**Variational Calculus and its Applications in Control Theory and Nano mechanics**
**Professor Sarthok Sircar**
**Department of Mathematics**
**Indraprastha Institute of Information Technology, Delhi**
**Lecture – 55**
**Constrained Optimization in Optimal Control Theory Part 1**

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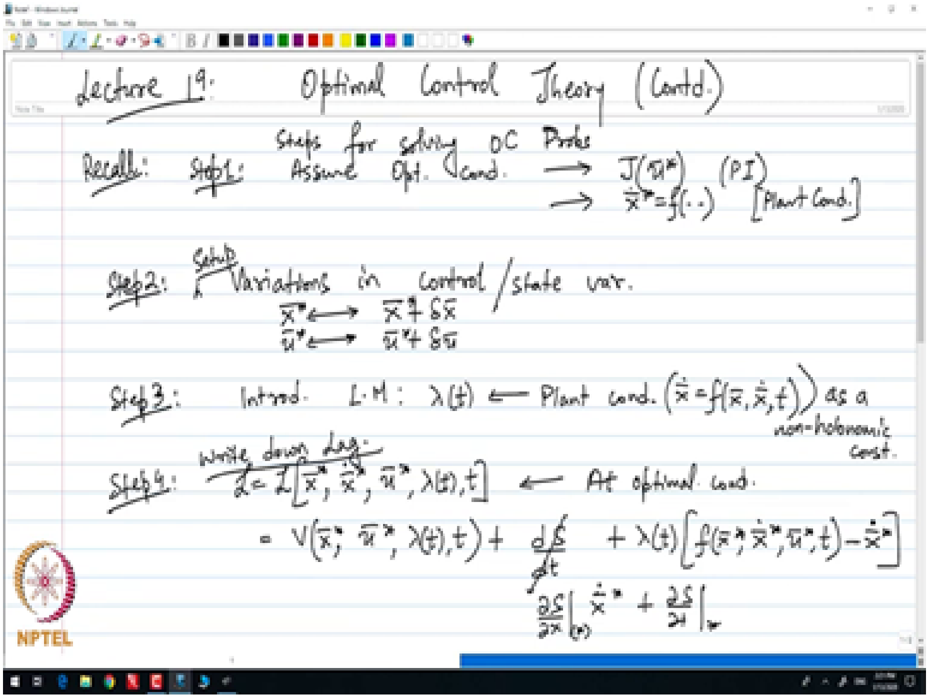


In today's lecture, we will continue solution methodology for Optimal Control Theory. Recall we were trying to note down the steps of the solution methodology for solving problems in optimal control theory via calculus of variation.

The step 1 was we assumed optimal conditions and we write down our performance index or the functional $J(u^{\star})$ at optimal values and we also write down our plant condition at the optimal value. Then the step 2 was we set up variations in control and state variables. Essentially, in functional as well as the plant condition wherever we have the state variables $\bar{x}^{\star}$ we replace it by $\bar{x}^{\star} + \delta\bar{x}$ and then, wherever we have control variable $\bar{u}^{\star}$ we replace it by $\bar{u}^{\star} + \delta\bar{u}$ the perturbed value of the control variable.

The step 3 was we write down, we write downthe functional for the optimal condition as well as the functional at the perturbed value. And further we introduce Lagrange multiplier $\lambda(t)$ . This multiplier introduces the plant condition $\dot{\bar{x}} = f(\bar{x}, \dot{\bar{x}}, t)$ non-holonomic constraint.

We will continue our discussion for the next set of steps.

Step 4 : we write down our integrand for the functional at the optimal condition and the functional at the perturbed conditions namely, the Lagrangian.So, we write down $L = L[\bar{x}^{\star}, \dot{\bar{x}}^{\star}, \bar{u}^{\star}, \lambda(t), t]$ at optimal conditions as well as at the perturbed condition. So, L becomes

$$L = V[\bar{x}^{\star}, \bar{u}^{\star}, \lambda(t), t] + \frac{\partial S}{\partial \bar{x}}\Big|_{\bar{x}^{\star}} \dot{\bar{x}}^{\star} + \frac{\partial S}{\partial t}\Big|_{\bar{x}^{\star}} + \lambda(t)\{f(\bar{x}^{\star}, \dot{\bar{x}}^{\star}, \bar{u}^{\star}, t) - \dot{\bar{x}}^{\star}\}$$

We have four terms in the Lagrangian. Then, we again write down the value of this Lagrangian for the perturbed variables. The perturbed variables are the perturbed state and control variables.

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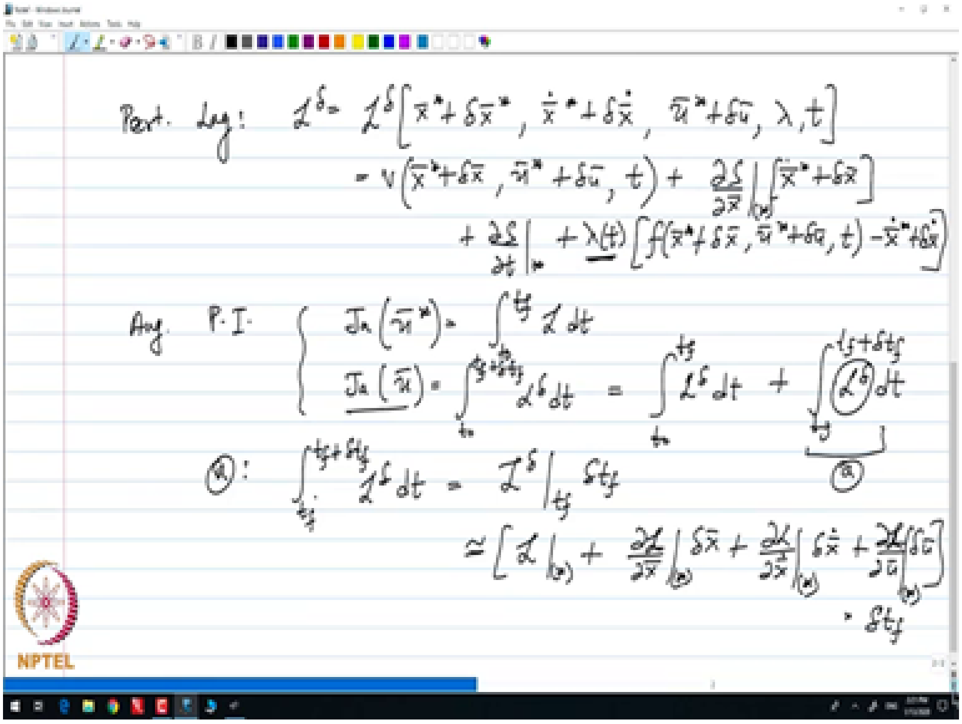


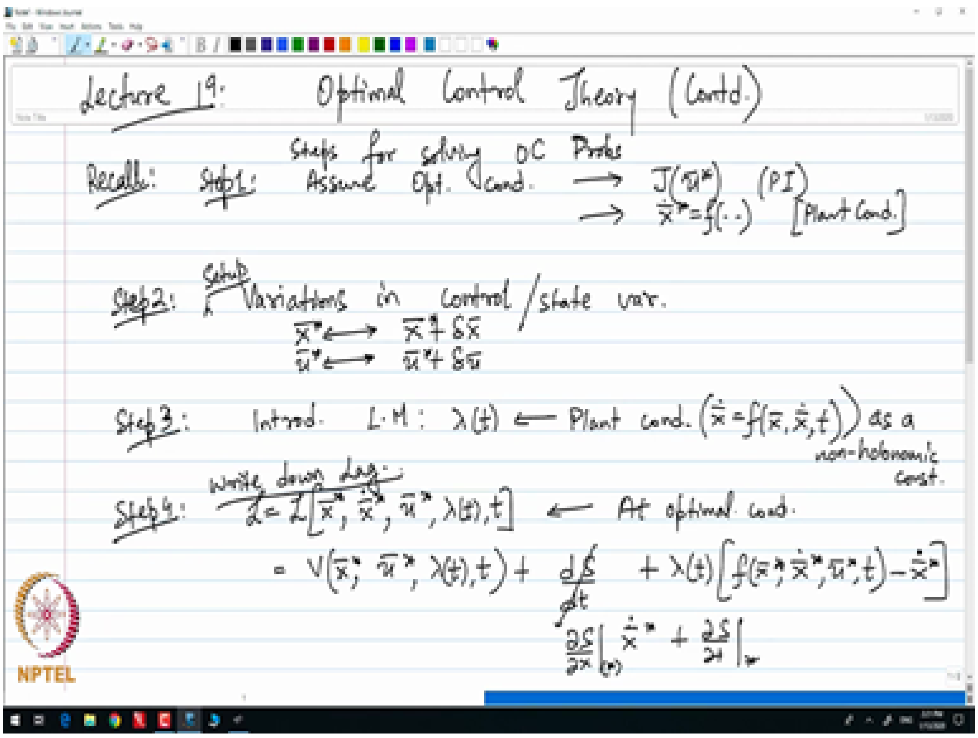

So, the perturbed Lagrangian is L

$$L^{\delta} = L^{\delta}\left[\bar{x}^{\star} + \delta\bar{x}^{\star}, \dot{\bar{x}}^{\star} + \delta\dot{\bar{x}}^{\star}, \bar{u}^{\star} + \delta\bar{u}^{\star}, \lambda, t\right]$$

$$= V\left[\bar{x}^{\star} + \delta\bar{x}, \bar{u}^{\star} + \delta\bar{u}, t\right] + \frac{\partial S}{\partial \bar{x}}\mid_{\bar{x}^{\star}}\left[\dot{\bar{x}}^{\star} + \delta\bar{x}\right] + \frac{\partial S}{\partial t}\mid_{\bar{x}^{\star}} + \lambda\left(t\right)\left\{f\left(\bar{x}^{\star} + \delta\bar{x}, \bar{u}^{\star} + \delta\bar{u}, t\right) - \dot{\bar{x}}^{\star} + \delta\dot{\bar{x}}\right\}$$

Then we have to write down the augmented performance index for the perturbed value. The augmented performance index or the functional at the original value or the optimal value is

$$J_a\left(\bar{u}^{\star}\right) = \int_{t_0}^{t_f} L dt$$

And the performance index at the perturbed value will be

$$J_a\left(\bar{u}\right) = \int_{t_0}^{t_f+\delta t_f} L^{\delta} dt = \int_{t_0}^{t_f} L^{\delta} dt + \int_{t_f}^{t_f+\delta t_f} L^{\delta} dt$$

Now, notice that in the second integral, the range of integration is very small namely, $\delta t_f$. So,I can write down the perturbed value of the Lagrangian at its optimal value. So, the second integral becomes

$$\int_{t_f}^{t_f+\delta t_f} L^{\delta} dt = L^{\delta}\mid_{t_f} \delta t_f$$

$$\approx \left[L\mid_{\bar{x}^{\star}} + \frac{\partial L}{\partial \bar{x}}\mid_{\bar{x}^{\star}} \delta\bar{x} + \frac{\partial L}{\partial \dot{\bar{x}}}\mid_{\bar{x}^{\star}} \delta\dot{\bar{x}} + \frac{\partial L}{\partial \bar{u}}\mid_{\bar{x}^{\star}} \delta\bar{u}\right] * \delta t_f$$

we use Taylor series approximation and expand perturbed value of the Lagrangian at its optimal value up to first order. Now I have to take the difference of the performance index with the original index or the optimal index. So, we are trying to write down our conditions for the first variation equal to 0.

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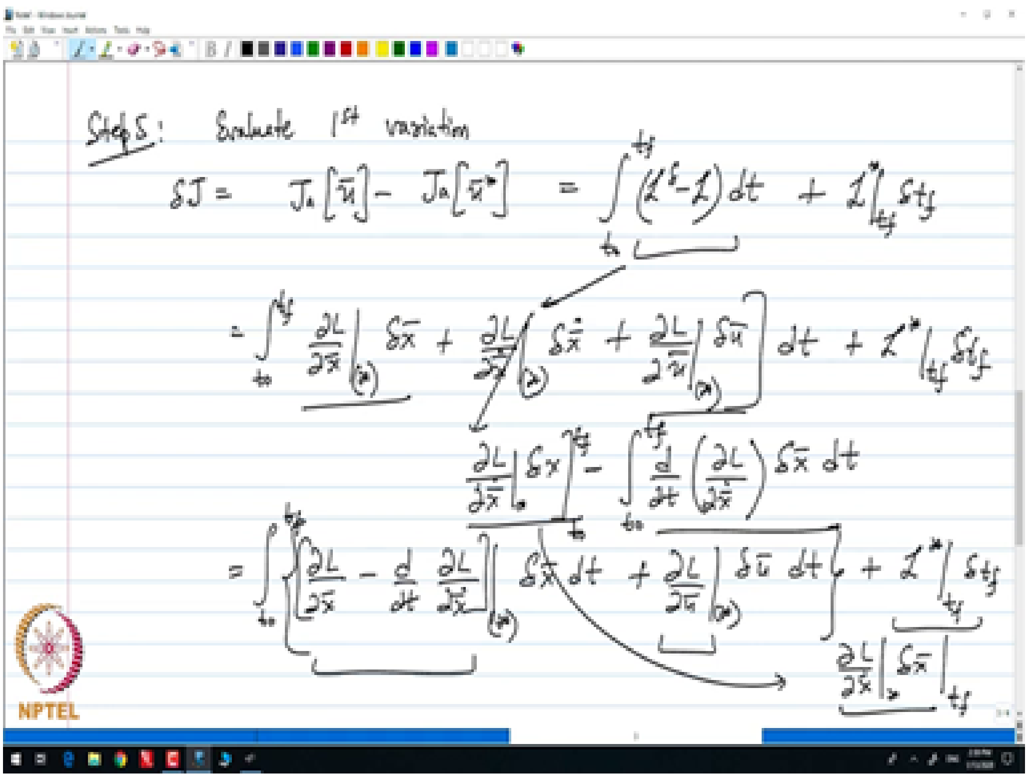

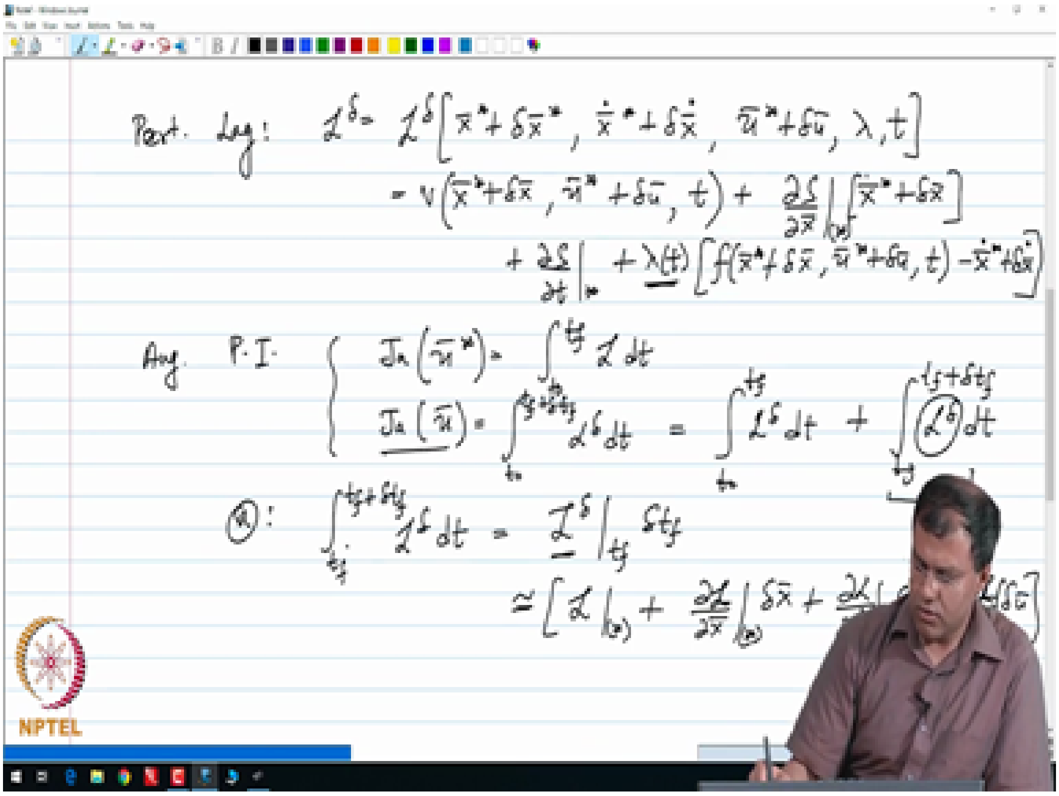


Step 5 : We evaluate the first variation using Taylor series and retain the first order terms. So, first variation $\delta J$ is

$$\delta J = J_a\left[\bar{u}\right] - J_a\left[\bar{u}^{\star}\right] = \int_{t_0}^{t_f}\left(L^{\delta} - L\right)dt + L^{\star}\left.\right|_{t_f}\delta t_f$$

Then we have to expand all these terms. Notice that the first integral we expand and see that the following terms arise.

$$\delta J = J_a\left[\bar{u}\right] - J_a\left[\bar{u}^{\star}\right] = \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \bar{x}}\left.\right|_{\bar{x}^{\star}}\delta\bar{x} + \frac{\partial L}{\partial \dot{\bar{x}}}\left.\right|_{\bar{x}^{\star}}\delta\dot{\bar{x}} + \frac{\partial L}{\partial \bar{u}}\left.\right|_{\bar{x}^{\star}}\delta\bar{u}\right]dt + L^{\star}\left.\right|_{t_f}\delta t_f$$

And then using integration by parts , we have

$$\int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{\bar{x}}}\left.\right|_{\bar{x}^{\star}}\delta\dot{\bar{x}}\right]dt = \frac{\partial L}{\partial \dot{\bar{x}}}\left.\right|_{\bar{x}^{\star}}\delta\bar{x}\left.\right|_{t_0}^{t_f} - \int_{t_0}^{t_f}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right)\delta\bar{x}dt$$

Then, we retain the integral terms all inside the integral and get the following :

$$\delta J = J_a\left[\bar{u}\right] - J_a\left[\bar{u}^{\star}\right] = \int_{t_0}^{t_f}\left[\left[\frac{\partial L}{\partial \bar{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right)\right]\left.\right|_{\bar{x}^{\star}}\delta\bar{x}dt + \frac{\partial L}{\partial \bar{u}}\left.\right|_{\bar{x}^{\star}}\delta\bar{u}dt\right] + L^{\star}\left.\right|_{t_f}\delta t_f + \frac{\partial L}{\partial \dot{\bar{x}}}\left.\right|_{\bar{x}^{\star}}\delta\bar{x}\left.\right|_{t_f}$$

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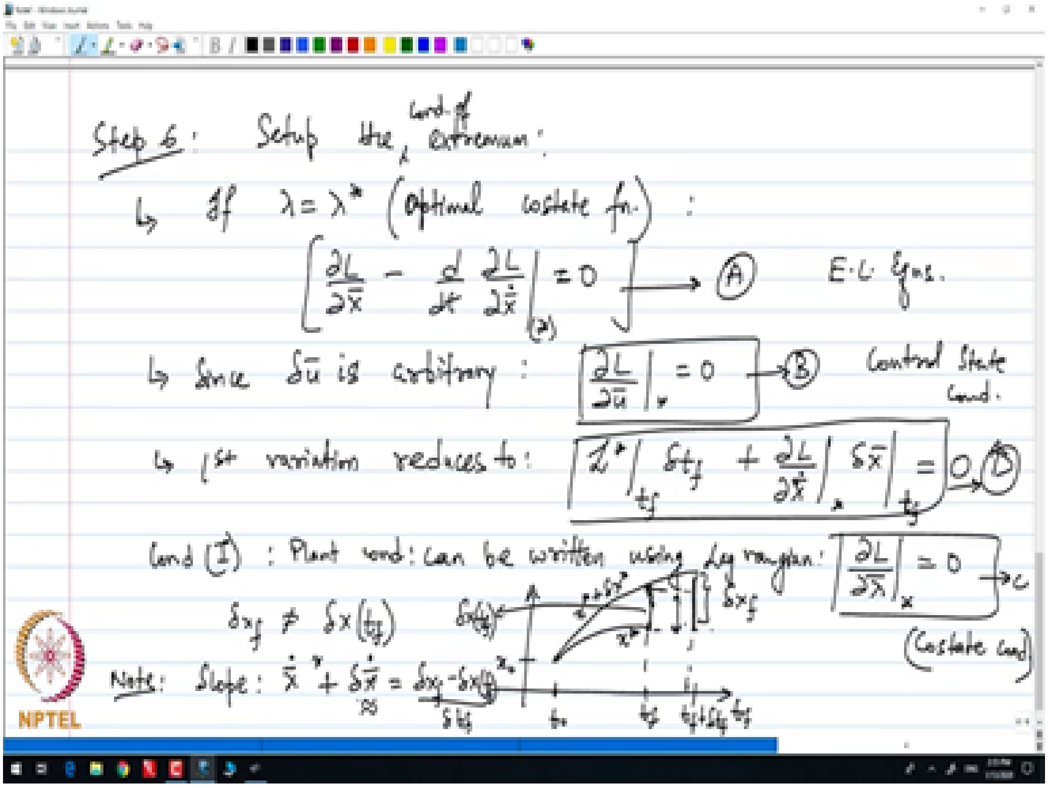


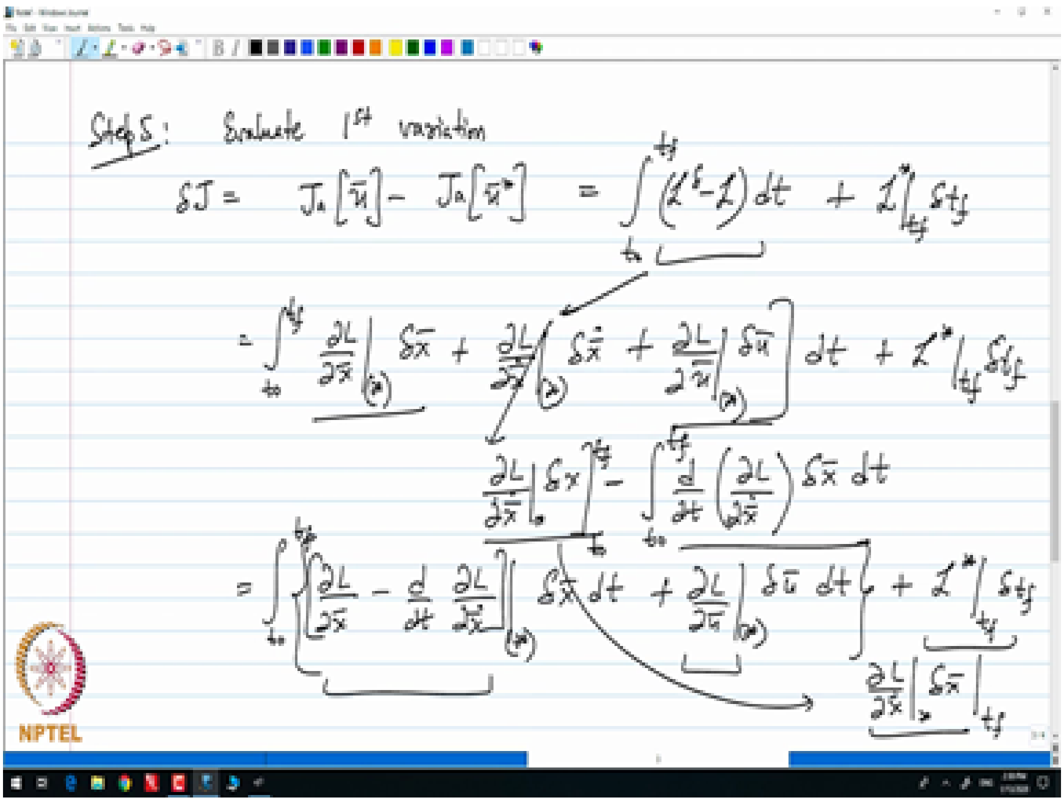


Step : 6 To Set up the conditions of the extremum. If we choose the Lagrange multiplier $\lambda = \lambda^{\star}$ which is optimal co-state function then Euler-Lagrange equations are retrieved back.

$$\frac{\partial L}{\partial \bar{x}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\bar{x}}}\right) |_{\bar{x}^\star} = 0 \qquad \textbf{(A)}$$

Since we have the optimal control condition, so, we have another constraint that comes through. Since variation in u, $\delta]baru$ is arbitrary we see that

$$\frac{\partial L}{\partial \bar{u}} |_{\bar{x}^\star} = 0 \qquad \textbf{(B)}$$

The first variation reduces to

$$L^\star |_{t_f} \delta t_f + \frac{\partial L}{\partial \dot{\bar{x}}} |_{\bar{x}^\star} \delta \bar{x} |_{t_f} = 0 \qquad \textbf{(D)}$$

We also have the plant condition which is the non-holonomic constraint and that can also be found using the Lagrange multiplier setup.So, condition (1) or the plant condition can be written using Lagrangian. So, we have

$$\frac{\partial L}{\partial \lambda} |_{\bar{x}^\star} = 0 \qquad \textbf{(C)}$$

Now,let me try to see how these curves look like. Suppose from $t_0$ to $t_f$, we have let us say $x_0$ to starting point for the optimal curve is as follows. So, this is my curve $x^\star$ and suppose the curve $x^\star$ is being perturbed, and it looks like the following.So, I am trying to highlight the fact that $\delta x_f \neq \delta x\,(t_f)$.

So, we have the following: we see that if this is time point $t_f$ and further I see that the gap between the optimal curve and the perturbed curve is going to denote as the variation in my state variable at time point $t_f$. So, this is the variation at $t_f$ and notice that if new time is $t_f + \delta t_f$ so, variation $\delta x_f$ is the perturbation from the optimal curve to the perturbed curve. So, there is a slight difference between the two quantities in particular we will see that they are not equal.

Then I am going to reformulate our relation $(D)$ noting the following. Note that slope of the optimal curve is : $\dot{\bar{x}}^\star + \delta\dot{\bar{x}} = \frac{\delta x_f - \delta x(t_f)}{\delta t_f}$ . we can rewrite this quantity as follows:

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$$\Rightarrow \delta x_f = \delta x(t_f) + \left[\dot{x} + \delta\dot{x}\right]_{*} \delta t_f$$

$$\approx \delta x(t_f) + \dot{x}^* \delta t_f$$

$$\Rightarrow \boxed{\delta x(t_f) = \delta x_f - \dot{x}^* \delta t_f} \longrightarrow (D_1)$$

Rewrite (D) using $(D_1)$: $\boxed{\left[\mathcal{L}^* - \frac{\partial L}{\partial \dot{x}}\Big|_{(*)} \dot{x}^*\right]_{t_f} \delta t_f + \frac{\partial L}{\partial \dot{x}}\Big|_{(*)}\Big|_{t_f} \delta x_f} = 0$

(E) : Natural B.C.S.

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**Step 6:** Setup the cond. of extremum:

↳ If $\lambda = \lambda^*$ (optimal costate fn.):

$$\left[\frac{\partial L}{\partial \bar{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{(*)} = 0\right] \longrightarrow (A) \quad \text{E-L Eqns.}$$

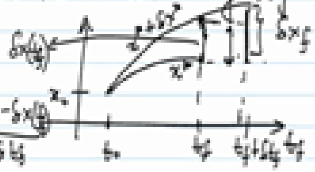
↳ Since $\delta\bar{u}$ is arbitrary: $\boxed{\frac{\partial L}{\partial \bar{u}}\Big|_{*} = 0} \rightarrow (B)$ Control State Cond.

↳ 1st variation reduces to: $\boxed{\mathcal{L}^*\Big|_{t_f} \delta t_f + \frac{\partial L}{\partial \dot{\bar{x}}}\Big|_{*} \delta\bar{x}\Big|_{t_f} = 0} \rightarrow (D)$

Cond (I) : Plant cond: can be written using Lagrangian: $\boxed{\frac{\partial L}{\partial \bar{\lambda}}\Big|_{*} = 0} \rightarrow C$

(Costate cond)

$$\delta x_f \neq \delta x(t_f)$$



Note: Slope: $\dot{x}^* + \delta\dot{x} \approx \frac{\delta x_f - \delta x(t_f)}{\delta t_f}$

we can rewrite this quantity as follows:

$$
\begin{aligned}
\delta x_f &= \delta x\left(t_f\right) + \left[\dot{\bar{x}}^{\star} + \delta \dot{\bar{x}}\right] \delta t_f \\
&\approx \delta x\left(t_f\right) + \dot{\bar{x}}^{\star} \delta t_f \\
\Rightarrow \quad \delta x\left(t_f\right) &= \delta x_f - \dot{\bar{x}}^{\star} \delta t_f
\end{aligned}
\qquad (D_1)
$$

Rewrite $(D)$ using the relation $(D_1)$ and we have the following:

$$
\left[L^{\star} - \frac{\partial L}{\partial \dot{\bar{x}}} \Big|_{\bar{x}^{\star}} \dot{\bar{x}}^{\star}\right] \Big|_{t_f} \delta t_f + \frac{\partial L}{\partial \dot{\bar{x}}} \Big|_{\bar{x}^{\star}} \Big|_{t_f} \delta x_f = 0 \qquad \textbf{(E)}
$$

So, $(E)$ is nothing but natural boundary condition. Now I have set up all first variation conditions, namely the Euler-Lagrange, the control condition as well as the natural boundary condition. Also, we have an extra condition in the form of a co-state condition which is nothing but the plant condition. It seems that this setup is quite complicated. So, let us slightly simplify this description by introducing our Hamiltonian formulation or we call as Pontryagin $H$ function.