Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture – 57

Constrained Optimization in Optimal Control Theory Part 3

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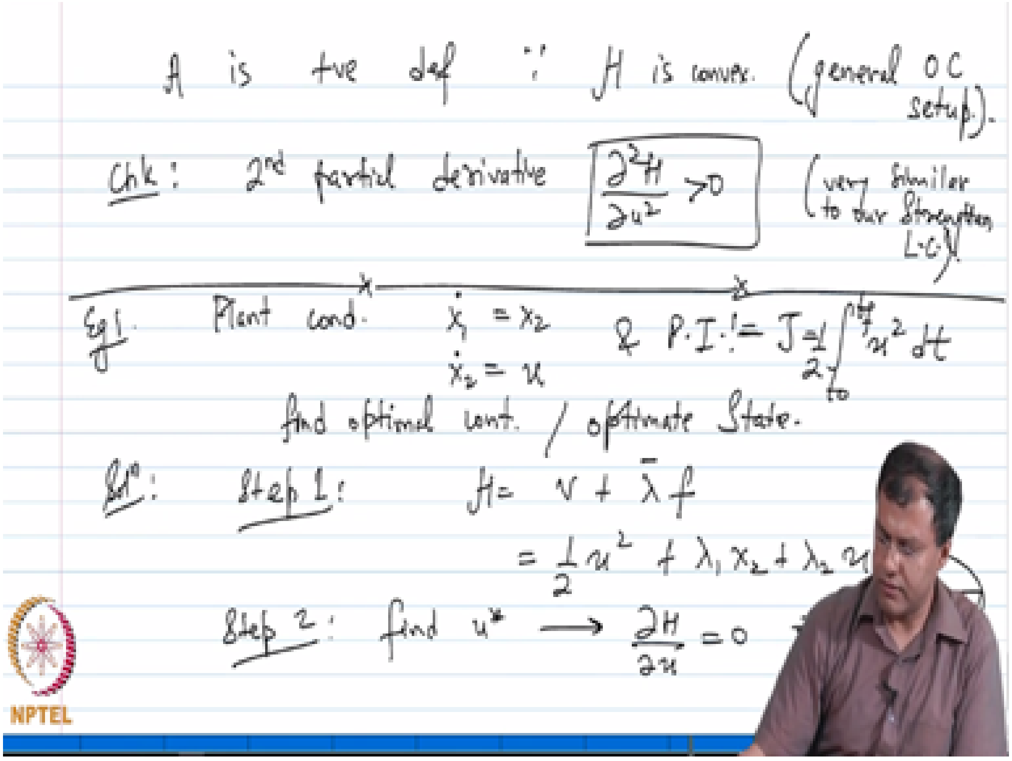

The first example that I have is the following: Let the plant condition is as follows:

$$\dot{x_1} = x_2 \quad ; \quad \dot{x_2} = u\,(t)$$

where $(\dot{\cdot})$ represents the time derivative. And performance index is given by

$$J = \tfrac{1}{2}\int_{t_0}^{t_f} u^2 dt$$

We need to find the optimal control and the optimal state. Basically we need to need to find u and x. Notice that this is a case where the cost function is set equal to 0. We will follow the step by step procedure namely the Pontryagin procedure.

Step 1 : we set up the Hamiltonian.

$$H = V + \bar{\lambda} f = \tfrac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$

Step 2 : Find $u^\star$ and we do that by $\frac{\partial H}{\partial u} = 0$ and after we do that we get $u = -\lambda_2$. And then we plug it back to get our optimal Hamiltonian where u is removed from this equation.

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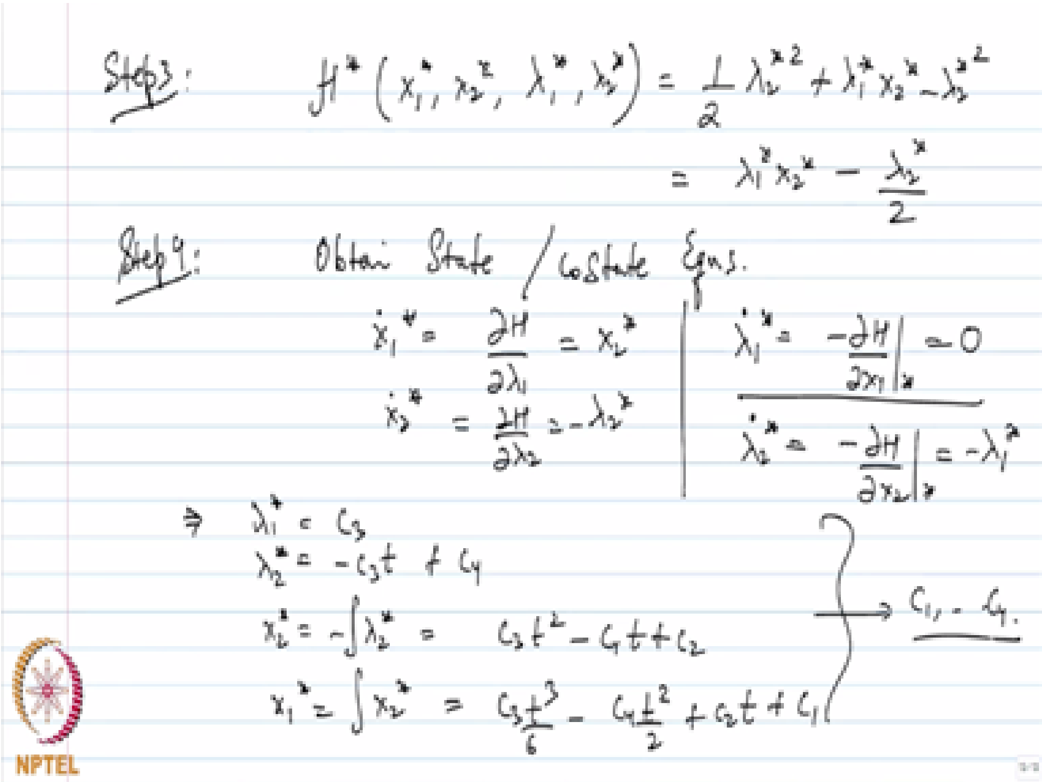

Step 3 : Find optimal Hamiltonian which is

$$H^\star\left(x_1^\star, x_2^\star, \lambda_1^\star, \lambda_2^\star\right) = \frac{1}{2}\lambda_2^{\star 2} + \lambda_1^\star x_2^\star - \lambda_2^{\star 2} = \lambda_1^\star x_2^\star - \frac{\lambda_2^{\star 2}}{2}$$

Step 4 : . Obtain state and co-state equations. We see the following:

$$\dot{x_1}^\star = \frac{\partial H}{\partial \lambda_1} = x_2^\star$$

$$\dot{x_2}^\star = \frac{\partial H}{\partial \lambda_2} = -\lambda_2^\star$$

$$\dot{\lambda_1}^\star = -\frac{\partial H}{\partial x_1} |_{\bar{x}^\star} = 0$$

$$\dot{\lambda_2}^\star = -\frac{\partial H}{\partial x_2} |_{\bar{x}^\star} = -\lambda_1^\star$$

So, we have four equations and 4 unknowns x_1 , x_2, λ_1, λ_2. We can start solving and we get the following:

$$\lambda_1^\star = c_3$$

$$\lambda_2^\star = -c_3 t + c_4$$

$$x_2^\star = -\int \lambda_2^\star = c_3 t^2 - c_4 t + c_2$$

$$x_1^\star = \int x_2^\star = c_3 \frac{t^3}{6} - c_4 \frac{t^2}{2} + c_2 t + c_1$$

Now we have a set of four unknowns c_1 to c_4 and we also, have, we have to see whether we have four equations to solve for four unknowns.

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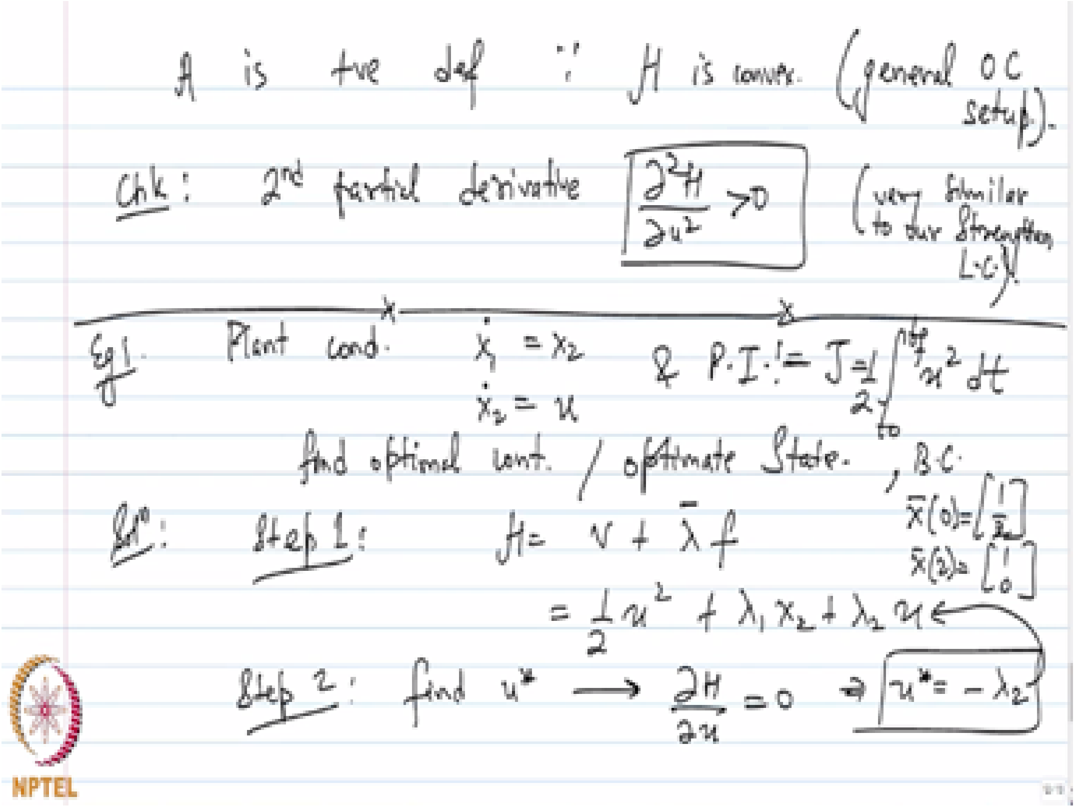

Notice that in this problem we have also specified some boundary conditions. The boundary condition are

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\bar{x}(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Essentially we have fixed boundary condition. So, these are set of vector boundary conditions of two constraints each. So, essentially we have four boundary conditions and we have four unknowns.

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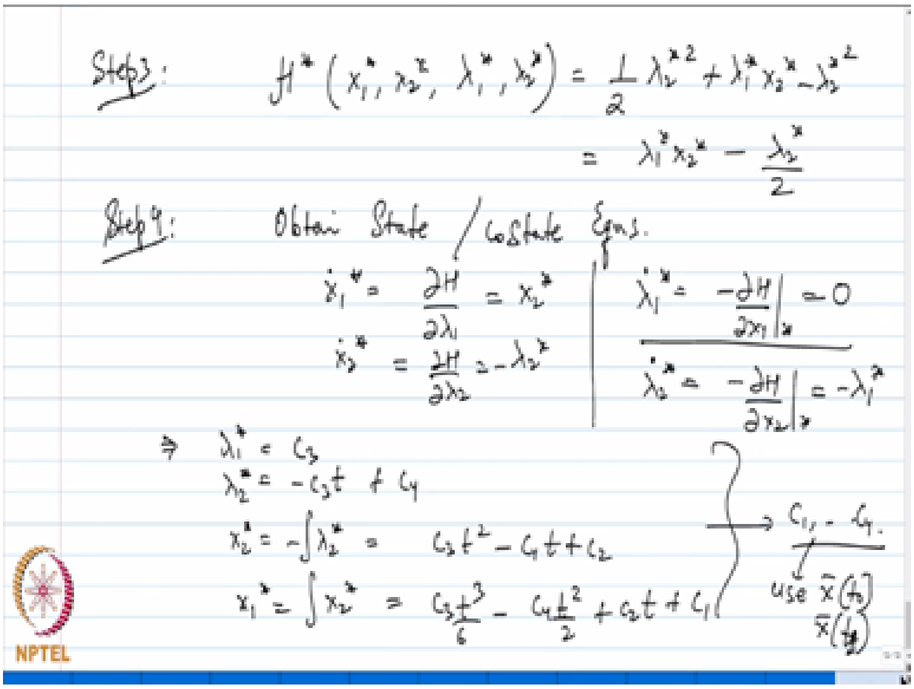

So, in order to find these unknowns use boundary condition $\bar{x}\,(t_0)$ and $\bar{x}\,(t_f)$. And we will find these constants. I will leave those evaluation of the constants to the students and look at another example.

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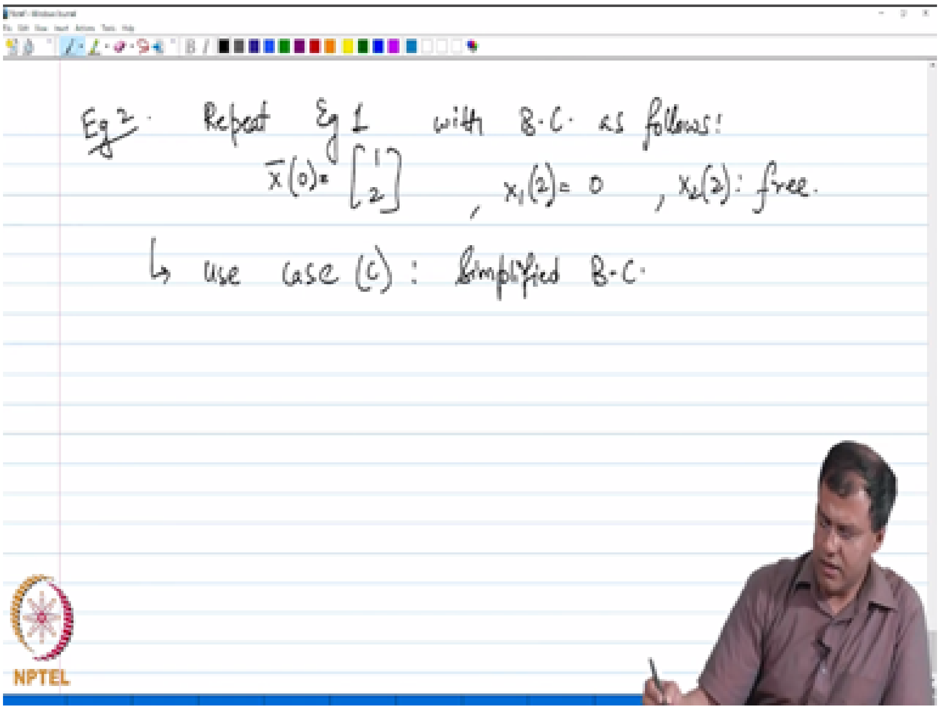

In the second example I am going to repeat example (1) but this time with boundary conditions as follows:

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ; \quad x_1(2) = 0 \;\; ; \;\; x_2(2) = free$$

It is a freely moving state variable. Notice that initial and the final time are 0 and 2. So, those are fixed. However final state x_2 is free so this is a case of fixed final time but free final state. So, we need to use case (c) in simplified boundary condition case.

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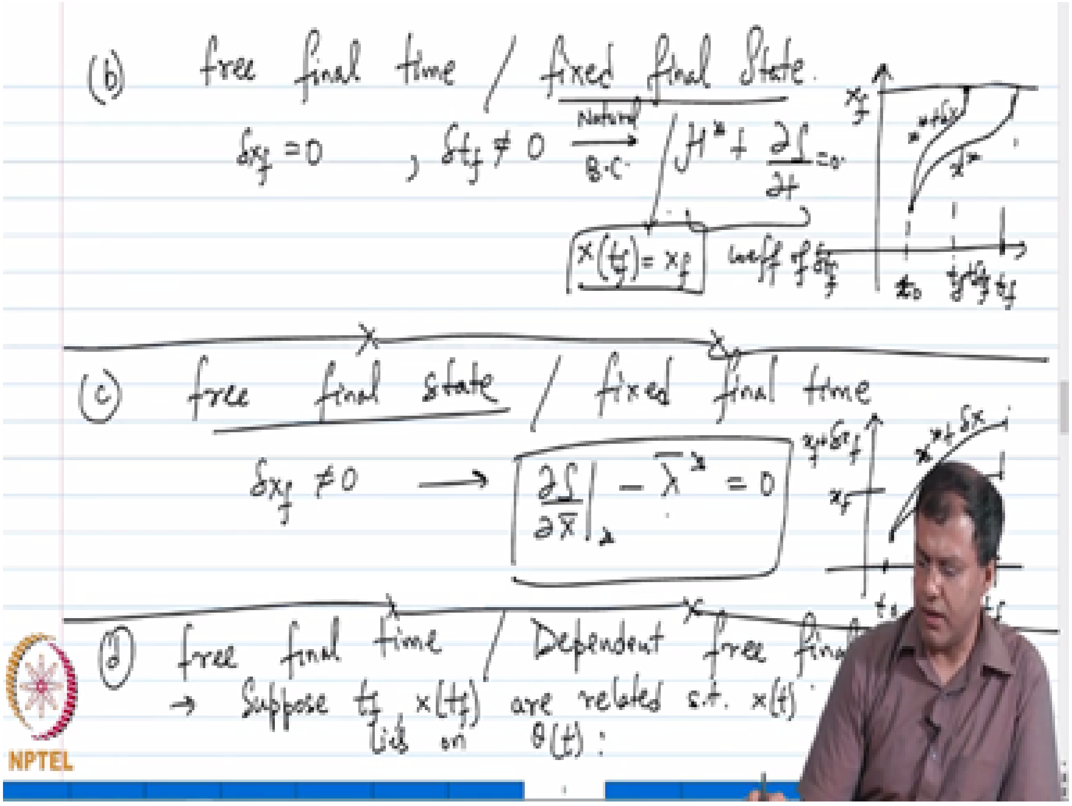

All we need to do is replace the fixed boundary condition with $\frac{\partial S}{\partial \bar{x}} = \bar{\lambda}^{\star}$

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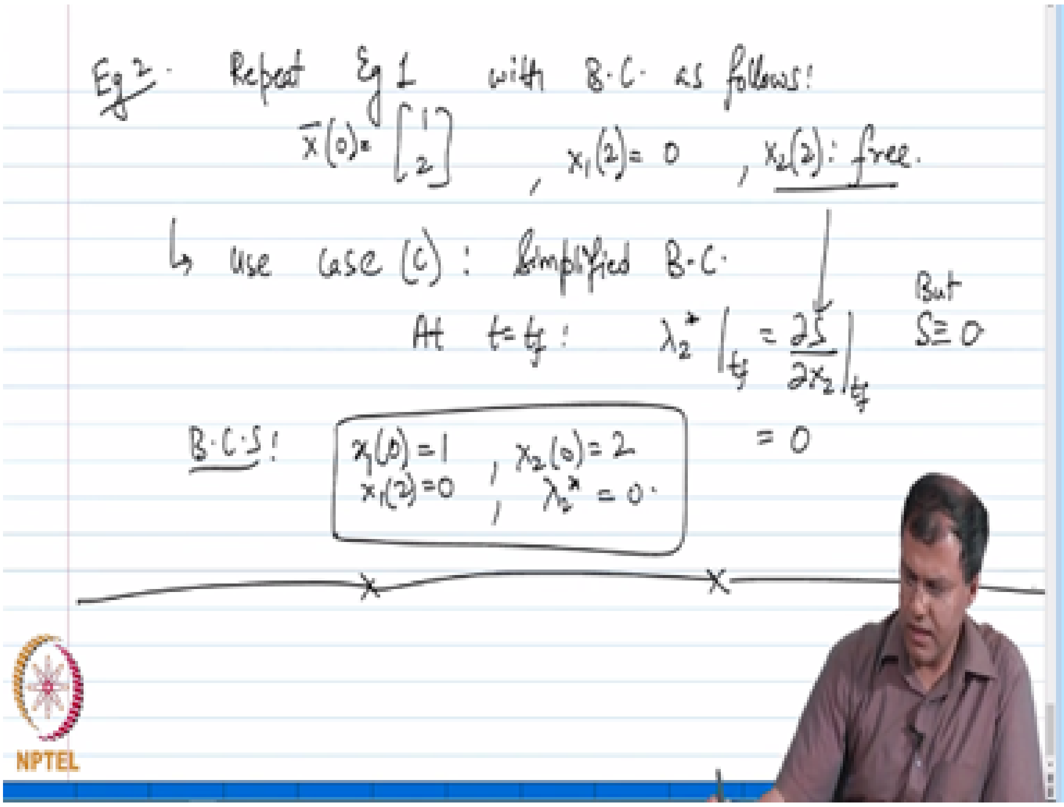

So, at $t = t_f$, we have an condition $\lambda_2^\star \mid_{t_f} = \frac{\partial S}{\partial x_2} \mid_{t_f}$ but cost function $S \equiv 0$. So, we get $\lambda_2^\star \mid_{t_f} = 0$.We have boundary conditions as follows:

$$x_1(0) = 1 \ \ , \ \ x_2(0) = 2 \ \ , \ \ x_1(2) = 0 \ \ , \ \ \lambda_2^\star = 0$$

These are four conditions to satisfy and we do the same solution method, we will get a solution in terms of four constants which are then resolved using these four conditions. Let us look at one more example.

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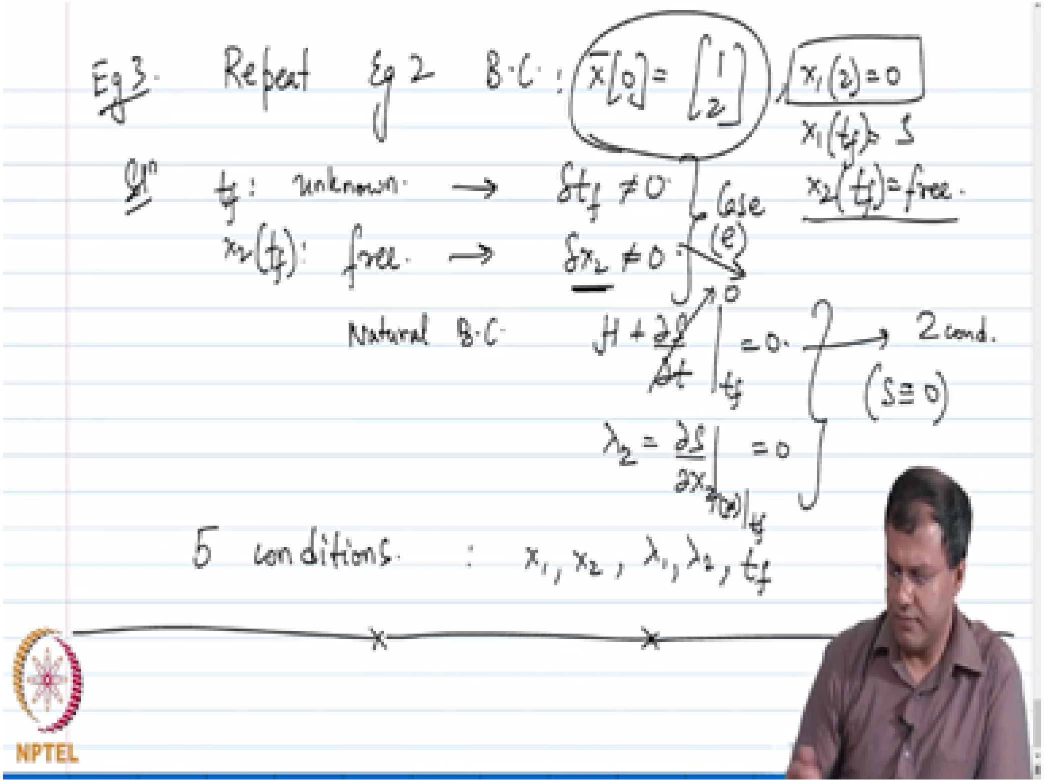

So, again in third example we repeat example (2) with the following boundary condition:

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ; \quad x_1(2) = 0 \;\; ; \;\; x_1(t_f) = 3 \;\; ; \;\; x_2(t_f) = free$$

Notice that second state variable at final time point is free and also, final time point t_f is not known. So, we do not know whether it is 2 or whether it is less than 2 or greater than 2.

So,in this problem we have t_f is unknown and also $x_2(t_f)$ is free which means that $\delta t_f \neq 0$ and also $\delta x_2 \neq 0$, this situation falls under simplified case (e) where we are able to vary final time as well as final state independent of each other and we see that the following set of natural boundary condition must hold.

$$H + \frac{\partial S}{\partial t} |_{t_f} = 0 \quad ; \quad \lambda_2 = \frac{\partial S}{\partial x_2} |_{\bar{x}^\star} |_{t_f}$$

However, $S \equiv 0$S which means that $\lambda_2 = 0$ and $\frac{\partial S}{\partial t} = 0$. So, these two will give me two conditions and we also have two boundary conditions coming at $t = 0$ and we have specified another condition $x_1(2)$.So, we have five conditions. And we have five unknowns namely x_1 , x_2 , λ_1 , λ_2 , and t_f . So, 5 unknowns, 5 conditions, the problem should be fully defined. Then let me look at one more example of the same example but with a different cost function.

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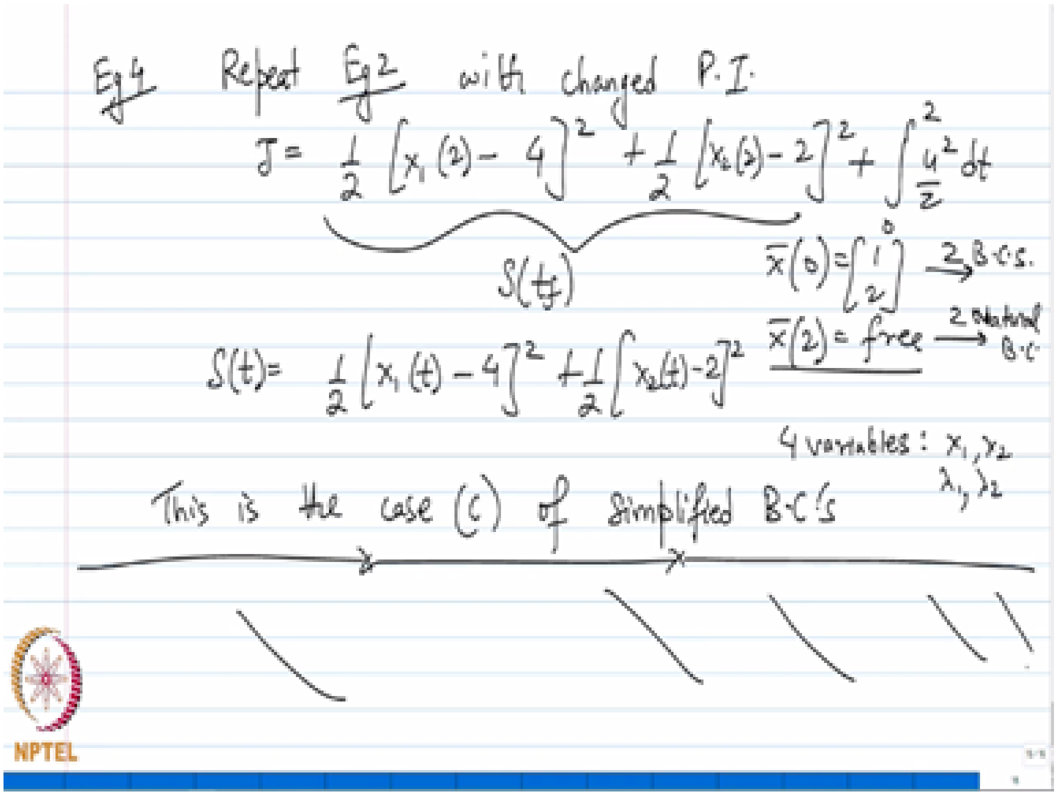

Repeat example (2) but with changed performance index. So, objective function or performance index is as follows:

$$J = \frac{1}{2}\left[x_1\left(2\right) - 4\right]^2 + \frac{1}{2}\left[x_2\left(2\right) - 2\right]^2 + \int_0^2 \frac{u^2}{2} dt$$

And boundary conditions are

$$\bar{x}\left(0\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ; \; \bar{x}\left(2\right) = free$$

So, both the variables are free. Notice that the quantity which is sitting outside the integral is $S\left(t_f\right)$.

Now we have a cost function $S\left(t\right)$ given by

$$S\left(t\right) = \frac{1}{2}\left[x_1\left(t\right) - 4\right]^2 + \frac{1}{2}\left[x_2\left(t\right) - 2\right]^2$$

Now again we use the same solution methodology but notice that we have two boundary conditions and we will get two natural boundary conditions from free boundary and of course we have four variables which are x_1, x_2,λ_1 and λ_2. My final time point two is known and so that should fully determine our system and this is the system where we have fixed time but free state. This is the case (c) of simplified boundary conditions

Let me wrap up this lecture by mentioning, that all the students have been able to get a feeling of how to solve the problems involving optimal controls and in the next lecture I am going to introduce another method to solve this optimal control problems namely via the Hamilton-Jacobi-Bellman equation and we are going to state the optimality criteria followed by we will look at certain methods of constraint optimization namely the method of penalty function and slack variables.