0 amplitude, we expect that these oscillations are almost unreliably small to be measured. So what I just said is that in this particular case when the frequency goes to infinity, this is not a case of practical interest because these frequencies are almost immeasurable. So, this is the case of zero amplitude limit and is of no practical interest because they cannot be measured without errors. So that is just one assumption. Let us now look at the same model of this oscillatory motion of these nanotubes via our Hamilton's equation or the Hamiltonian formulation.

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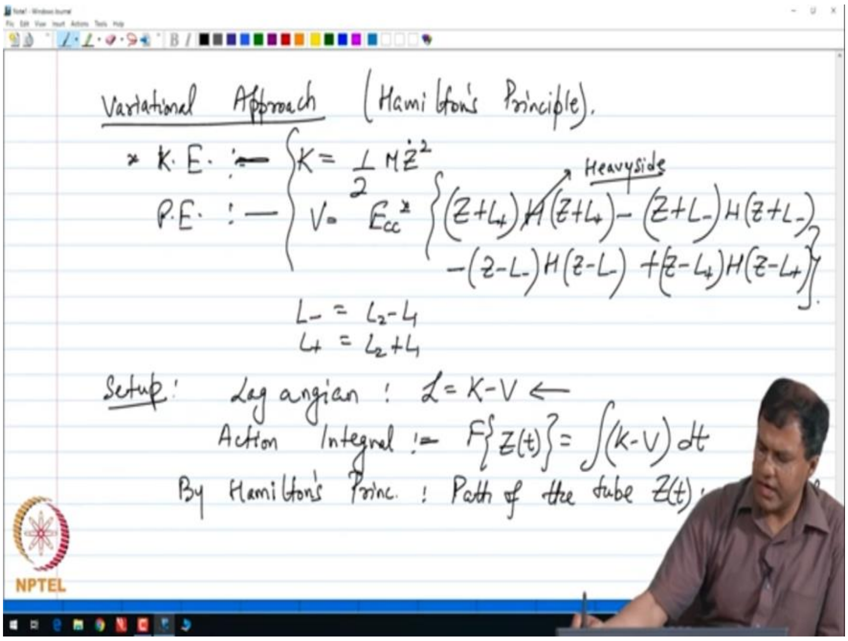

So, now, let us look at the variational approach. So, let us try to answer question three using our variational approach. So, that is via my Hamilton's principle. So, now, to write down my Hamiltonian, we have to look at what is my kinetic energy and what is my potential energy of the problem. So, my kinetic energy of the problem is given by

$$
\begin{aligned}
& K = \frac{1}{2} M \dot{Z}^2 \\
\text{and P.E. is,} \quad & V = E^*_{cc}[(Z+L_+)H(Z+L_+) - (Z+L_-)H(Z+L_-) \\
& \quad - (Z-L_-)H(Z-L_-) + -(Z-L_+)H(Z-L_+)]
\end{aligned}
$$

So this is the potential energy completely written in a single expression using my Heaviside function. Then we set up the Lagrangian of the system so, $L = K - V$ and then we set up the action integral which is

$$
FZ(t) = \int (K - V) dt
$$

So, that is my action integral and my Hamilton's principle says that the optimal solution to the action integral will be given by the principle of least action or the extremal to this action integral. So, by Hamilton's principle I have the path of the tube Z of t is the extremal level of the action, so the path of the tube is the extremal of the action. So, then next we have to extremize the action integral and we look at now, the Hamilton's eq, let us begin with our Euler-Lagrange equation. But even before that, notice that my Lagrangian is independent of variable t, which means that we can very well use our Beltrami identity to write, to reduce our Euler-Lagrange.

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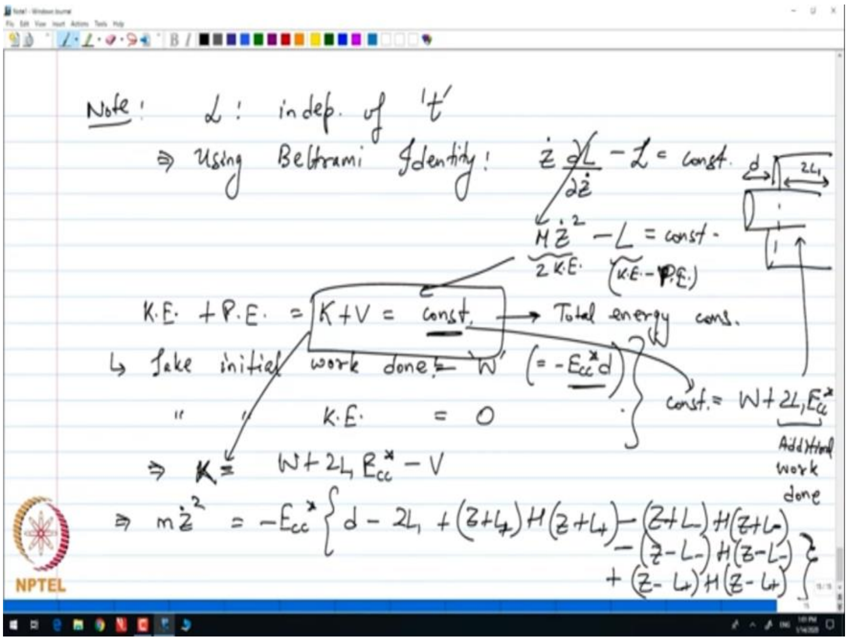

So, notice my Lagrangian, L is independent of the independent variable t, or from here, if I use my Beltrami identity I see that, I am going to use the following,

$$\dot{Z}\frac{\partial L}{\partial \dot{Z}} - L = \text{const.} => M\dot{Z}^2 - L = \text{const.}$$

Now first quantity is 2 times the kinetic energy by our definition and L is kinetic energy minus the potential energy and then we rearrange this relation we get that, $K + V = \text{const.}$ or what I am getting is the Beltrami identity is telling me again that the net energy is conserved in the system so which means that the total energy is conserved. So, I can also find this constant on my right-hand side of this relation. So, so if I take my initial work, W which is eventually $-E^*_{cc}d$ depends on how much is the extrusion length, so if I take my initial work done W, then work done to extrude the inner tube by a distance. So, take my initial kinetic energy to be 0 then the above using this my above constant, this constant comes out to be $W + 2L_1E^*_{cc}$ because of the potential of the system when it is not accelerating, because in my case 3 that is the work done by the system, so this quantity is the additional work done by the system in extruding out from the equilibrium position. So, this is the additional work done. Note that when we have two nanotubes, so let me draw this figure here of the inner and the outer. So, we first extrude by a distance d and also before even extruding by distance d, it also has to be extruded by a distance 2L1, which is the length of the inner nanotube, so that is the additional work done because of this following diagram. So, my constant is the following which I have written. So now, from this relation, I see that my kinetic energy is

$$\begin{aligned} & K = W + 2L_1E^*_{cc} - V \\ => \quad & m\dot{Z}^2 = -E^*_{cc}[d - 2L_1 + (Z+L_+)H(Z+L_+) - (Z+L_-)H(Z+L_-) \\ & \quad - (Z-L_-)H(Z-L_-) + -(Z-L_+)H(Z-L_+)] \end{aligned}$$

So, these are my four terms and from here.

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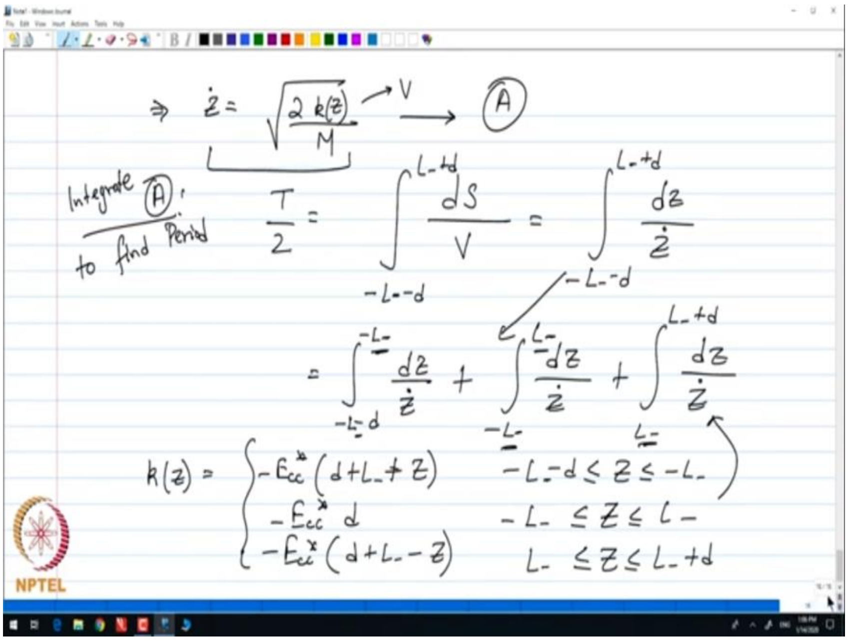

From this whole giant relation, I see that my

$$\dot{Z} = \sqrt{\frac{2k(z)}{M}}$$

and of course, this is my equation of motion, I can integrate it once and find out the time period of oscillation but that is going to be very cumbersome. Instead we look at the time period in a different way. Notice that from A, I am not going to integrate A and find out the respective times of the motion of this particle, I can definitely integrate A to find time period, what I cannot do is to find the position as a function of time because of the complexity of the square root function.

$$\begin{aligned}
\frac{T}{2} &= \int_{-L_- - d}^{L_- + d} \frac{ds}{V} \\
&= \int_{-L_- - d}^{L_- + d} \frac{dZ}{\dot{Z}} \\
&= \int_{-L_- - d}^{-L_-} \frac{dZ}{\dot{Z}} + \int_{-L_-}^{L_-} \frac{dZ}{\dot{Z}} + \int_{L_-}^{L_- + d} \frac{dZ}{\dot{Z}}
\end{aligned}$$

So let me specifically write down my function

$$f(x) = \begin{cases} -E_{cc}^*(d + L_- + Z), & -L_- - d \leq Z \leq -L_- \\ -E_{cc}^*(d), & -L_- \leq Z \leq L_- \\ -E_{cc}^*(d + L_- - Z), & L_- \leq Z \leq L_- + d \end{cases}$$

and let me write down the final answer after integration.

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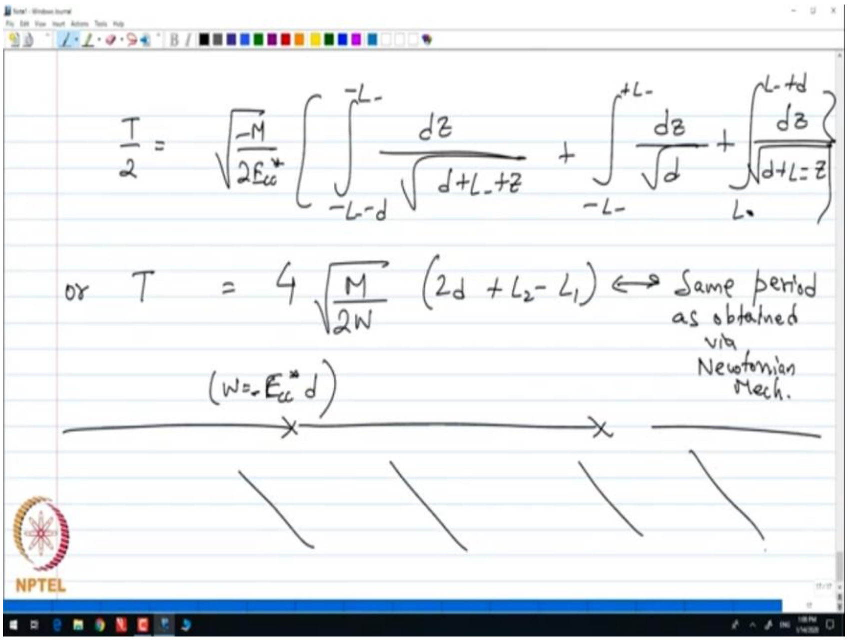

$$\frac{T}{2} = \sqrt{\frac{-M}{2E^*_{cc}}}\left[\int_{-L_- -d}^{-L_-} \frac{dz}{\sqrt{d+L_- +Z}} + \int_{-L_-}^{L_-} \frac{dz}{\sqrt{d}} + \int_{L_-}^{L_- +d} \frac{dz}{\sqrt{d+L_- -Z}}\right]$$

Or, $\quad T = 4\sqrt{\dfrac{M}{2W}}(2d + L_2 - L_1)$

So that completes my description in the Hamiltonian description of these nano rod oscillators and I hope that the students have learned in this course, starting from the basic theory of calculus of variations to how to apply this theory in optimal control problems and finally, how to look at the motion of carbon nanorods in various situations. So, I hope the students are going have learned quite a few tricks to solve the problems involving calculus of variations with specific applications in optimal control and Nanomechanics. So, thank you very much for listening and thanks a lot.