Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture – 58

Constrained Optimization in Optimal Control Theory Part 4

In this second half of this lecture discourse on Optimal Control Theory and solution of optimal controls via calculus of variations, I am going to look at problems involving constrained optimization and mainly with the introduction of the Hamilton-Jacobi-Bellman equation.

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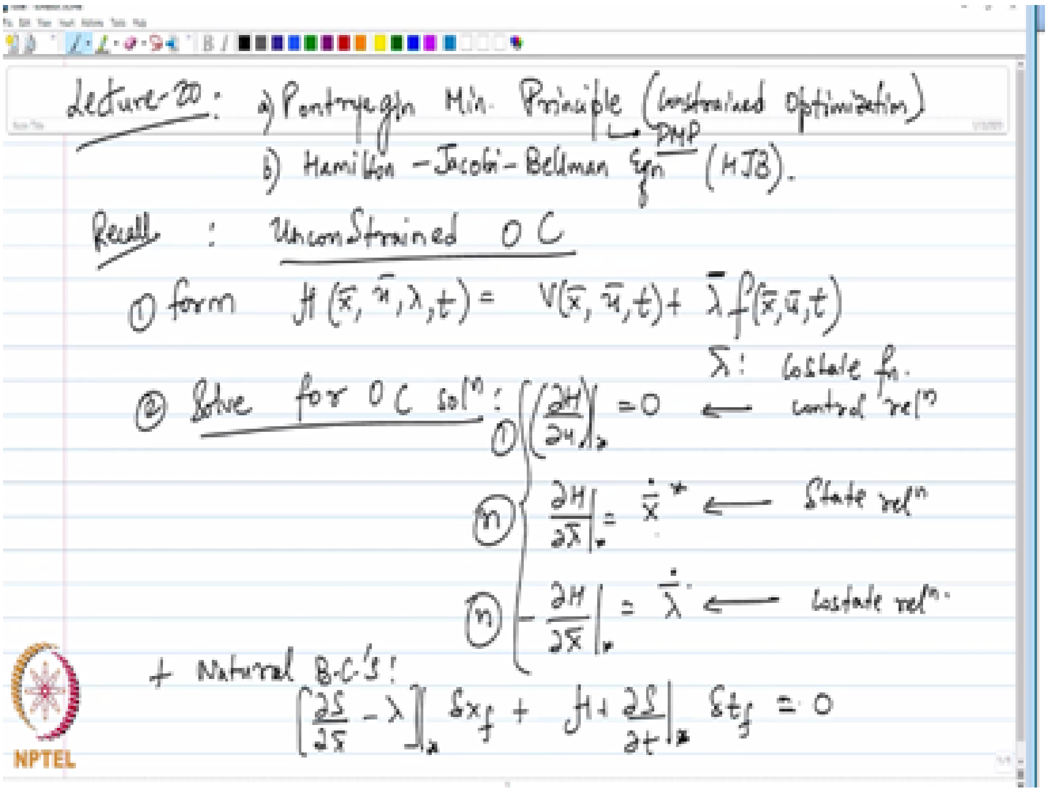

Today's topic of discussion will be the following. We are going to look at The Pontryagin Minimum Principle, also known as the constrained optimization . This rule is denote by rule PMP. Also, we are going to discuss derivation as well as the use of Hamilton-Jacobi-Bellman equation. Notice that this equation will be very similar similar to Hamilton-Jacobi equation.

Let us recall some of the concepts that we have covered in the last class. Mainly we have done unconstrained optimization that is unconstrained minimization via the introduction of the Hamiltonian or the Pontryagin H function.So, far we have done unconstrained optimal control and we are going to recall our solution methodology. We form the Hamiltonian

$$H\left(\bar{x}, \bar{u}, \lambda, t\right) = V\left(\bar{x}, \bar{u}, t\right) + \bar{\lambda} f\left(\bar{x}, \bar{u}, t\right)$$

where $\bar{\lambda}$ is co-state function or the Lagrange Multiplier. And we solve for the optimal control problem. We solve for the solution of the form

$$\frac{\partial H}{\partial \bar{u}} \mid_{\bar{x}^\star} = 0$$

So, this is for control relation. And then we also had $2n$ equations

$$\frac{\partial H}{\partial \lambda} \mid_{\bar{x}^\star} = \dot{\bar{x}}^\star$$

This equation is state relation which gives me state vector relation.

$$-\frac{\partial H}{\partial \bar{x}} \mid_{\bar{x}^\star} = \dot{\bar{\lambda}}^\star$$

This equation is co-state relation which gives me Lagrange Multiplier. Notice we have one relation given for the control, n relations for the state relations and n equations for the co-state relations. And we had the natural boundary conditions which are

$$\left[\frac{\partial S}{\partial \bar{x}} - \lambda\right] \mid_{\bar{x}^\star} \delta x_f + H + \frac{\partial S}{\partial t} \mid_{\bar{x}^\star} \delta t_f = 0$$

So, this was in summary of the total methodology of solving our unconstrained optimal control problem.

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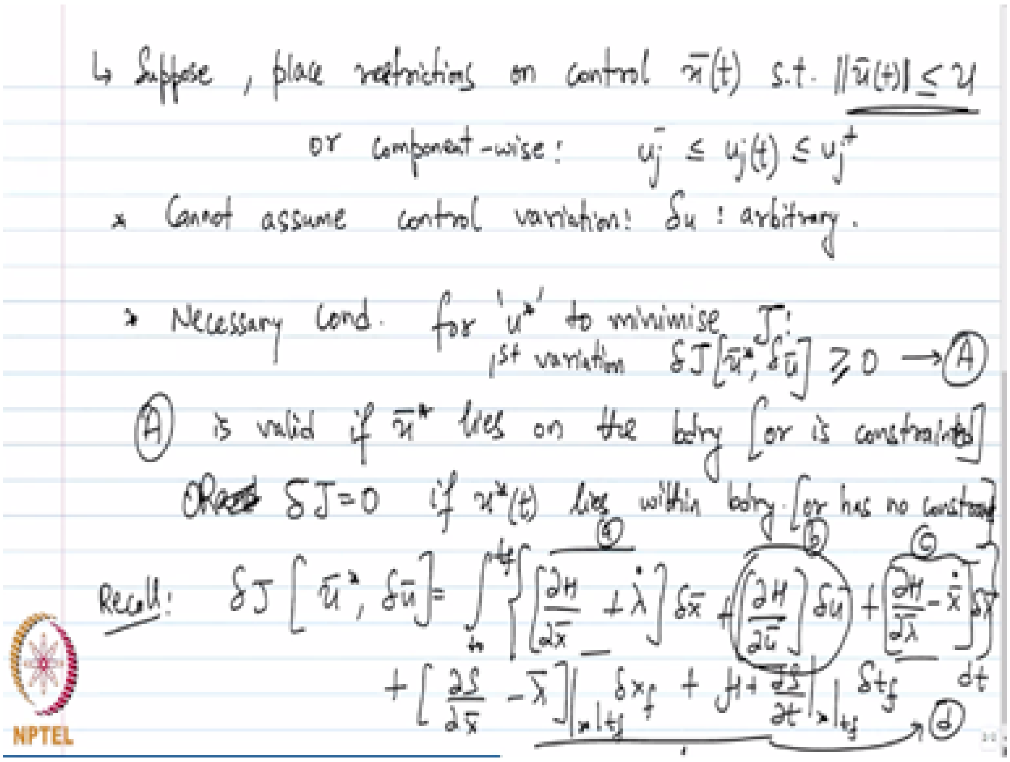

Now, suppose we place restrictions on controls function $\bar{u}(t)$ such that $|\bar{u}(t)| \leq u$ or component-wise what we see is the following:

$$u_j^- \leq u_j(t) \leq u_j^+$$

Now, we cannot assume that the control variation δu is arbitrary.Notice that so far we have been optimizing performance index or functional by setting the first variation equal to zero. The moment we have a constraint on the control variable, we cannot set the equality condition but certainly we would like the first variation to be non negative. So, the necessary condition for $u^\star$ to minimize J will be

$$\delta J[\bar{u}^\star, \delta \bar{u}] \geq 0 \qquad \textbf{(A)}$$

We can now break down the condition (A) into two parts. Suppose we fall back into unconstrained case or optimal u is found by taking the derivative or it is within the constraint then this inequality reduces to the strict equality. Otherwise if the the minimal u or the minimal control is found on the boundary then you have to solve the inequality.So , condition (A) is valid if optimal control $\bar{u}^{\star}$ lies on the boundary or is constrained. OR $\delta J = 0$ if $u^{\star}(t)$ lies within the boundary or has no constraints.

Let us look at this sort of a problem. Recall that first variation was as follows:

$$\delta J\left[\bar{u}^{\star}, \delta \bar{u}\right] = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial H}{\partial \bar{x}} + \dot{\lambda}\right] \delta \bar{x} + \left[\frac{\partial H}{\partial \bar{u}}\right] \delta \bar{u} + \left[\frac{\partial H}{\partial \lambda} - \dot{\bar{x}}\right] \delta \bar{\lambda} \right\} dt + \left[\frac{\partial S}{\partial \bar{x}} - \bar{\lambda}\right] |_{\bar{x}^{\star}}|_{t_f} \delta x_f + H + \frac{\partial S}{\partial t} |_{\bar{x}^{\star}}|_{t_f} \delta t_f$$

Let us call the above relation as (1) . Recall that when we were setting the first variation equal to zero we had four condition set equal to zero. So, now again we are going to set all but one condition equal to zero that is this quantity which I am circling. Because no longer the optimal Hamiltonian can be found by differentiating with respect to u because the derivative on the boundary does not exist. So, we are including the boundary values of the constraint variable u.

Let me call this relation (a), this is relation (b), and this is relation (c), and these boundary condition are relation (d).

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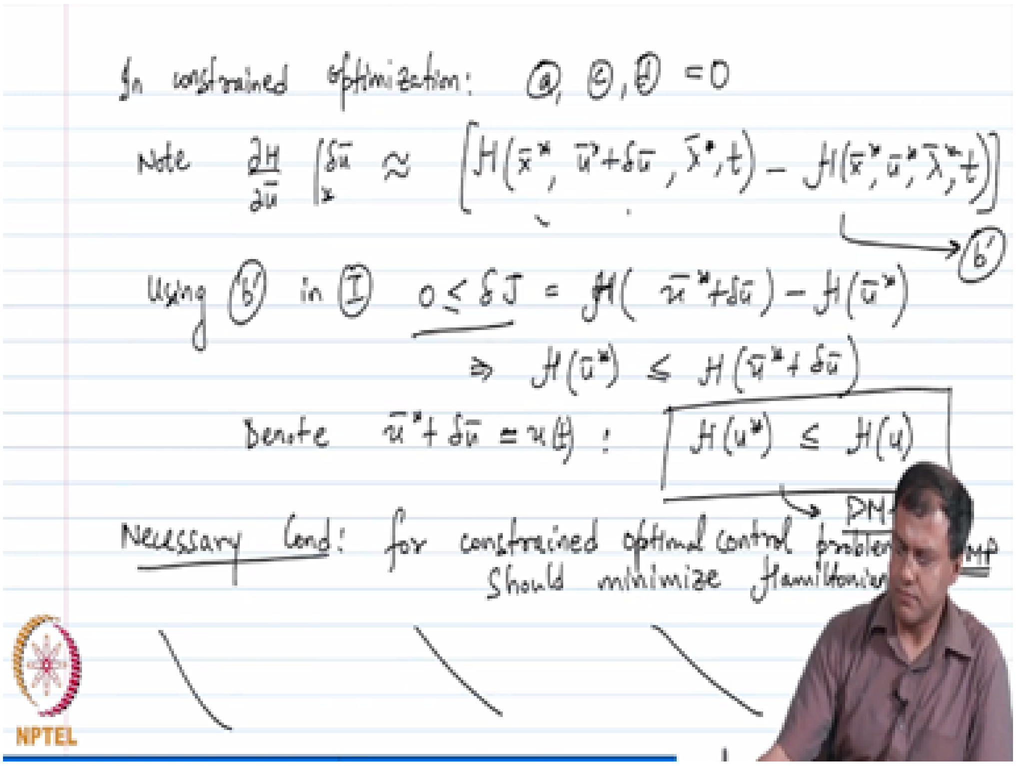

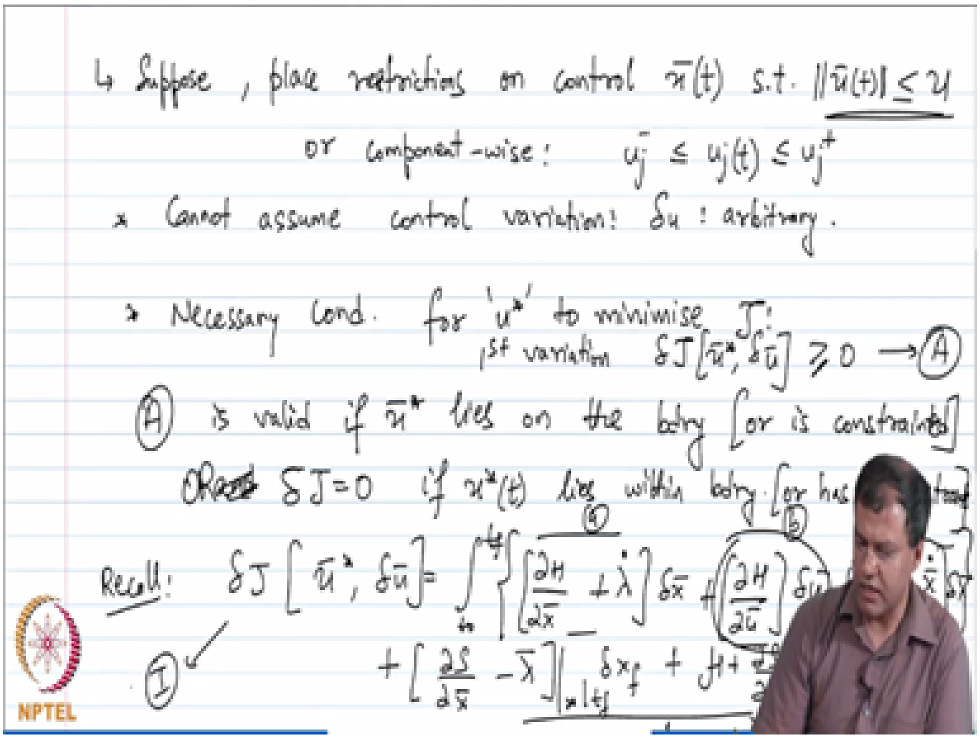

So, in constrained optimization we set relation (a),(c),(d) equal to zero.And note that

$$\frac{\partial H}{\partial \bar{u}} \mid_{\bar{x}^\star} \delta\bar{u} \approx H\left(\bar{x}^\star, \bar{u}^\star + \delta\bar{u}, \bar{\lambda}^\star, t\right) - H\left(\bar{x}^\star, \bar{u}^\star, \bar{\lambda}^\star, t\right) \qquad \textbf{(b')}$$

Using (b') in (1) we see that

$$\delta J = H\left(\bar{u}^\star + \delta\bar{u}\right) - H\left(\bar{u}^\star\right)$$

We have to satisfy that $\delta J \geq 0$. This is to begin with we had and from here we get

$$H\left(\bar{u}^\star\right) \leq H\left(\bar{u}^\star + \delta\bar{u}\right)$$

So, the final argument is that the Hamiltonian value at the optimal condition for the constrained optimization should be such that it gives me the minimum value in the neighborhood of the optimal control. Let me denote $\bar{u}^\star + \delta\bar{u} = u\left(t\right)$ and we see that minimum condition has changed into an inequality that

$$H\left(\bar{u}^\star\right) \leq H\left(u\right)$$

And this exactly Pontryagin Minimum Principle. Let me just state the necessary condition that we have found for the constrained control problem.The necessary condition for constrained optimal control problem should minimize the Hamiltonian and that isPontryagin Minimum Principle states. Now, let us look at the solution methodology .

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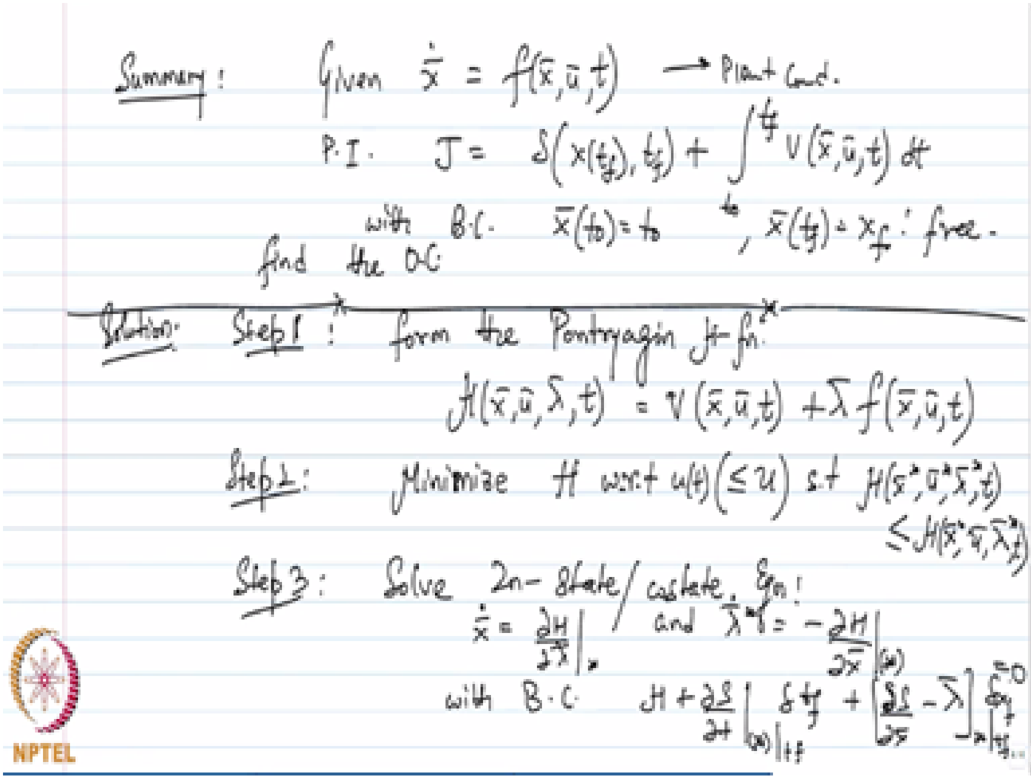

The summary is as follows: We start with a statement. Given plant condition is $\dot{\bar{x}} = f(\bar{x}, \bar{u}, t)$ and performance index is given to be

$$J = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(\bar{x}, \bar{u}, t)\, dt$$

with boundary condition $\bar{x}(t_0) = t_0$, $\bar{x}(t_f) = x_f$ and this could be free boundary condition. Find the optimal control.

The solution method is as follows. We will see that the only difference from our previous method for that previous description in case of unconstrained optimization is we are going to replace our equality, where we optimize our control function with an inequality which minimizes control Hamiltonian.

Step 1 : Form the Pontryagin H function.

$$H\left(\bar{x}, \bar{u}, \bar{\lambda}, t\right) = V(\bar{x}, \bar{u}, t) + \bar{\lambda} f(\bar{x}, \bar{u}, t)$$

Step 2 : Minimize H with respect to $u(t)\,(\leq u)$ such that

$$H\left(\bar{x}^\star, \bar{u}^\star, \bar{\lambda}^\star, t\right) \leq H\left(\bar{x}^\star, \bar{u}, \bar{\lambda}^\star, t\right)$$

we have to minimize by looking at the boundary values. The interior values are found as we have done for unconstrained case.

Step 3 : Solve $2n$ state and co-state equations given by

$$\dot{\bar{x}} = \frac{\partial H}{\partial \lambda}\Big|_{\bar{x}^\star} \quad ; \quad \dot{\bar{\lambda}}^\star = \frac{\partial H}{\partial \bar{x}}\Big|_{\bar{x}^\star}$$

with the boundary condition

$$H + \frac{\partial S}{\partial t}\Big|_{\bar{x}^\star}\Big|_{t_f} \delta t_f + \left[\frac{\partial S}{\partial \bar{x}} - \bar{\lambda}\right]\Big|_{\bar{x}^\star}\Big|_{t_f} \delta x_f = 0$$

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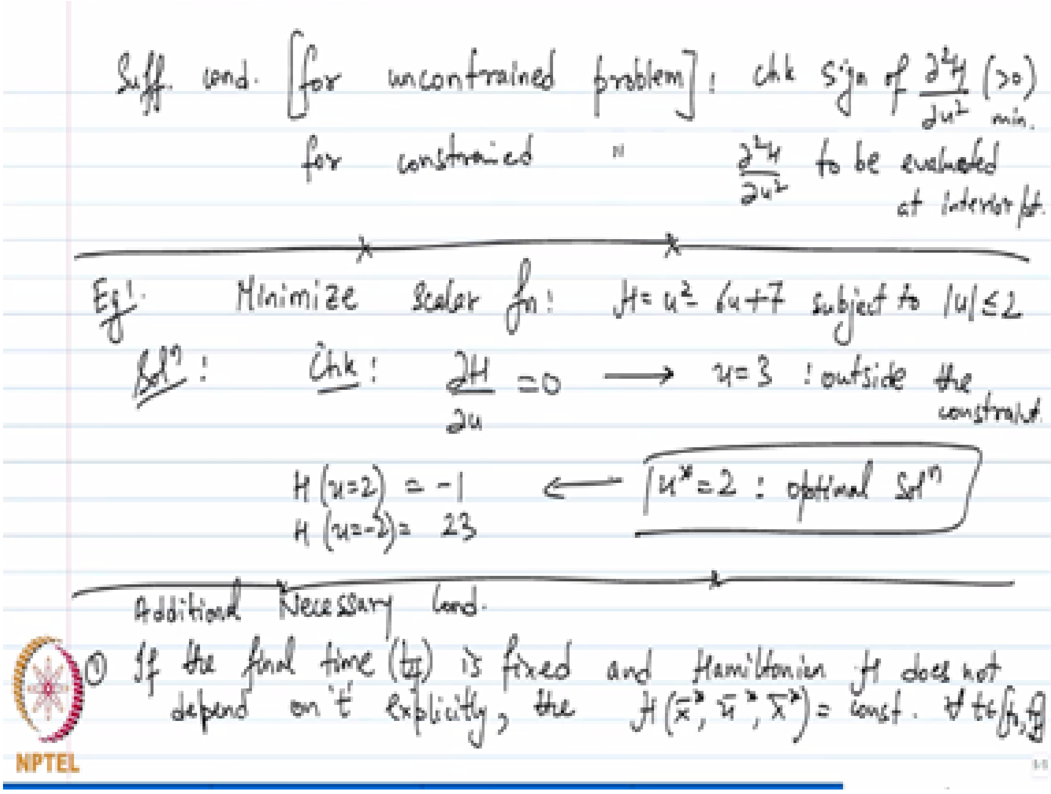

The sufficient condition for unconstrained problem was to check the sign of $\frac{\partial^2 H}{\partial u^2}$ and if this is positive then we are guaranteed minimum. And that was possible only for the unconstrained problem. If u is found in the interior of the boundary we can still take this second derivative otherwise we will again have to resort to the Pontryagin function. At the boundary values these derivatives are not available and these are not to be evaluated. So, for constrained problem $\frac{\partial^2 H}{\partial u^2}$ to be evaluated at interior points. We cannot justify the second derivatives for the boundary points.

Let us look at some examples. I have a very simple example to begin with. We need to minimize the scalar function $H = u^2 - 6u + 7$ subject to constrained $|u| \leq 2$.

Check that $\frac{\partial H}{\partial u} = 0$ gives us $u = 3$ that is not a viable solution because it exceeds our constraint. So, $u = 3$ is outside the constraint. Then we have to check the boundary values. Notice that

$$H\left(u = 2\right) = -1 \quad ; \quad H\left(u = -2\right) = 23$$

So, $u^\star = 2$ is optimal solution. So, certainly we will not get the derivative equal to zero at $u^\star = 2$ but it gives optimal for the constrained problem. Let me wrap up the discussion of this part of the constrained optimization before giving examples by mentioning two other points.The first point is we have two additional necessary conditions.The first condition is if the final time t_f and the Hamiltonian H does not depend on t explicitly then $H\left(\bar{x}^\star, \bar{u}^\star, \bar{\lambda}^\star\right)$ is a constant for all $t \in [t_0, t_f]$.

Now students should immediately recognize this additional necessary condition as version of Beltrami identity because if the Hamiltonian is independent of the independent variable then we know that from the Beltrami identity the Hamiltonian becomes the constant. So, this is nothing but the Beltrami identity.

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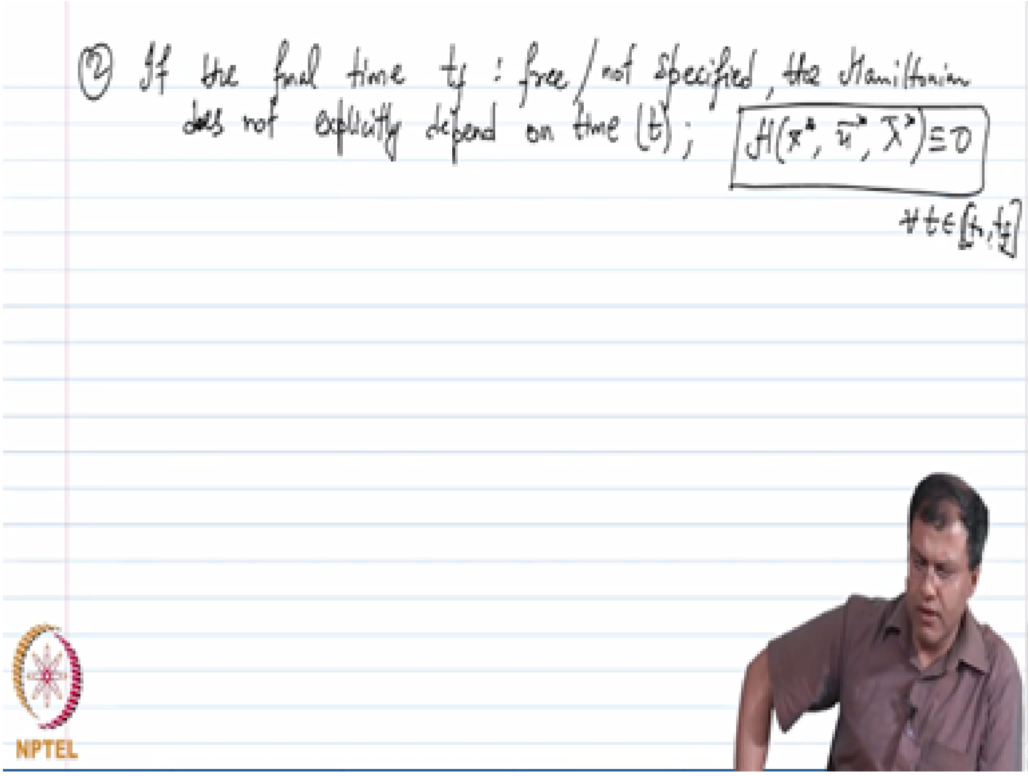

And the seocnd condition is if the final time t_f is free and not specified, the Hamiltonian does not explicitly depend on time t then we have $H\left(\bar{x}^{\star}, \bar{u}^{\star}, \bar{\lambda}^{\star}\right) = 0$ for all $t \in [t_0, t_f]$.

Now we are going to continue our discussion on this constrained optimization by looking at another form of the optimal control equations and then we will look at some examples.