Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 06
Introduction –Euler Lagrange Equations Part-6

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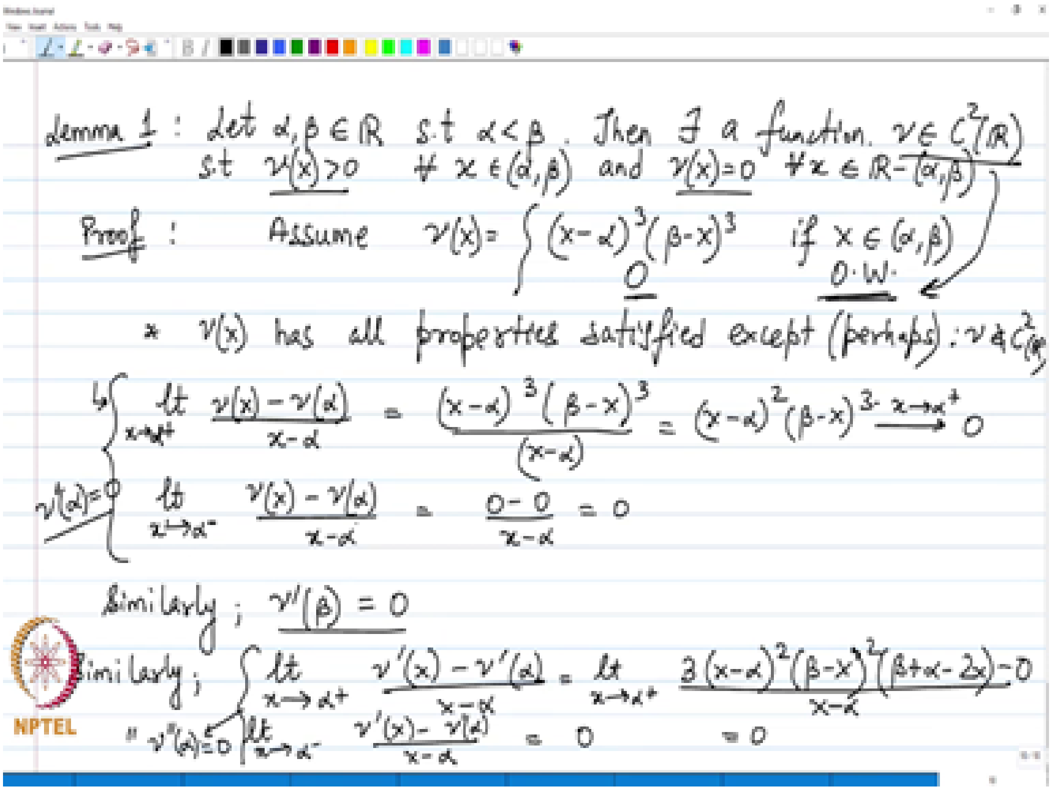

So, the result is broken down into 2 smaller results. Lets denotes first result by lemma 1, the first lemma says that let $\alpha, \beta \in \mathbb{R}$ s.t $\alpha < \beta$, then $\exists$ a function $\nu \in C^2(\mathbb{R})$ (i.e ν is second order differentiable over the entire real axis) s.t $\nu(x) > 0 \ \ \forall\, x \in (\alpha, \beta)$ and $\nu(x) = 0 \,\forall \in \mathbb{R} - (\alpha, \beta)$

So, all this lemma says is that, for any given real interval, I can always construct a second order differentiable function, which is positive inside the interval and vanishes outside the interval. And the proof of this lemma is straightforward, by assuming the following function.

$$\text{Assume} \qquad \nu(x) = \begin{cases} (x-\alpha)^3(\beta-x)^3 & x \in (\alpha,\beta) \\ 0 & otherwise \end{cases}$$

We can see that ν is positive inside the interval
$\Rightarrow$ $\nu(x)$ has all properties satisfied except (perhaps) $\nu \notin C^2(R)$

we just need to figure out what is the first and the second derivative of this function inside the interval (α, β). So, as I just said; ν of x has all properties satisfied, except perhaps ν is not second order differentiable. Well, we are not sure unless and until we have shown that it is second order differentiable.

So, let us now calculate the first and the second derivative of this function. So, the first derivative we can go ahead by evaluating the necessary limits.

$$\lim_{x\to\alpha^+}\frac{\nu(x)-\nu(\alpha)}{x-\alpha}=\lim_{x\to\alpha^+}\frac{(x-\alpha)^3(\beta-x)^3}{(x-\alpha)}=\lim_{x\to\alpha^+}(x-\alpha)^2(\beta-x)^3\to 0$$

$$For \;\; \nu(x)=0\lim_{x\to\alpha^+}\frac{\nu(x)-\nu(\alpha)}{x-\alpha}=\frac{0-0}{x-\alpha}=0$$

i.e $\nu^{'}(\alpha)$ exist and $\nu^{'}(\alpha)=0$, Similarly, $\nu^{'}(\beta)=0$
What we are trying to show is that ν is indeed differentiable up to second order at the boundary points, because in the interior, nu is a polynomial. So, that will be definitely differentiable. Outside the interval, it is 0. So, certainly differentiable, but perhaps not at the boundary points.

we have showed that it is first order differentiable, at the boundary points. So, similarly, Let us look at the second derivative.

$$\nu^{''}(\alpha)=\lim_{x\to\alpha^+}\frac{\nu^{'}(x)-\nu^{'}(\alpha)}{x-\alpha}=\lim_{x\to\alpha^+}\frac{3(x-\alpha)^2(\beta-x)^2(\beta+\alpha-2x)-0}{(x-\alpha)}=0$$

Similarly $\lim_{x\to\alpha^-}\nu^{'}(x)$ exist and equal to zero i.e $\nu^{''}(\alpha)$ exist and $\nu^{''}(\alpha)=0$, Similarly, $\nu^{''}(\beta)=0$

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Similarly $\nu''(\beta) = 0$

$$\nu'(x) = \begin{cases} 3(x-\alpha)^2(\beta-x)^2(\beta+\alpha-2x) & x \in (\alpha,\beta) \\ 0 & \text{o.w.} \end{cases}$$

$$\nu''(x) = \begin{cases} 6(x-\alpha)(\beta-x)\left[(x-\alpha)^2 + (\beta-x)^2 - 3(x-\alpha)(\beta-x)\right] & x \in (\alpha,\beta) \\ 0 & \text{o.w.} \end{cases}$$

$\nu(x) \in C^2(\mathbb{R}) \longleftarrow$ "Choice of perturbation fn."

From (I): for small ϵ: sign of $J(\hat{y}) - J(y)$ is determined by the sign of $\delta J(\eta, y)$, unless $\boxed{\delta J(\eta,y) = 0 \ \forall \eta \in H}$

$\downarrow$ necessary cond. for local max.

(Analogue case of $\nabla f = 0$)

From (II):
$$\int_{x_0}^{x_1} \underbrace{\eta'}_{2^{nd}} \underbrace{\frac{\partial f}{\partial y'}}_{1^{st}} dx = \underbrace{\eta \frac{\partial f}{\partial y'}\Big|_{x_0}^{x_1}}_{0} - \int_{x_0}^{x_1} \eta \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] dx \quad \text{(II')}$$

$\Rightarrow$ (II) (II'):
$$\delta J(\eta, y) = \int_{x_0}^{x_1} \eta \underbrace{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]}_{E} dx = 0 \quad \text{(III)}$$

Define: $E : [x_0, x_1] \to \mathbb{R}$ by $E = \frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]$

(III) $= \langle \eta, E \rangle = \int \eta E \, dx = 0$ (for extremal)

We have

$$\nu'(x) = \begin{cases} 3(x-\alpha)^2(\beta-x)^2(\beta+\alpha-2x) & x \in (\alpha,\beta) \\ 0 & otherwise \end{cases}$$

$$\nu''(x) = \begin{cases} 6(x-\alpha)(\beta-x)\left[(x-\alpha)^2+(\beta-x)^2-3(x-\alpha)(\beta-x)\right] & x \in (\alpha,\beta) \\ 0 & otherwise \end{cases}$$

Hence $\nu(x) \in C^2(\mathbb{R})$ and that concludes the proof of our first lemma. So essentially what we have shown is that ν is a choice of our perturbation function.

Let us go back to the slide, where we are going to use the lemma. So, this is the choice of our perturbation function. So, let us go back to slide, 2 slides back. So, we are trying to show that this holding, this integral of this quantity holding, well set equal to 0 implies that this function E is 0. And we are showing it via contradiction, by finding a particular value of η and in the first result we have shown that ν satisfies all the properties of the perturbation function. That will be our choice.

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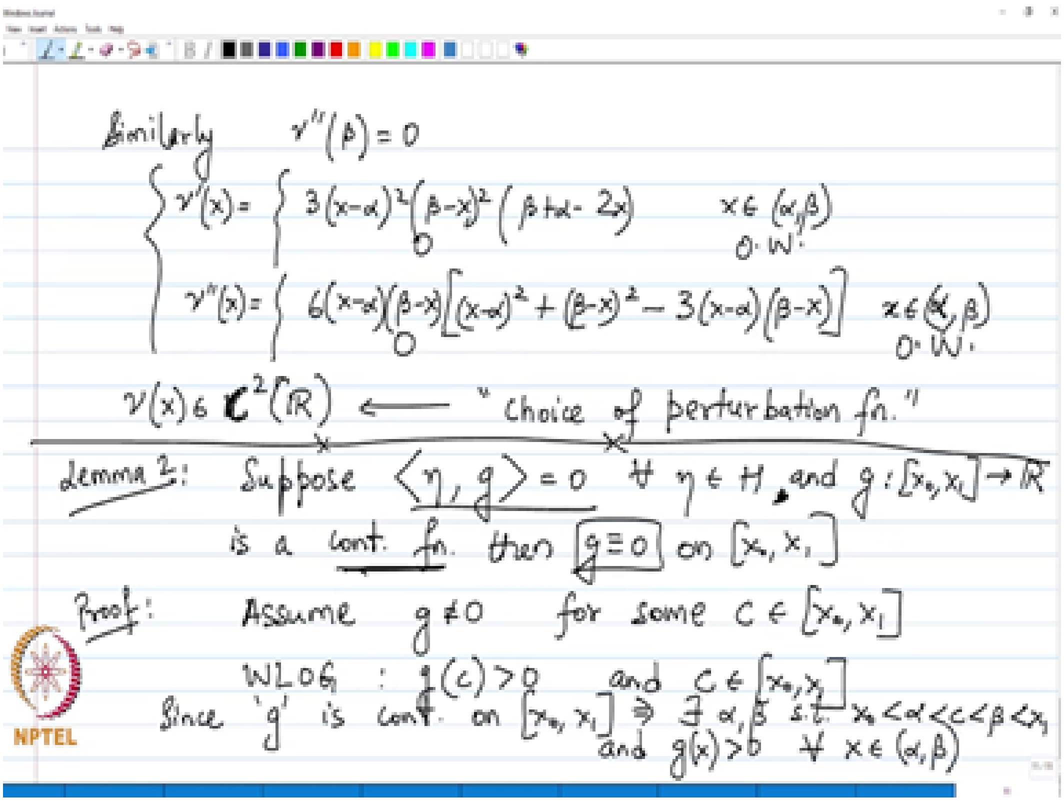

Lemma 2: Suppose $\langle \eta, g \rangle = 0 \; \forall \, \eta \in H$ and $g : [x_o, x_1] \to \mathbb{R}$ is a continous function then $g \equiv 0$ on $[x_o, x_1]$

Again, the proof is very straightforward. So, without loss of generality, we will show some contradiction, we assume that g $\neq 0$ for some C $\in [x_o, x_1]$ and then without loss of generality, let us further assume $g(c) > 0$(You could always assume $g(c) < 0$) and c $\in [x_o, x_1]$ Since g is continous on $[x_o, x_1] \Rightarrow \exists \, \alpha$ s.t $x_o < \alpha < c < \beta < x_1$ and $g(x) > 0 \; \forall \, x \in (\alpha, \beta)$

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By Lemma 1 : $\exists$ a function $\nu \in C^2[x_0, x_1]$ s.t. $\nu > 0$ $\forall\, x \in (\alpha, \beta)$ and $\nu(x) = 0$ $\forall\, x \in [x_0, x_1] - (\alpha, \beta)$

$\Rightarrow \nu \in H \leftarrow$ set of perturb. fns.

and $\langle \nu, g \rangle = \int_{x_0}^{x_1} \nu g \, dx = \int_{\alpha}^{\beta} \nu g \, dx > 0$

$(\Rightarrow \Leftarrow)$

Similarly $\nu''(\beta) = 0$

$$\nu'(x) = \begin{cases} 3(x-\alpha)^2(\beta-x)^2(\beta+\alpha-2x) & x \in (\alpha, \beta) \\ 0 & \text{o.w.} \end{cases}$$

$$\nu''(x) = \begin{cases} 6(x-\alpha)(\beta-x)\left[(x-\alpha)^2 + (\beta-x)^2 - 3(x-\alpha)(\beta-x)\right] & x \in (\alpha, \beta) \\ 0 & \text{o.w.} \end{cases}$$

$\nu(x) \in C^2(\mathbb{R}) \longleftarrow$ "Choice of perturbation fn."

Lemma 2: Suppose $\boxed{\langle \eta, g \rangle = 0}$ $\forall\, \eta \in H$, and $g : [x_0, x_1] \to \mathbb{R}$ is a cont. fn. then $\boxed{g \equiv 0}$ on $[x_0, x_1]$

Proof: Assume $g \neq 0$ for some $c \in [x_0, x_1]$

WLOG : $g(c) > 0$ and $c \in [x_0, x_1]$

Since 'g' is cont. on $[x_0, x_1] \Rightarrow \exists\, \alpha, \beta$ s.t. x_0
and $g(x) > 0$ $\forall\, x$

So, by lemma 1, $\Rightarrow \exists$ a function $\nu \in C^2[x_o, x_1]$ s.t $\nu > 0 \; \forall \; x \in (\alpha, \beta)$ and $\nu(x) = 0 \; \forall \; x \in [x_o, x_1] - (\alpha, \beta)$
$\Rightarrow \nu \in H$ (Set of perbutation function) and
$\langle \nu, g \rangle = \int_{x_o}^{x_1} \nu g dx = \int_{\alpha}^{\beta} \nu g dx > 0$(because outside this interval ν is identically 0), which is a contradiction because we have assumed that that this particular integration must be equal to 0

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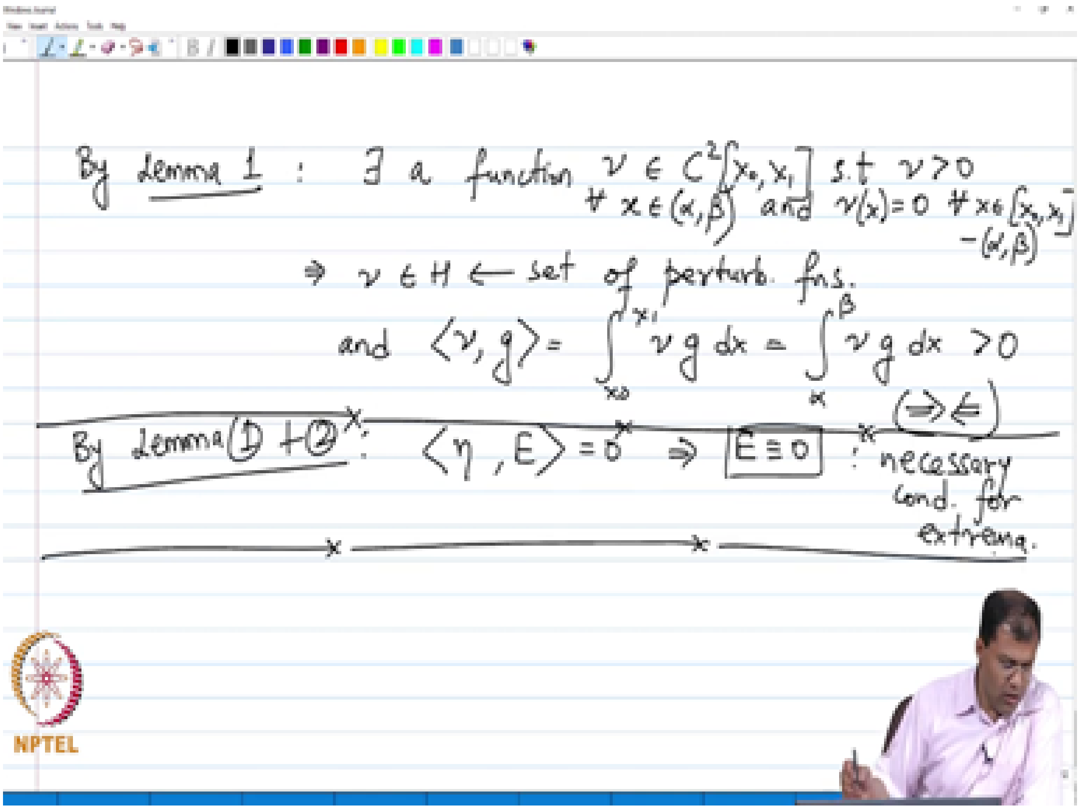

So, by lemma 1 and 2: $\langle \eta, E \rangle = 0 \Rightarrow E \equiv 0$ which is our necessary condition for extrema.

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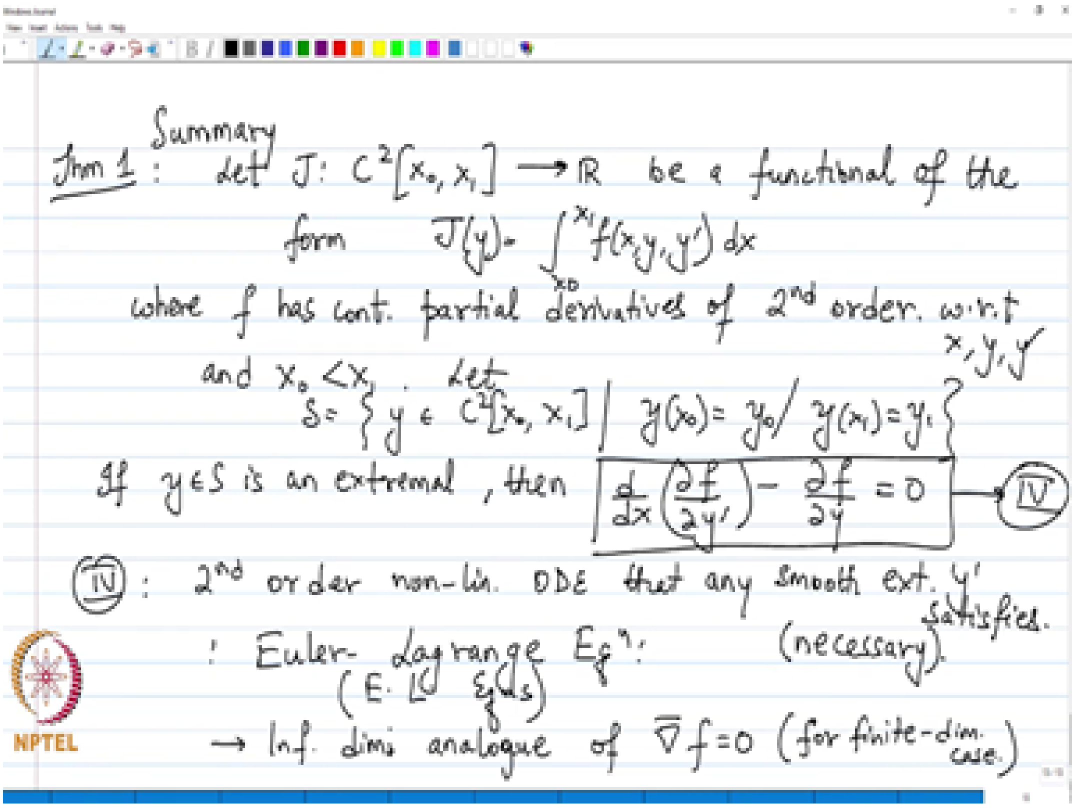

So, let us now recap in the form of a theorem which is the most important theorem of this entire course, summary.

From **I**: Let $C^2[x_o, x_1] \to \mathbb{R}$ be a functional of the form $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'})dx$
where f has continuous partial derivatives of 2^{nd} order w.r.t x, y, $y^{'}$ and $x_o < x_1$ and
Let S $= \{y \in C^2[x_o, x_1] | y(x_o) = y_o, y(x_1) = y_1\}$

Now, the result says if $y \in S$ is an extremal, it must satisfy the equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \qquad \textbf{IV}$$

This is the most important results(**IV**), which we are going to look at in more depth, over the next several lectures.

The equation **IV** is a second order non-linear ODE, that any smooth extremal y satisfies and that is also the necessary condition and is known as the famous Euler Lagrange Equation (E.L equations).

Note that this is the infinite dimensional analog of $\bar{\bigtriangledown} f = 0$, which was the condition for the finite dimensional case.

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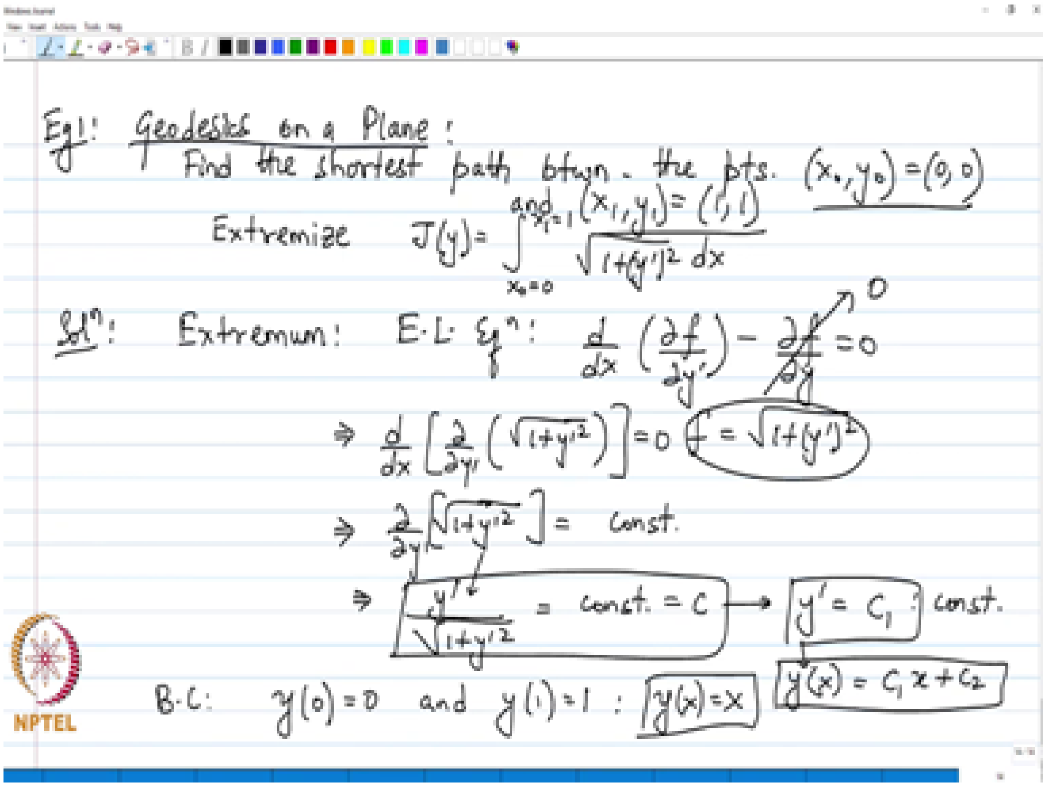

So, my first example involves finding the Geodesics over a plane, Again, students please recall that Geodesics on a plane will involve finding the shortest. So, this particular problem has been described in the previous lecture, it involves finding the shortest path between the points. Let me now fix some points. Let me say that my first point $(x_o, y_o) = (0, 0)$ and my second point $(x_1, y_1) = (1, 1)$ So, I am just fixing some points randomly.

So, we have to extremize. Given these fixed points, we have to extremize J(y) which is the functional from $x_o = 0$ to $x_1 = 1$ of the total arclength of the curve $J(y) = \int_{x_o}^{x_1} \sqrt{1 + (y')^2}dx$

To find the extremum, It must satisfy the Euler Lagrange Equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

$$\Rightarrow \frac{d}{dx}\left[\frac{\partial}{\partial y'}(\sqrt{1 + (y')^2})\right] = 0$$

As f is independent of y, therefore $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow \frac{\partial}{\partial y'}\left[\sqrt{1 + (y')^2}\right] = Constant$$

$$\Rightarrow \frac{y'}{\sqrt{1 + (y')^2}} = Constant = C$$

$$\Rightarrow y' = C_1 = constant \Rightarrow y(x) = C_1 x + C_2$$

B.C: y(0) = 0 and y(1) = 1, we have y(x) = x

So, the Geodesics on a plane come out to be a straight line.

With the necessary boundary condition, we can eliminate all the involved constants to come at a particular solution to the problem. Here we have shown that the extremal is the straight line. We have not mentioned anything whether this extremal is maximum or minimum. We will later on show that this is a minimum of the problem, when we talk about the sufficient conditions of the functional, for finding the extremal of the functional.

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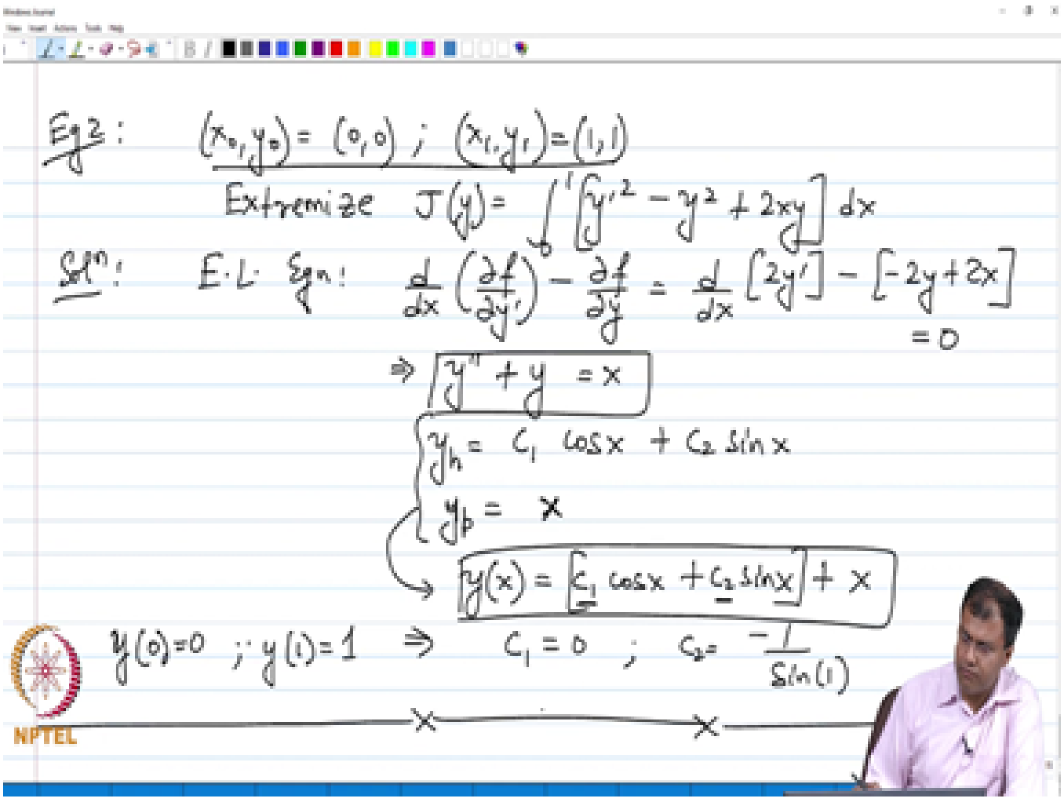

Now, let us look at another **example**. let us fix the boundary points $(x_o, y_0) = (0, 0)$, $(x_1, y_1) = (1, 1)$ and we have to find the extremal of the functional J(y),

$$J(y) = \int_0^1 \left[y'^2 - y^2 + 2xy \right] dx$$

Solution: E.L equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = \frac{d}{dx}[2y'] - [-2y + 2x] = 0$$

$$\Rightarrow y'' + y = x \qquad \text{(this is a non-homogenous ODE)}$$

$$\Rightarrow y_c = C_1 \cos x + C_2 \sin x \;\; and \;\; y_p = x$$

$$\Rightarrow y(x) = [C_1 \cos x + C_2 \sin x] + x$$

With B.C y(0) = 0, y(1) = 1 $\Rightarrow C_1 = 0, \;\; C_2 = -\frac{1}{\sin 1}$

So, I get a particular extremal, out of the family of extremals, that is how we do a typical simple case study of finding the extremal to any functional.

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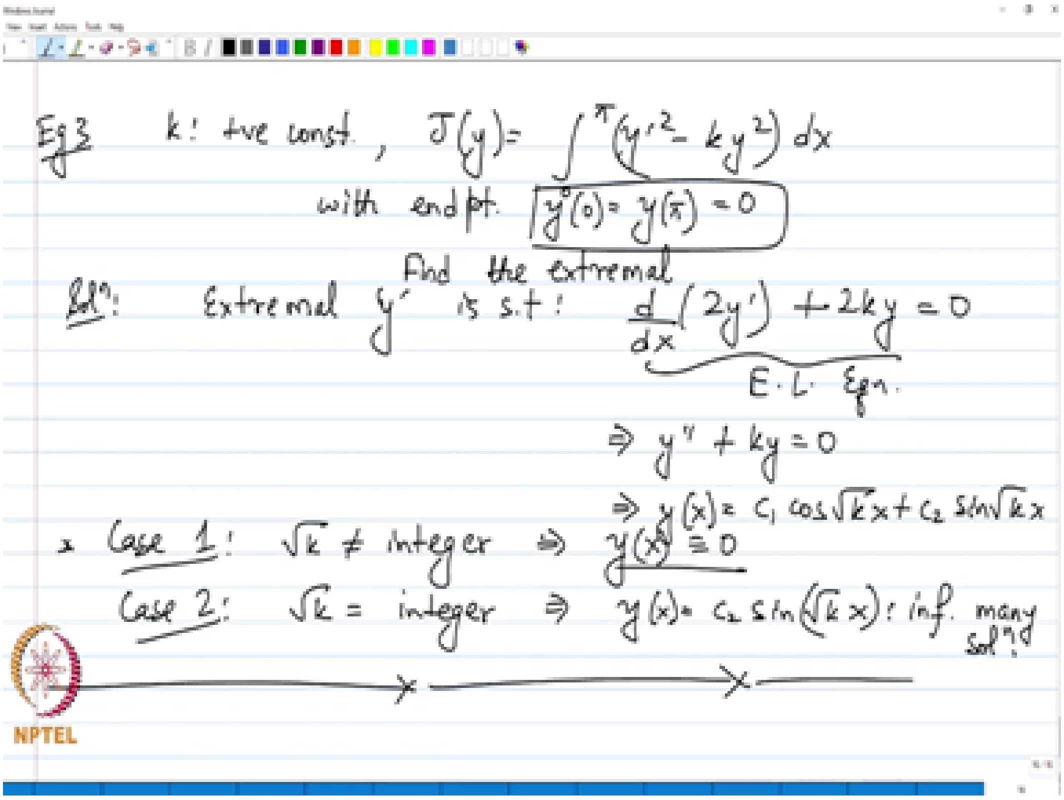

Let us quickly wrap up this lecture by giving another quick example. Let us say I have a positive constant k and J is a functional such that

$$J(y) = \int_0^{\pi} (y'^2 - ky^2)dx$$

with end points $\quad$ y(0) = y(π) = 0, $\quad$ we have to find the Extremal

So, the extremal y is s.t it must satisfy the Euler Lagrange equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$

$$\Rightarrow \frac{d}{dx}(2y') + 2ky = o$$

$$\Rightarrow y'' + ky = o \qquad \text{(a homogenous ODE)}$$

$$\Rightarrow y(x) = C_1 \cos\sqrt{k}x + C_2 \sin\sqrt{k}x$$

Now we suppose two cases

Case 1: $\sqrt{k} \neq$ integer $\Rightarrow y(x) \equiv 0$

Case 2: $\sqrt{k} =$ integer $\Rightarrow y(x) = C_2 \sin(\sqrt{k}x)$

So, I have infinitely many solutions in this case, depending on the value C_2.

So, wraps up our discussion in this lecture. And in the next lecture, I am going to talk about further several cases of Euler Lagrange Equations. And in particular, I am going to do 4 case studies or 4 specific cases of the solution to the Euler Lagrange Equation. Thank you very much for listening. Thanks a lot.