Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 59
Constrained Optimization in Optimal Control Theory Part 5

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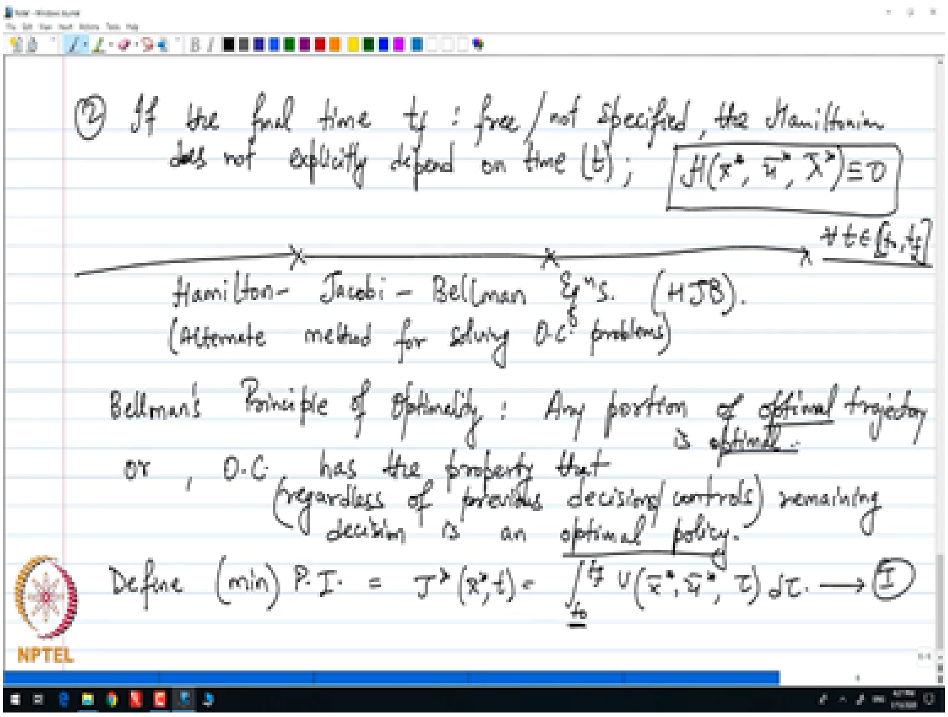

So, the form that I would like to highlight now next is the solution of the Hamilton-Jacobi-Bellman equations. Students who have attended my previous lectures will realize that these equations are similar to the Hamilton-Jacobi equation which used to find the generating function. And through the generating function we use to find the extremal curves via taking certain derivatives.

This time we will see that our generating function in our Hamilton-Jacobi-Bellman equation is the functional itself. Before we start let me just state the principal of Hamilton-Jacobi-Bellman optimality condition. The optimality condition by Hamilton-Jacobi-Bellman says that any portion of the optimal curve is optimal or in other words it does not depend what the solution was at previous times but the time that follow which satisfies the Hamilton-Jacobi-Bellman equation will always be optimal in the future time. So, this is an alternate method for solving optimal control problems.

Let me start with the the Bellman's principle of optimality which says that any portion of optimal trajectory is optimal. As I said the meaning of this statement is it does not depend on the previous times whatever the solution was, you just need to look at the future times and the solution to this equation will guarantee optimality for the solution at the future time.

Or the optimal control has the property that regardless of previous decisions or controls the remaining decision is an optimal policy.So, let us start our basic setup of the problem. We define the minimal performance index to be

$$J^{\star}\left(\bar{x}^{\star}, t\right) = \int_{t}^{t_f} V\left(\bar{x}^{\star}, \bar{u}^{\star}, \tau\right) d\tau \quad \textbf{(1)}$$

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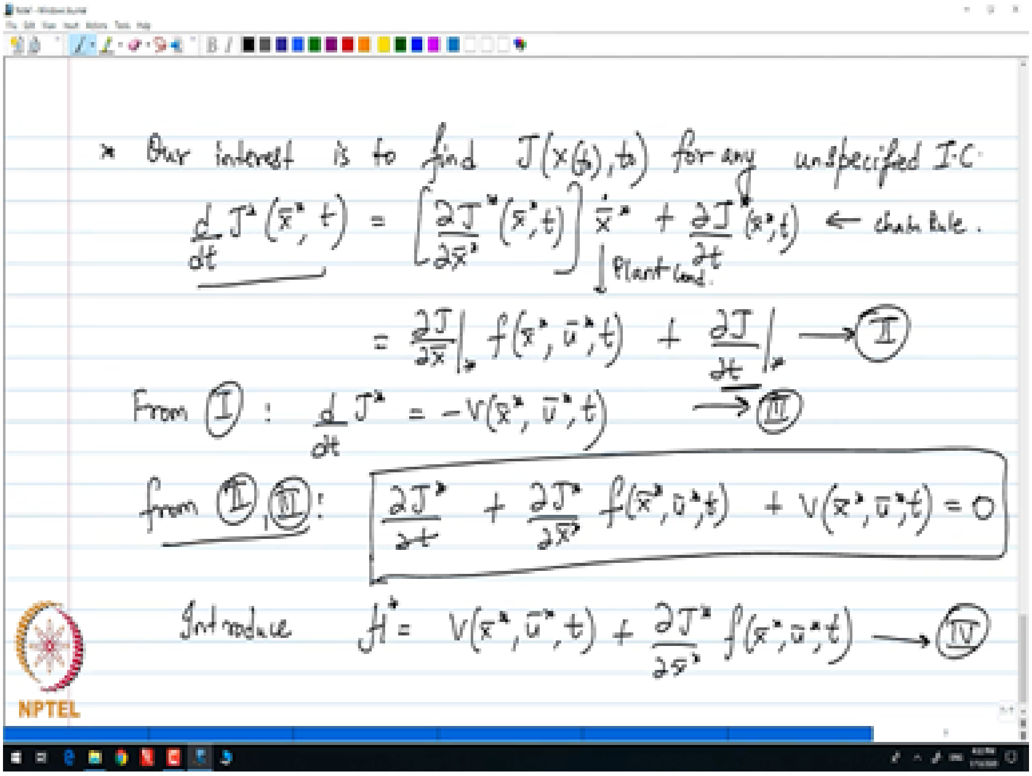

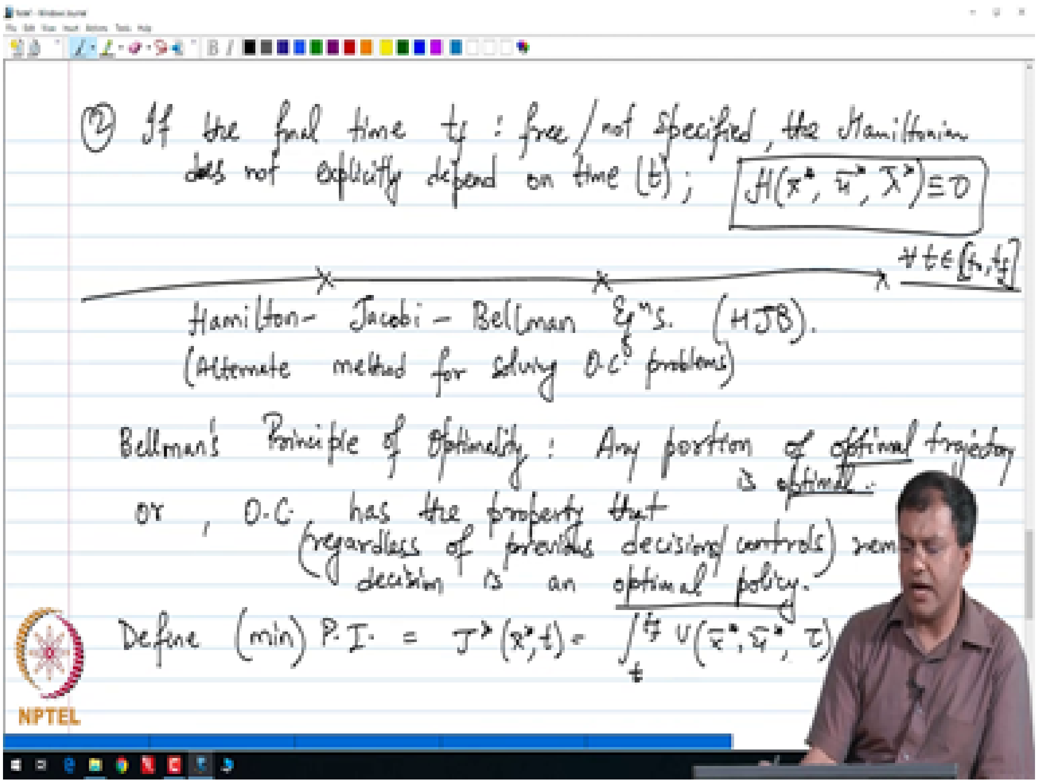

Our interestis to find $J\left(x\left(t_0\right), t_0\right)$ for any unspecified initial condition. Before I start describing the problem note that I can always find

$$\frac{\mathrm{d}}{\mathrm{d}t} J^{\star}\left(\bar{x}^{\star}, t\right) = \left[\frac{\partial J^{\star}}{\partial \bar{x}^{\star}}\left(\bar{x}^{\star}, t\right)\right] \dot{\bar{x}}^{\star} + \frac{\partial J^{\star}}{\partial t}\left(\bar{x}^{\star}, t\right)$$

$$= \frac{\partial J}{\partial \bar{x}} \mid_{\bar{x}^{\star}} f\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) + \frac{\partial J}{\partial t} \mid_{\bar{x}^{\star}} \quad \textbf{(2)}$$

From (1) we see that

$$\frac{\partial J^{\star}}{\partial t} = -V\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) \quad \textbf{(3)}$$

From (2) and (3) we see the following equation

$$\frac{\partial J^{\star}}{\partial x} + \frac{\partial J^{\star}}{\partial \bar{x}^{\star}} f\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) + V\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) = 0$$

This equation is nothing but Hamilton-Jacobi-Bellman equation that we were after.Further let us simplify this equation by introducing the Hamiltonian H function

$$H^{\star} = V\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) + \frac{\partial J^{\star}}{\partial \bar{x}^{\star}} f\left(\bar{x}^{\star}, \bar{u}^{\star}, t\right) \quad \textbf{(4)}$$

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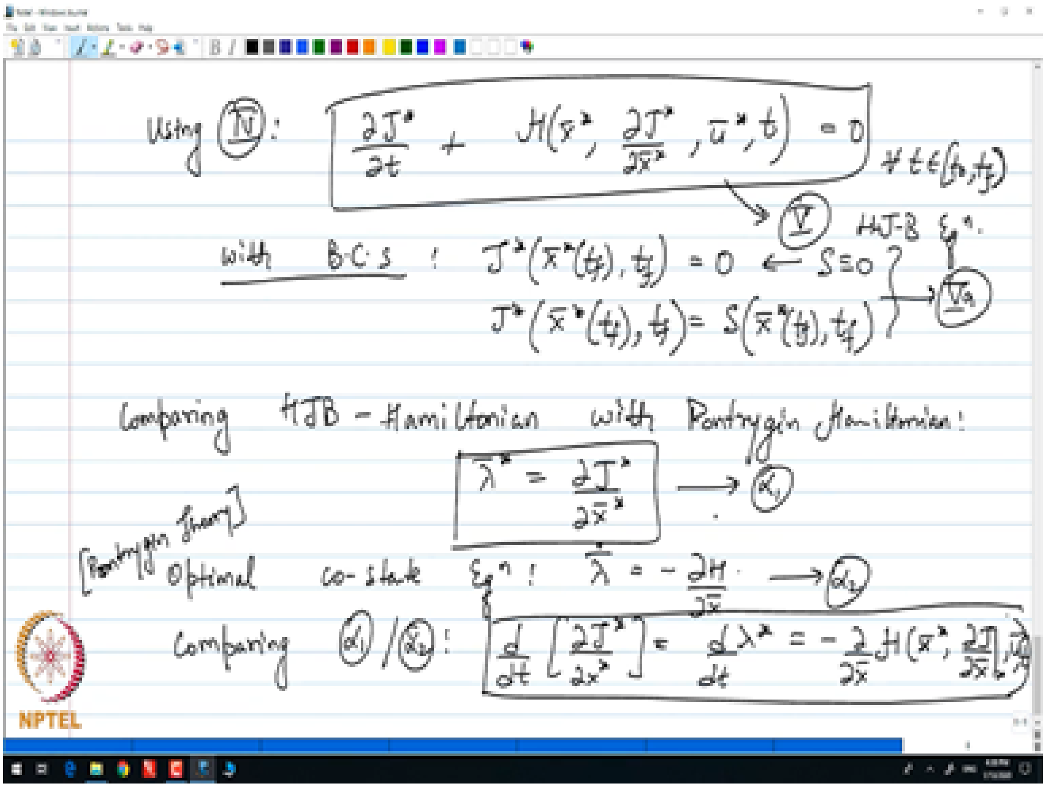

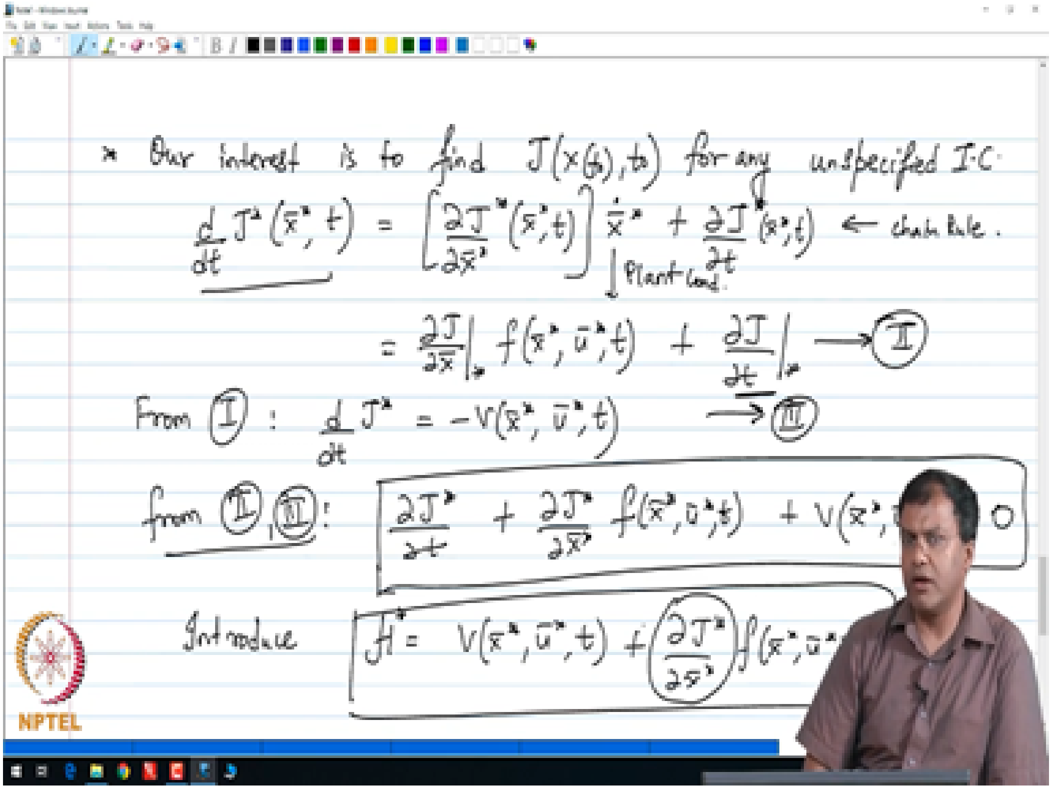

Using (4), we have

$$\frac{\partial J^\star}{\partial t} + H\left(\bar{x}^\star, \frac{\partial J^\star}{\partial \bar{x}^\star}, \bar{u}^\star, t\right) = 0 \quad \forall \;\; t \in [t_0, t_f) \qquad \textbf{(5)}$$

This is the form of HJB equation that we will be using in our examples. There is also the set the boundary conditions as follows

$$J^\star\left(\bar{x}^\star\left(t_f\right), t_f\right) = 0 \qquad if \;\; S \equiv 0 \qquad \text{otherwise}$$

$$J^\star\left(\bar{x}^\star\left(t_f\right), t_f\right) = S\left(\bar{x}^\star\left(t_f\right), t_f\right) \qquad \textbf{(5a)}$$

So, we still retain the form of the performance index without the cost function because the cost function appears in the form of a boundary condition.So, we can compare this HJB equation with our regular Hamiltonian formulation or the Pontryagin Hamiltonian function. So, comparing HJB Hamiltonian with the Pontryagin Hamiltonian and we see that

$$\bar{\lambda}^\star = \frac{\partial J^\star}{\partial \bar{x}^\star} \qquad (\alpha_1)$$

And optimal co-state equation is

$$\dot{\bar{\lambda}} = -\frac{\partial H}{\partial \bar{x}} \qquad (\alpha_2)$$

this is from Pontryagin theory. Now, comparing results in (α_1) and (α_2) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial J^\star}{\partial x^\star}\right] = \frac{\mathrm{d}\lambda^\star}{\mathrm{d}t} = -\frac{\partial}{\partial \bar{x}} H\left(\bar{x}^\star, \frac{\partial J}{\partial \bar{x}}\mid_{\bar{x}^\star}, \bar{u}^\star, t\right)$$

Let us summarize solution methodology and then we will look at an example. So, the solution methodology for HJB setup is as follows:

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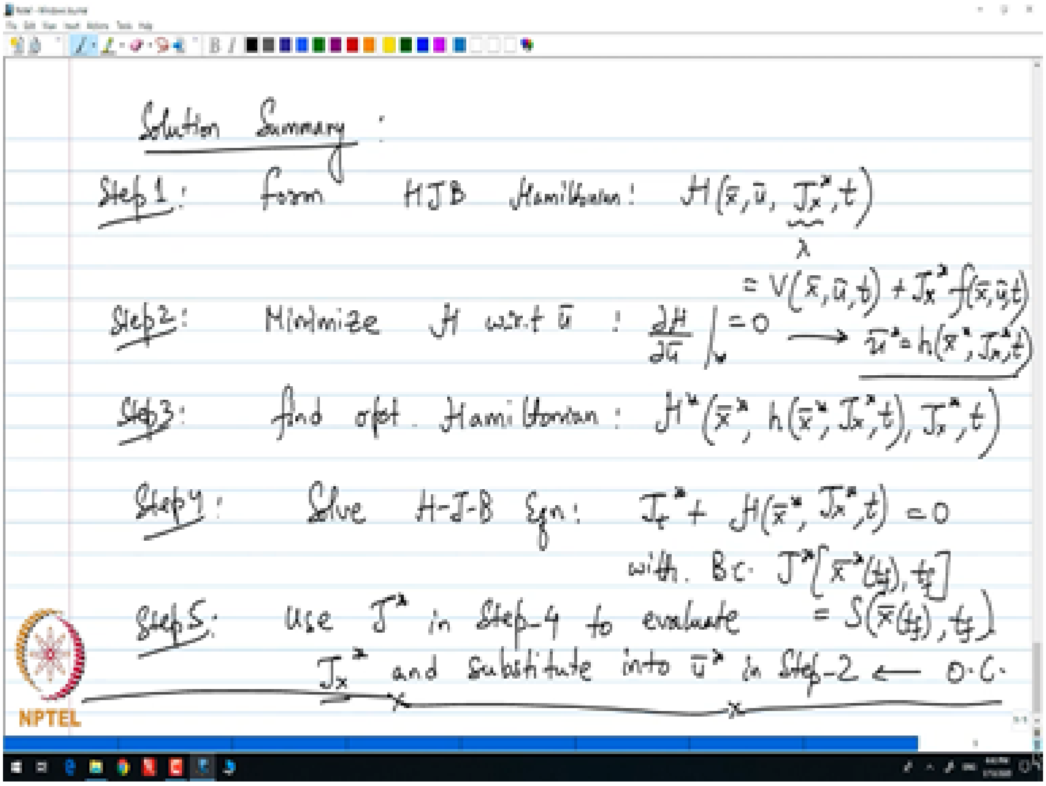

Solution summary is as follows:

Step 1 : Form the HJB Hamiltonian.

$$H\left(\bar{x}, \bar{u}, J_x^\star, t\right) = V\left(\bar{x}, \bar{u}, t\right) + J_x^\star f\left(\bar{x}, \bar{u}, t\right)$$

Step 2 : Minimize H with respect to $\bar{u}$ by setting $\frac{\partial H}{\partial \bar{u}} \mid_{\bar{x}^\star} = 0$

From here we see that $\bar{u}^\star = h\left(\bar{x}^\star, J_x^\star, t\right)$

Step 3 : Find optimal Hamiltonian $H^\star\left(\bar{x}^\star, h\left(\bar{x}^\star, J_x^\star, t\right), J_x^\star, t\right)$

Step 4 : Solve the HJB equation which is

$$J_t^\star + H\left(\bar{x}^\star, J_x^\star, t\right) = 0 \qquad \text{with boundary condition}$$

$$J^\star\left[\bar{x}^\star\left(t_f\right), t_f\right] = S\left(\bar{x}\left(t_f\right), t_f\right)$$

Step 5 : Use $J^\star$ in step (4) to evaluate $J_x^\star$ and substitute into $\bar{u}^\star$ in step (2) to give optimal control solution.

So, that will complete the description of the problem. So, let us look at example of this solution methodology.

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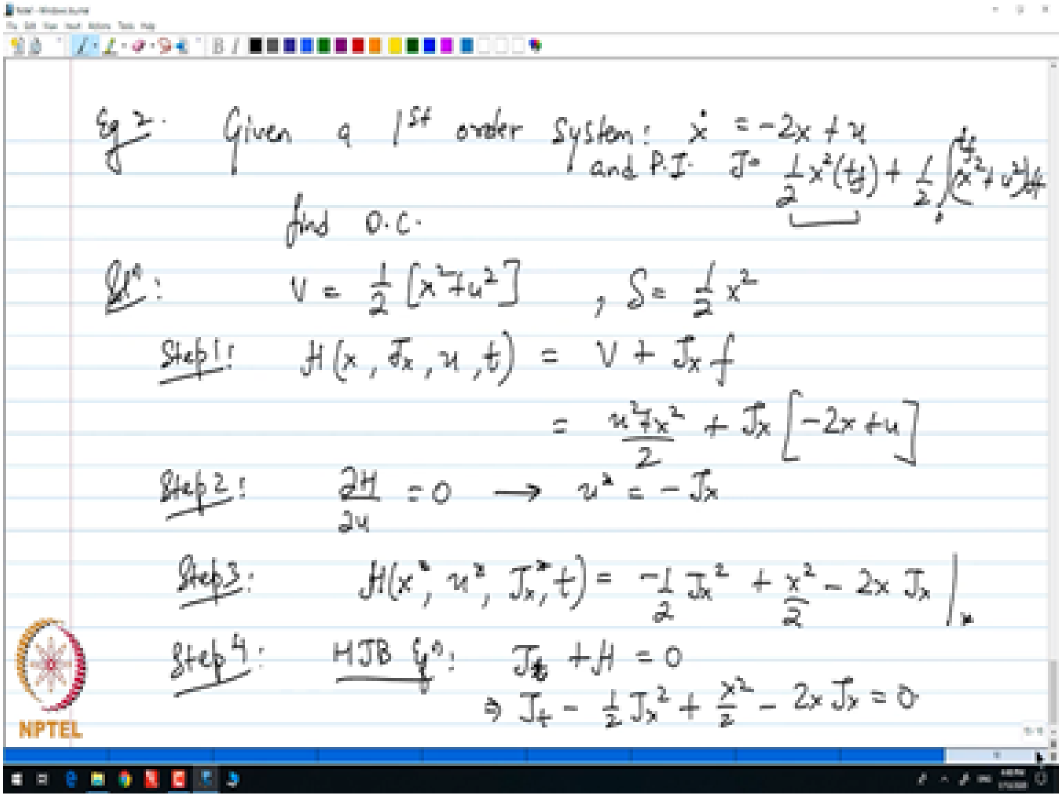

This is example (2): Given a first order system $\dot{x} = -2x + u$ and performance index is

$$J = \frac{1}{2}x^2\left(t_f\right) + \frac{1}{2}\int_0^{t_f}\left(x^2 + u^2\right)dt$$

Find the optimal control. We need to find the optimal control for the setup, it is a one variable problem. we use a step wised solution methodology. Firstly we have to identify the various components here.

$$V = \frac{1}{2}\left[x^2 + u^2\right] \quad ; \quad S = \frac{1}{2}x^2$$

Step 1 : We have to write the Hamiltonian

$$H\left(\bar{x}, \bar{u}, J_x^{\star}, t\right) = V + J_x f$$

$$= \frac{u^2+x^2}{2} + J_x\left[-2x + u\right]$$

Step 2 : We take the partial derivative of H with respect to u.

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^{\star} = -J_x$$

Step 3 : Using the solution found in step two, I just replace optimal value of $u^{\star}$ in our Hamiltonian.

$$H\left(x^{\star}, u^{\star}, J_x^{\star}, t\right) = -\frac{1}{2}J_x^2 + \frac{x^2}{2} - 2xJ_x \mid_*$$

Step 4 : Write down HJB equation. The HJB equation will be

$$J_t + H = 0$$

$$\Rightarrow J_t - \frac{1}{2}J_x^2 + \frac{x^2}{2} - 2xJ_x = 0$$

Note that we have to solve for the unknown J in this problem. In our previous discussion, we have shown that one method of solving Hamiltonian-Jacobi equation namely, the separation of variable. We are going to use that idea and separate time and space this time. Assuming that these can be separated, so, that is a major assumption may not hold all the time.

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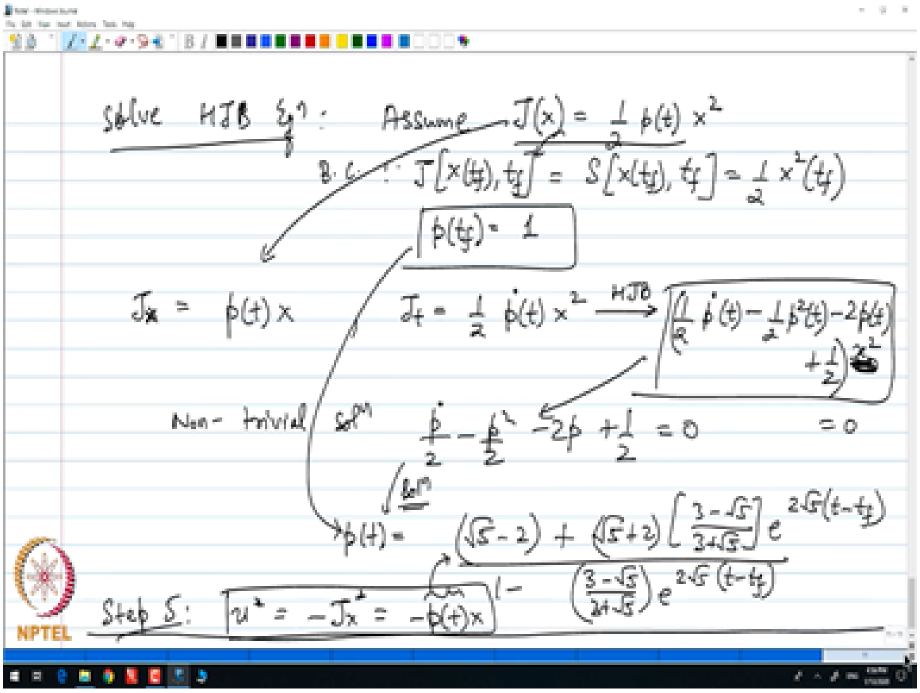

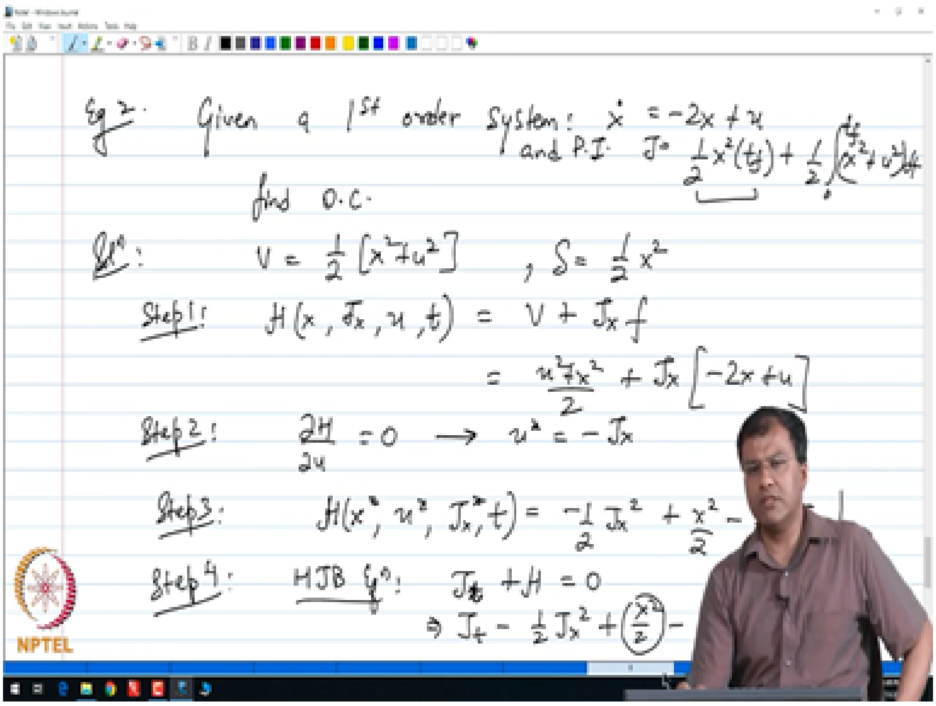

Now, solve HJB equation . Assume $J(x) = \frac{1}{2}p(t)x^2$

And the boundary condition is as follows :

$$J\left[x\left(t_f\right), t_f\right] = S\left[x\left(t_f\right), t_f\right] = \tfrac{1}{2}x^2\left(t_f\right)$$

Now if we plug J_x into the boundary condition we get a boundary condition $p\left(t_f\right) = 1$. So, at time point $t = t_f$ the separable form must satisfy $p\left(t_f\right) = 1$. Notice that this form of $J\left(x\right)$ will give us

$$J_x = p\left(t\right)x \quad ; \quad J_t = \tfrac{1}{2}\dot{p}\left(t\right)x^2$$

And when we plug it into HJB equation we get the following equation:

$$\left(\tfrac{1}{2}\dot{p}\left(t\right) - \tfrac{1}{2}p^2\left(t\right) - 2p\left(t\right) + \tfrac{1}{2}\right)x^2 = 0$$

Now we cannot have $x = 0$ as the solution because we will then get a trivial solution so, from here the only non-trivial solution will be given by the following ODE

$$\tfrac{\dot{p}}{2} - \tfrac{p^2}{2} - 2p + \tfrac{1}{2} = 0$$

And this is a standard first order non linear but constant coefficient ODE and let me write down the solution to this problem.Students are expected to have a background in solution methods of first order ODEs. we see that the solution is given by the following function

$$p\left(t\right) = \frac{\left(\sqrt{5}-2\right)+\left(\sqrt{5}+2\right)\left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]e^{2\sqrt{5}\left(t-t_f\right)}}{1-\left[\frac{3-\sqrt{5}}{3+\sqrt{5}}\right]e^{2\sqrt{5}\left(t-t_f\right)}}$$

We have found that this function p is going to certainly satisfy the boundary conditions . And students are requested to check that this is indeed the solution.

Step 5 : The final step is to get the optimal control.The optimal control $u^\star$ is given by

$$u^\star = -J_x^\star = -p\left(t\right)x$$

where $p\left(t\right)$ is given by the following expressions above.

Note that as $t \to \infty$ function p goes to a constant value which is $\sqrt{5} - 2$ or optimal control will be of course a purely a function of x. So, that is just an observation and that is how we solve this class of problem. so, solve the Hamilton-Jacobi-Bellman equation and find the optimal control by taking the partial derivative of the solution with respect to x that will give me optimal value of the control function.

Now, let me finally introduce two more methods of constrained optimization so, what happens if we now solve constrained problems where the constraint itself are inequalities as we had seen earlier. So, how to setup the Hamilton-Jacobi-Bellman equation? We will see that we are going to introduce an extra variable. And hence a non-holonomic constraint which will absorb the inequality constraints.