Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 60
Constrained Optimization in Optimal Control Theory Part 6

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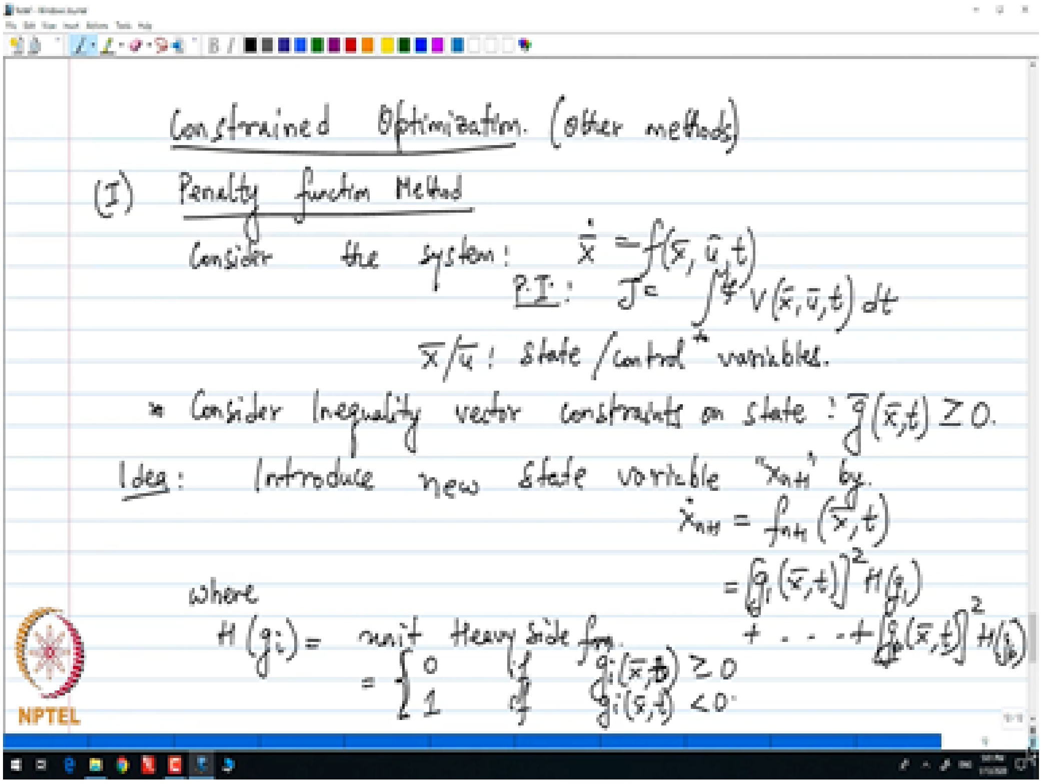

I am going to describe two methods of constrained optimization.The first method is the Penalty function method. We will see that we are going to change the inequality into an equality constrained by introducing the so called Penalty function or in the language of calculus of variation introducing a non-holonomic constrained or a differential constrained.

Consider the system given by $\dot{\bar{x}} = f(\bar{x}, \bar{u}, t)$ and the performance index which is given by $J = \int_{t_0}^{t_f} V(\bar{x}, \bar{u}, t)\, dt$, where $\bar{x}$ and $\bar{u}$ are usual state and control variables. And now we look at some inequality constraints. So consider inequality vector constraints on state variables $\bar{g}(\bar{x}, t) \geq 0$. The idea of this Penalty method is as follows : we will introduce a new state variable x_{n+1} by

$$\begin{aligned}\dot{x}_{n+1} &= f_{n+1}(\bar{x}, t) \\ &= [g_1(\bar{x}, t)]^2 H(g_1) + ... + [g_p(\bar{x}, t)]^2 H(g_p)\end{aligned}$$

where $H(g_i)$ are unit Heavy side function defined as

$$H = \begin{cases} 0 & \text{if } g_i(\bar{x}, t) \geq 0 \\ 1 & \text{if } g_i(\bar{x}, t) < 0 \end{cases}$$

Notice that when constraint is negative we are taking a square of the constraint so right hand side will always be positive which means that right hand side will never be less than 0. We will have the minimal value of the right hand side is 0.

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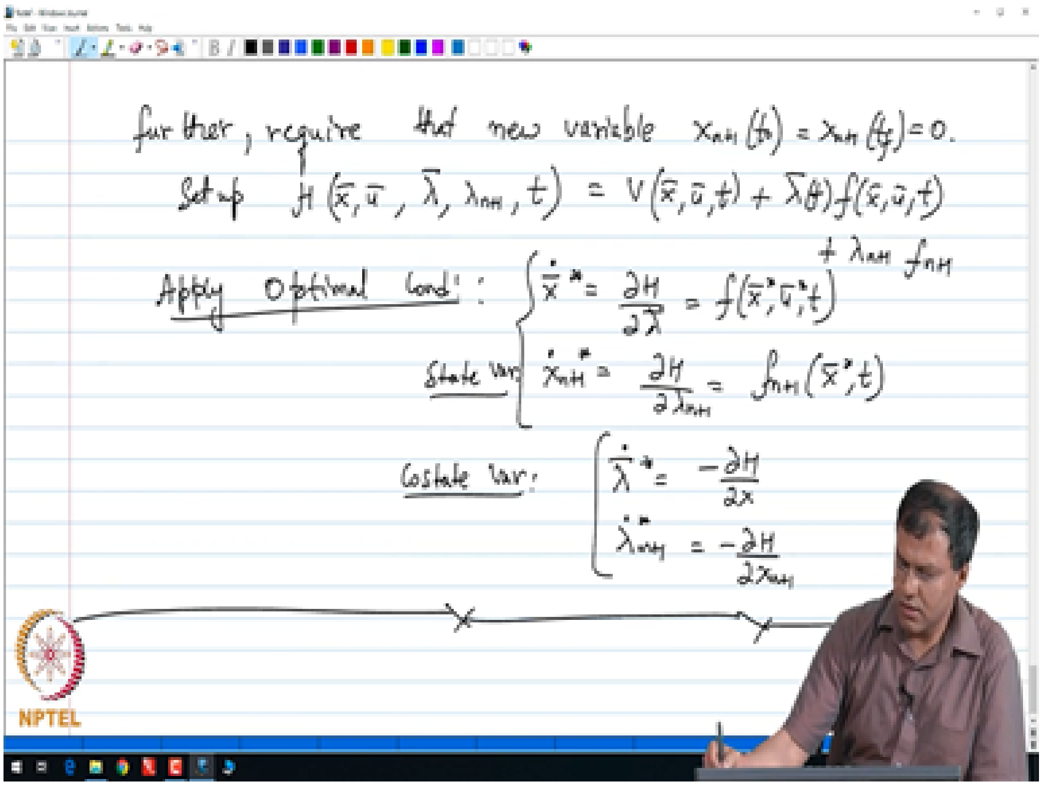

Further require that the new variable has a boundary condition $x_{n+1}(t_0) = x_{n+1}(t_f) = 0$. Now, we setup the Hamiltonian

$$H\left(\bar{x}, \bar{u}, \bar{\lambda}, \lambda_{n+1}, t\right) = V\left(\bar{x}, \bar{u}, t\right) + \bar{\lambda}\left(t\right) f\left(\bar{x}, \bar{u}, t\right) + \lambda_{n+1} f_{n+1}$$

And then we apply optimality condition or Hamilton's equation, we have to solve the following set of equations

$$\dot{\bar{x}}^{\star} = \frac{\partial H}{\partial \bar{\lambda}} = f\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) \quad ; \quad \dot{\bar{x}}^{\star}_{n+1} = \frac{\partial H}{\partial \lambda_{n+1}} = f_{n+1}\left(\bar{x}^{\star}, t\right)$$

And then for co-state variable we also have $n+1$ equations. The co- state variables are

$$\dot{\bar{\lambda}}^{\star} = -\frac{\partial H}{\partial \bar{x}} \quad ; \quad \dot{\lambda}^{\star}_{n+1} = -\frac{\partial H}{\partial \bar{x}_{n+1}}$$

So, we have $2n+2$ equations to be solved. Notice that the only dependence of H on x_{n+1} is via λ_{n+1} so H does not explicitly contain x_{n+1} as such. So let me just highlight this penalty function method with an example.

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Eg 3 : Consider 2nd order sys. : $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, $J(\bar{x}, u) = \frac{1}{2}\int_{t_0}^{t_f} (x_1^2 + u^2)\, dt$

t_f : free

final state : $x(t_f)$ is free

Control fn. '$u(t)$' is constrained as $-1 \le u(t) \le 1$ $\forall t \in [t_0, t_f]$

State variable 'x_2' is constrained : $|x_2| \le 3$ $\forall t \in [t_0, t_f]$

Sol^n : State constraints : $\begin{cases} x_2 + 3 \ge 0 \rightarrow g_1 \ge 0 \\ 3 - x_2 \ge 0 \rightarrow g_2 \ge 0 \end{cases}$

Step 1 Setup Hamiltonian : $H = \underbrace{\frac{x_1^2}{2} + \frac{u^2}{2}}_{V} + \underbrace{\lambda_1 x_2 + \lambda_2 u}_{\bar{\lambda}.\bar{f}}$
$+ \lambda_3 \left[(x_2+3)^2 H(x_2+3) + (3-x_2)^2 H(3-x_2) \right]$

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Constrained Optimization. (Constraint on State variable)

(I) Penalty function Method

Consider the system : $\dot{\bar{x}} = f(\bar{x}, \bar{u}, t)$

P.I. : $J = \int_{t_0}^{t_f} V(\bar{x}, \bar{u}, t)\, dt$

$\bar{x}/\bar{u}$: state / control variables.

* Consider Inequality vector constraints on state : $\bar{g}(\bar{x}, t) \ge 0$.

Idea : Introduce new state variable 'x_{n+1}' by.

$\dot{x}_{n+1} = f_{n+1}(\bar{x}, t)$
$= [g_1(\bar{x}, t)]^2 H(g_1) + \dots + [g_p(\bar{x}, t)]^2 H(g_p)$

where

$H(g_i)$ = unit Heavy Side fn.
$= \begin{cases} 0 & \text{if } g_i(\bar{x}, t) \ge 0 \\ 1 & \text{if } g_i(\bar{x}, t) < 0 \end{cases}$

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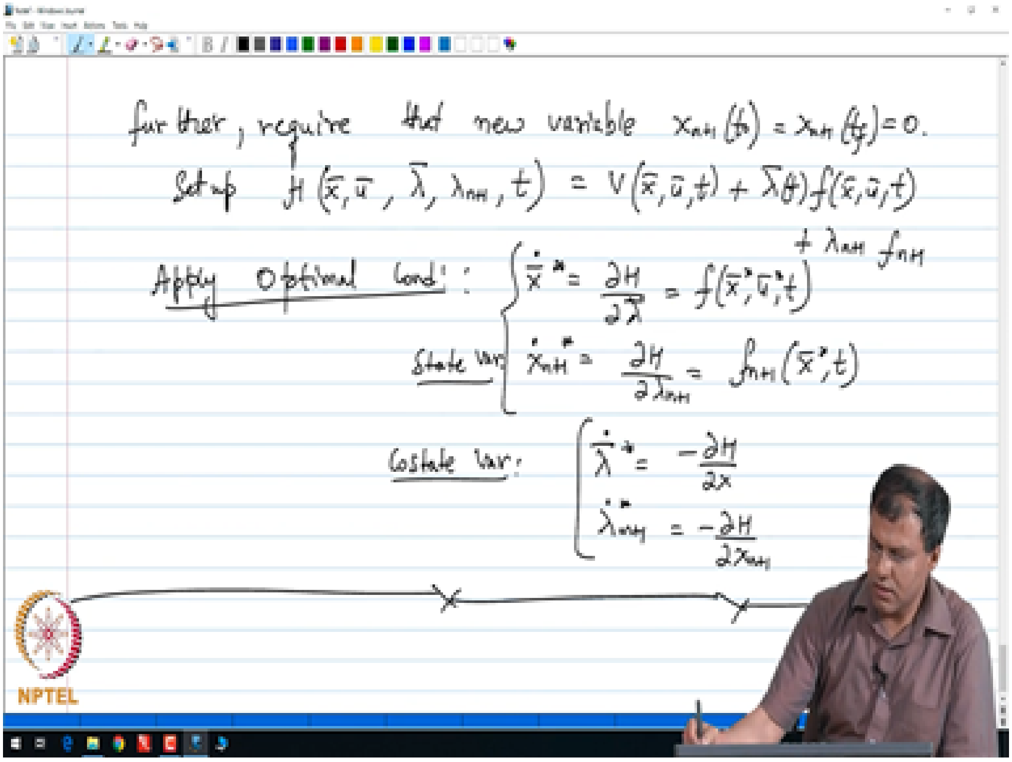

The example is consider the second order system given by

$$\dot{x_1} = x_2$$

$$\dot{x_2} = u$$

This is a similar system that we had solved in our previous example and this time performance index is

$$J(\bar{x}, u) = \frac{1}{2}\int_{t_0}^{t_f}\left(x_1^2 + u^2\right)dt$$

And further, it is given that the final point t_f is free and the final state $x(t_f)$ is also free.

And the control function $u(t)$ is constrained as $-1 \le u(t) \le 1 \;\; \forall \;\; t \in [t_0, t_f]$. And further the state variable x_2 is constrained given by $|x_2| \le 3 \;\; \forall \;\; t \in [t_0, t_f]$. Notice that state constraints are as follows :

$x_2 + 3 \ge 0$ let me call this constraint as g_1 so $g_1 \ge 0$

$3 - x_2 \ge 0$ let me call this constraint as g_2 so $g_2 \ge 0$

Essentially these are constrained problems where we are putting constrained on the state variables. In my previous discussion where we were discussing the Pontryagin minimum principle, we were putting constrained on the control variable.

Step 1 : We setup the Hamiltonian.

$$H = \frac{x_1^2}{2} + \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 u + \lambda_3\left[(x_2+3)^2 H(x_2+3) + (3-x_2)^2 H(3-x_2)\right]$$

where $H(x_2+3)$ and $H(3-x_2)$ are the unit Heavy Side function and then we have to setup the equations for state variables and co state variables. We will have $2n+2$ equations.

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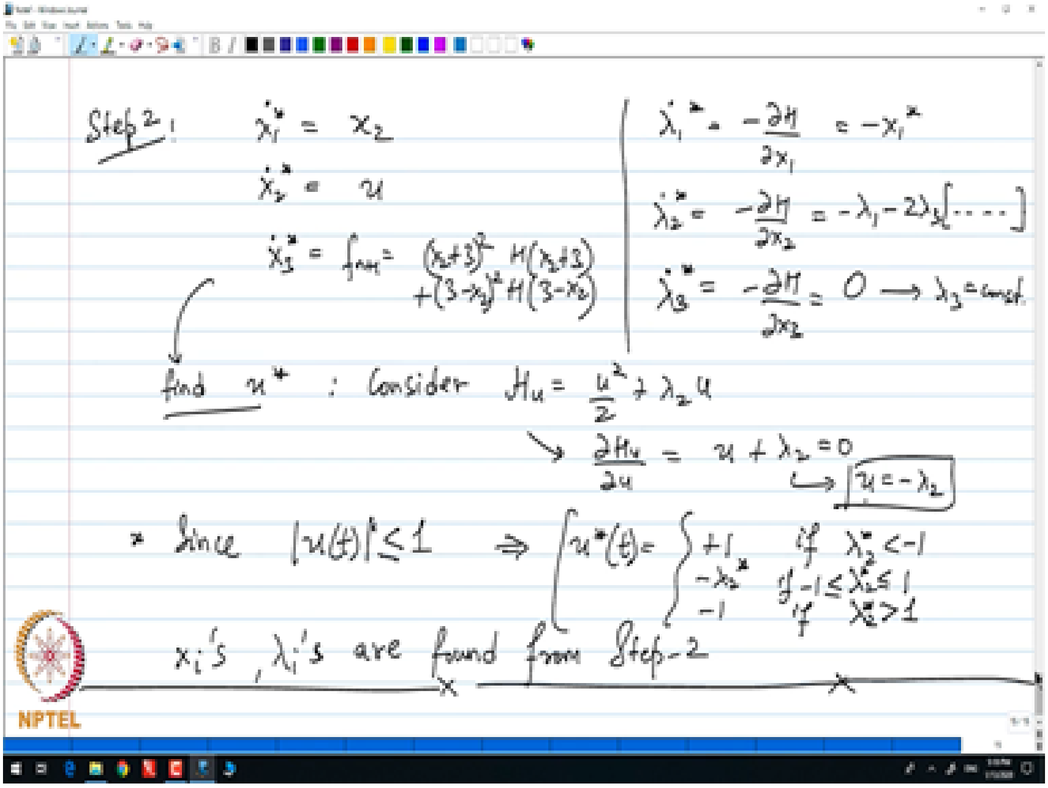

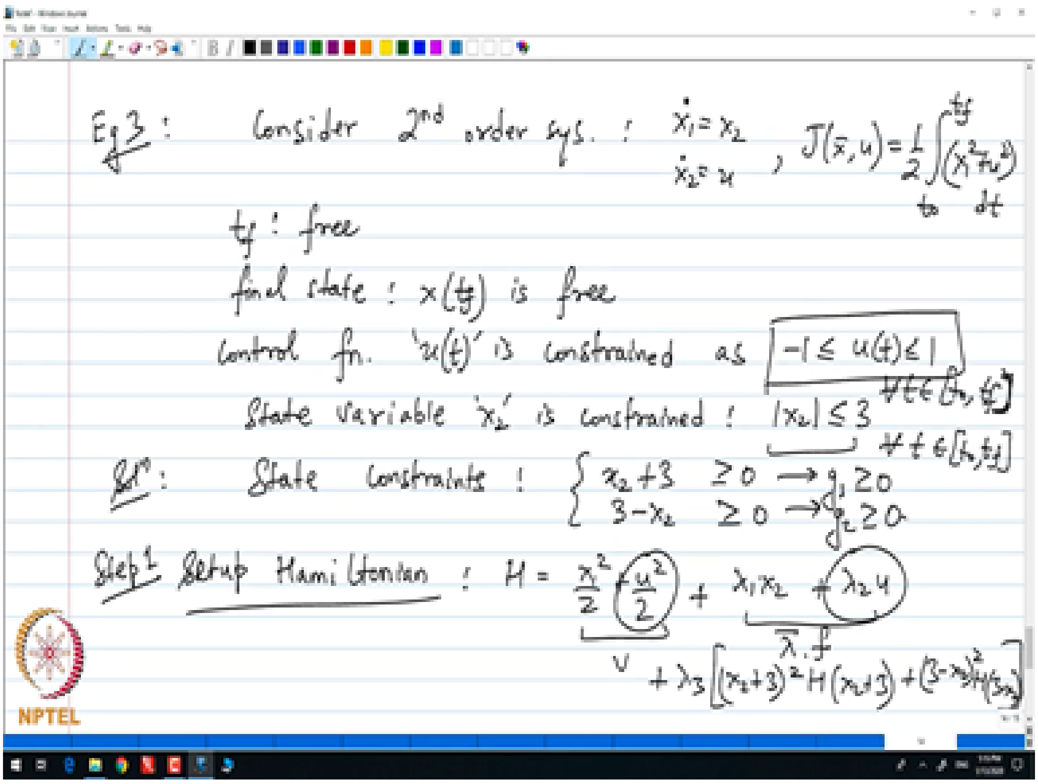

Step 2: $\dot{x}_1^\star = x_2 \quad ; \quad \dot{x}_2^\star = u \quad ;$

$$\dot{x}_3^\star = f_{n+1} = (x_2+3)^2 \, H\,(x_2+3) + (3-x_2)^2 \, H\,(3-x_2)$$

$$\dot{\lambda}_1 = -\tfrac{\partial H}{\partial x_1} = -x_1^\star \quad ;$$

$$\dot{\lambda}_2 = -\tfrac{\partial H}{\partial x_2} = -\lambda_1 - 2\lambda_3 \left[\tfrac{\mathrm{d} f_{n+1}}{\mathrm{d} x_2}\right] \quad ;$$

$$\dot{\lambda}_3 = -\tfrac{\partial H}{\partial x_3} = 0 \qquad \Rightarrow \quad \lambda_3 = \text{constant}$$

Then we have to solve these $2n+2$ equations and I have $2n+2$ constraints x_1 to x_3 and λ_1 to λ_3, this is should be solvable. Find the optimal control $u^\star$. let us look at the Hamiltonian again notice that in the Hamiltonian u appears here as I have circled and u also appears here which means that if we were to minimize or maximize the Hamiltonian with respect to this function u, we only need to look at the circled quantity. So to find $u^\star$ consider the modified Hamiltonian

$$H_u = \tfrac{u^2}{2} + \lambda_2 u$$

From here when we take the derivative of H_u with respect to u we see that

$$\tfrac{\partial H_u}{\partial u} = u + \lambda_2 = 0 \qquad \Rightarrow \quad u = -\lambda_2$$

Since $|u\,(t)| \leq 1$

$$\Rightarrow \quad u^\star\,(t) = \begin{cases} 1 & \text{if}\, \lambda_2^\star < -1 \\ -\lambda_2^\star & \text{if} -1 \leq \lambda_2^\star \leq 1 \\ -1 & \text{if}\, \lambda_2^\star > 1 \end{cases}$$

We have six unknown and six equations the unknowns are x_1 , x_2 , x_3 and λ_1 , λ_2 , λ_3 and from here when we plug $u = -\lambda_2$ provided it is not on the boundary,the control is found right away from our solution. I would just end the discussion in this example by saying that x_i's and λ_i's are found from step (2) that should be the end of the discussion in this example. So that highlights our penalty function method where the penalty is placed on the state variables. Now, we are going to wrap up our discussion by highlighting yet another method.

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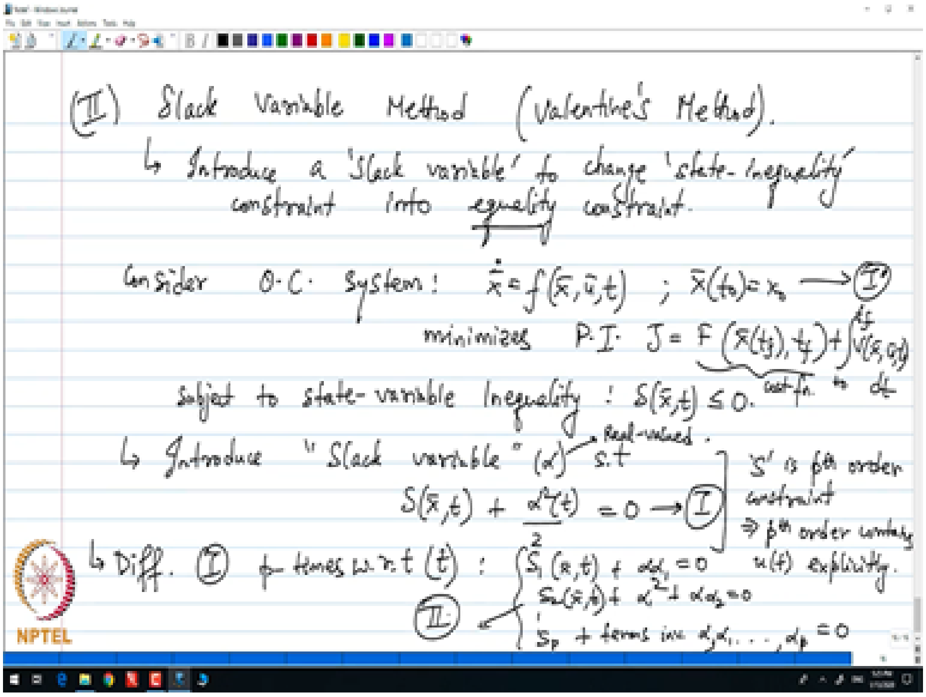

The final method namely the Slack variable method also known as the Valentine's method. So I am going to discuss this Valentines method. We introduce a Slack variable to change the state inequality constraint into an equality constraint. Consider the optimal control system given by

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}, t) \quad ; \quad \bar{x}(t_0) = x_0 \qquad \textbf{(1')}$$

which minimizes the performance index

$$J = F(\bar{x}(t_f), t_f) + \int_{t_0}^{t_f} V(\bar{x}, \bar{u}, t)\, dt$$

subject to the state variable inequality given by $S(\bar{x}, t) \le 0$. Now I am going to change this inequality constrained into an equality constrained by introducing a new variable known as the slack variable. So introduce slack variable (α) such that

$$S(\bar{x}, t) + \frac{\alpha^2(t)}{2} = 0 \qquad \textbf{(1)}$$

Notice that α is real valued function and $\alpha^2(t)$ will always be positive which means $S = -\frac{\alpha^2(t)}{2}$ or $S \le 0$ only when $\alpha = 0$. So this new equality certainly satisfies the original inequality with this new variable. Our assumption here is that S is differentiable up to some orders, let us say the constraint is differentiable up to p orders.So, S is the p^{th} order constraint such that p^{th} order contains $u(t)$ explicitly. Since it is a p^{th} order constraint means that S is smoothly differentiable up to order p. Differentiating (1) p-times with respect to independent variable t ,we get the following set of relations:

$$S_1(\bar{x}, t) + \alpha\alpha_1 = 0$$

$$S_2(\bar{x}, t) + \alpha_1^2 + \alpha\alpha_2 = 0$$

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$$S_p(\bar{x}, t) + \text{terms involving } \alpha_1, \alpha_2, ..., \alpha_p = 0 \qquad \textbf{(2)}$$

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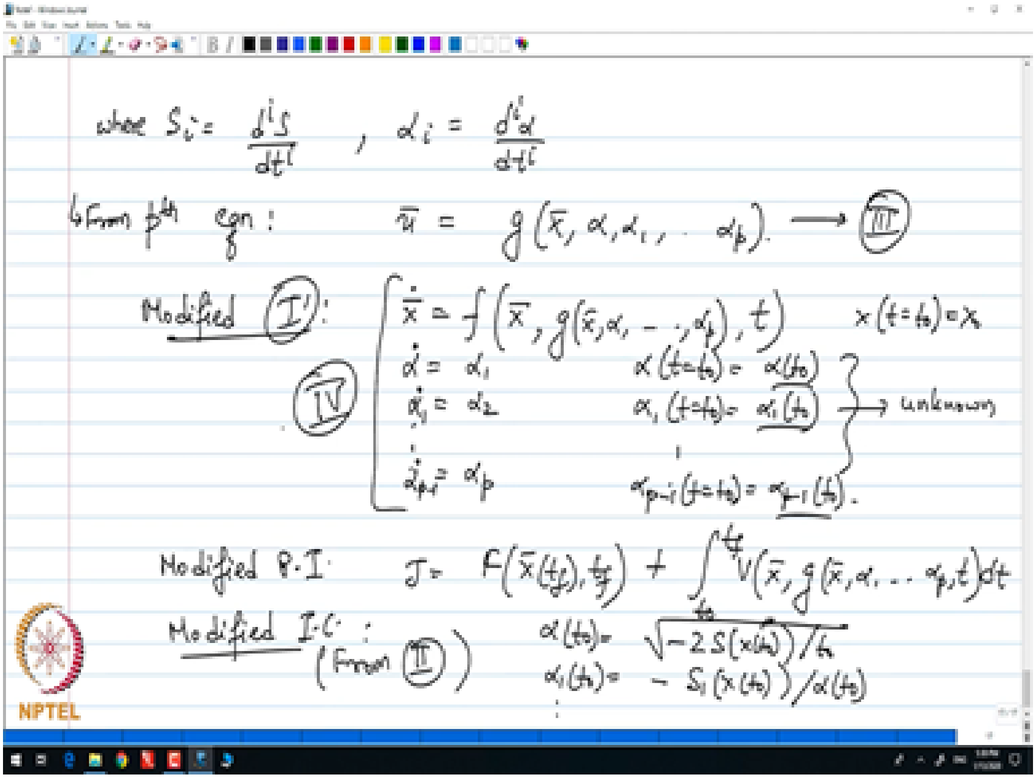

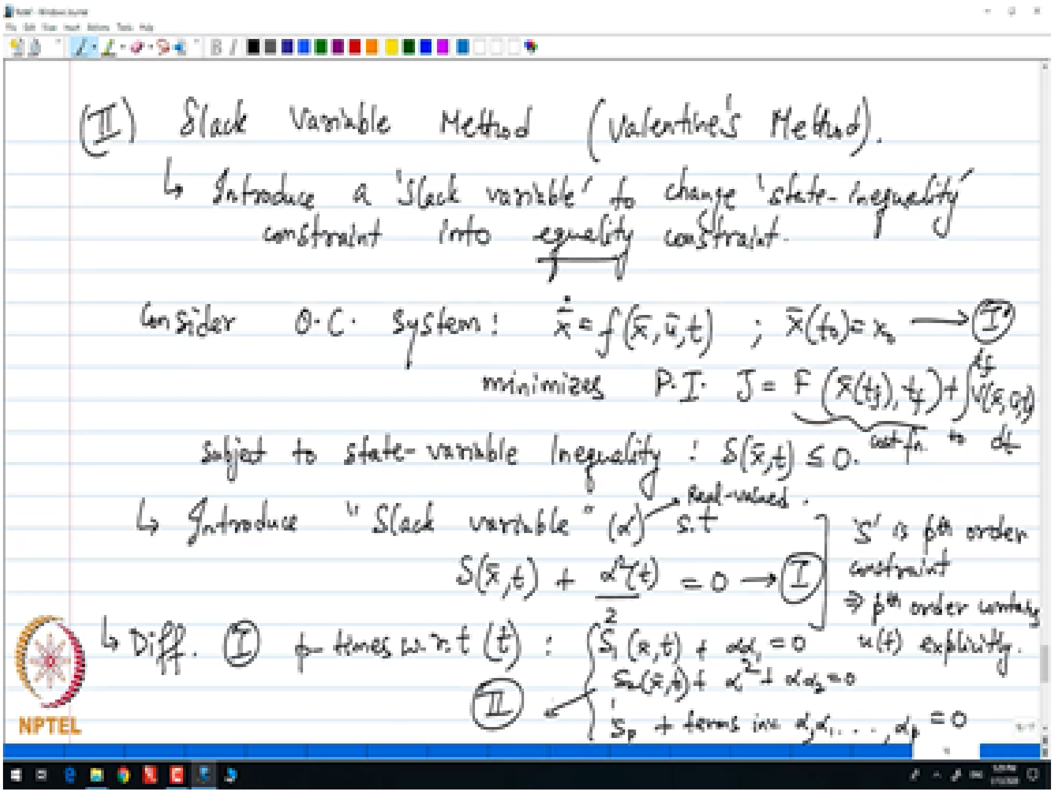

where $\quad S_i = \frac{\partial^i S}{\partial t^i} \quad ; \quad \alpha_i = \frac{\partial^i \alpha}{\partial t^i}$

From p^{th} equation

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$$\bar{u} = g\left(\bar{x}, \alpha, \alpha_1, ..., \alpha_p\right) \qquad \textbf{(3)}$$

The modified plant condition or modified equation $(1')$ is

$$\dot{\bar{x}} = f\left(\bar{x}, g\left(\bar{x}, \alpha, \alpha_1, ..., \alpha_p\right), t\right) \quad ; \qquad x\left(t = t_0\right) = x_0$$

$$\dot{\alpha} = \alpha_1 \quad ; \qquad \alpha\left(t = t_0\right) = \alpha\left(t_0\right)$$

$$\dot{\alpha}_1 = \alpha_2 \quad ; \qquad \alpha_1\left(t = t_0\right) = \alpha_1\left(t_0\right)$$

.

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$$\dot{\alpha}_{p-1} = \alpha_p \quad ; \qquad \alpha_{p-1}\left(t = t_0\right) = \alpha_{p-1}\left(t_0\right) \qquad \textbf{(4)}$$

So finally, modified performance index is as follows

$$J = F\left(\bar{x}\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} V\left(\bar{x}, g\left(\bar{x}, \alpha, \alpha_1, ..., \alpha_p\right), t\right) dt$$

and modified initial condition from (2) are as follows

$$\alpha\left(t_0\right) = \sqrt{\frac{-2S(x(t_0))}{t_0}}$$

$$\alpha_1\left(t_0\right) = \frac{-S_1(x(t_0))}{\alpha(t_0)} \quad \text{and so on .}$$

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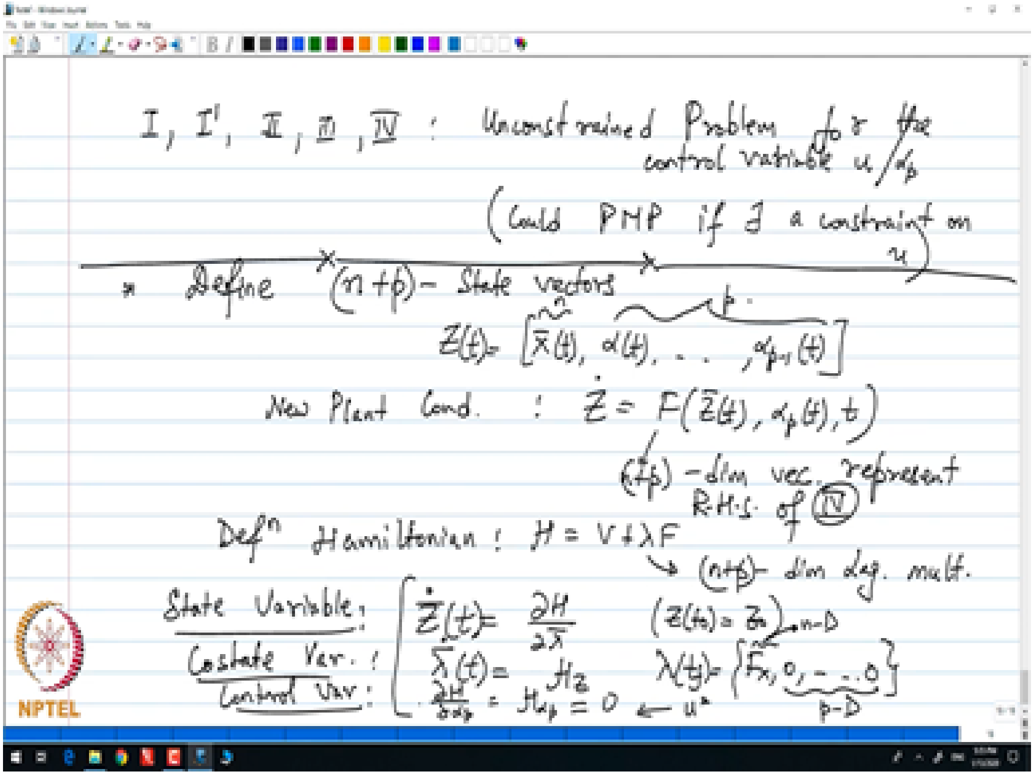

All the relations (1),(1′),(2),(3) and (4) they are going to describe unconstrained problem for the control variable u or α_p whatever it is. Because the p^{th} equation will be given α_p and that will directly give the function u. And we could apply the Pontryagin minimum principle if there is a constraint on u . Let me just summarize how we setup the problem. Define $(n+p)$ state vectors as

$$Z\left(t\right)=\left[\bar{x}\left(t\right),\alpha\left(t\right),...,\alpha_{p-1}\left(t\right)\right]$$

We define new plant condition as $\dot{Z} = F\left(\bar{z}\left(t\right),\alpha_p\left(t\right),t\right)$ where F is an $(n+p)$ dimensional vector representing the RHS of condition (4). And finally define Hamiltonian $H = V + \lambda F$ where λ is $(n+p)$ dimensional Lagrange multiplier.

And finally let me complete description by writing down the equations to solve we have state variables in the form of a vector equation as

$$\dot{Z}\left(t\right)=\frac{\partial H}{\partial \lambda}$$

with given initial condition $Z\left(t_0\right) = z_0$ and co- state variables are as follows $\dot{\lambda} = -H_z$. Also we have that $\lambda\left(t_f\right) = \{F_x, 0, 0, ..., 0\}$. Notice that only the first n components of λ will be non-zero, the rest are all set equal to zero. And control variable is given by

$$\frac{\partial H}{\partial \alpha_p}=H_{\alpha_p}=0$$

And from here we can get control solution $u^{\star}$. So that completesthe description of Slack variable method which completely converts the inequality into an equality and then converts original system into a slightly lengthier system but system which can be treated as an unconstrained optimization problem. In this lecture I am going to end my discussion on the optimal control theory and mention that students are requested to look at the text I have highlighted in my introductory video on the reference for more problems. And in the next lecture I am going to start with another application of calculus of variation namely looking at the motion of carbon Nano rods.