Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 53
Conjugate Points / Jacobi Accessory Equations / Introduction to Optimal Control Theory
Part 5

Now, one last topic of discussion that remains in the analysis of the second variation is that what are the saddle points. See we have touched upon the topic of the extrema being maxima or minima, but how about an extrema of being a saddle points. So what is the criteria of detecting saddle points? The answer lies in extending Morse lemma from the finite dimensional to the infinite dimensional case namely the solution lies in determining what is the Morse index for the infinite dimensional function space.

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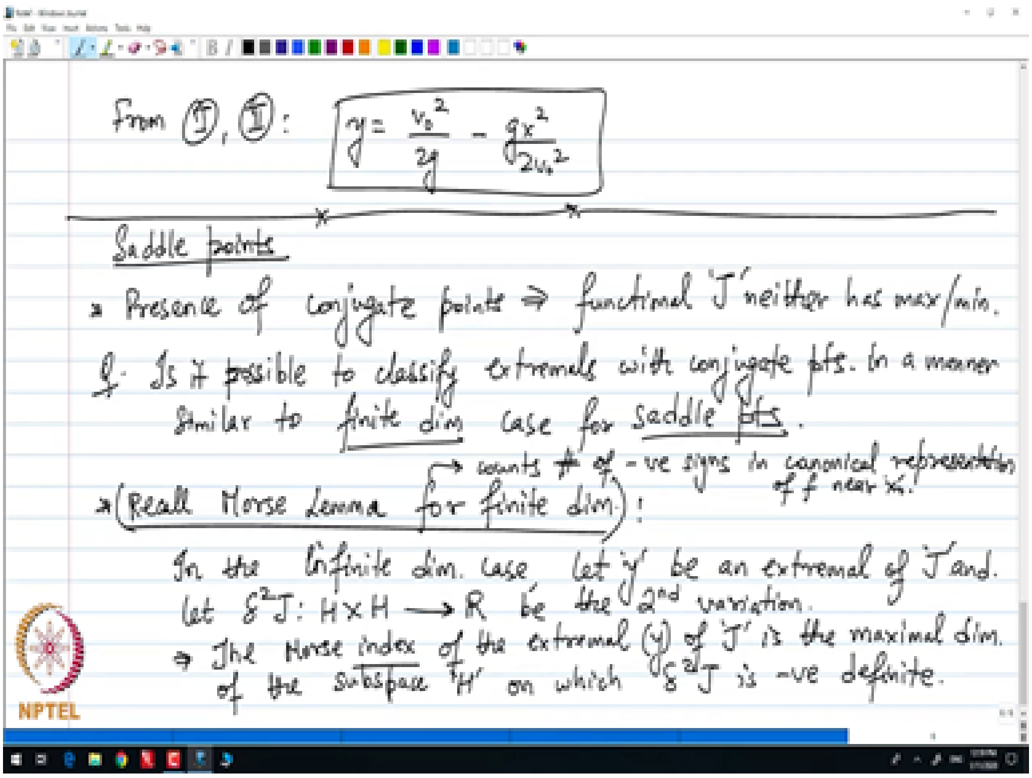

So let us look at the saddle points. The presence of conjugate points implies that the functional J neither has maxima nor has minima.

Question : Is it possible to classify extremals with conjugate points in a manner similar to the finite dimensional case for saddle points.

What did we do for finite dimensional case for saddle points, we looked at the Morse lemma and we calculated the Morse index, if the index is not one of the extremals, that is $\lambda = n$ or $\lambda = 0$ in the finite dimensional case then we are guaranteed to have saddle points. So recall, the Morse lemma for the finite dimensional case. It counts the number of negative signs in the canonical representation of f near the point x_0.

In the infinite dimensional case, let $'y'$ be an extremal of the functional $'J'$ and let $\delta^2 J : H \times H \to \mathbb{R}$ be the second variation, where H is Hilbert space.

It implies that the Morse index of the extremal $'y'$ of $'J'$ is the maximal dimension of the subspace H on which the second variation $\delta^2 J$ is negative definite.

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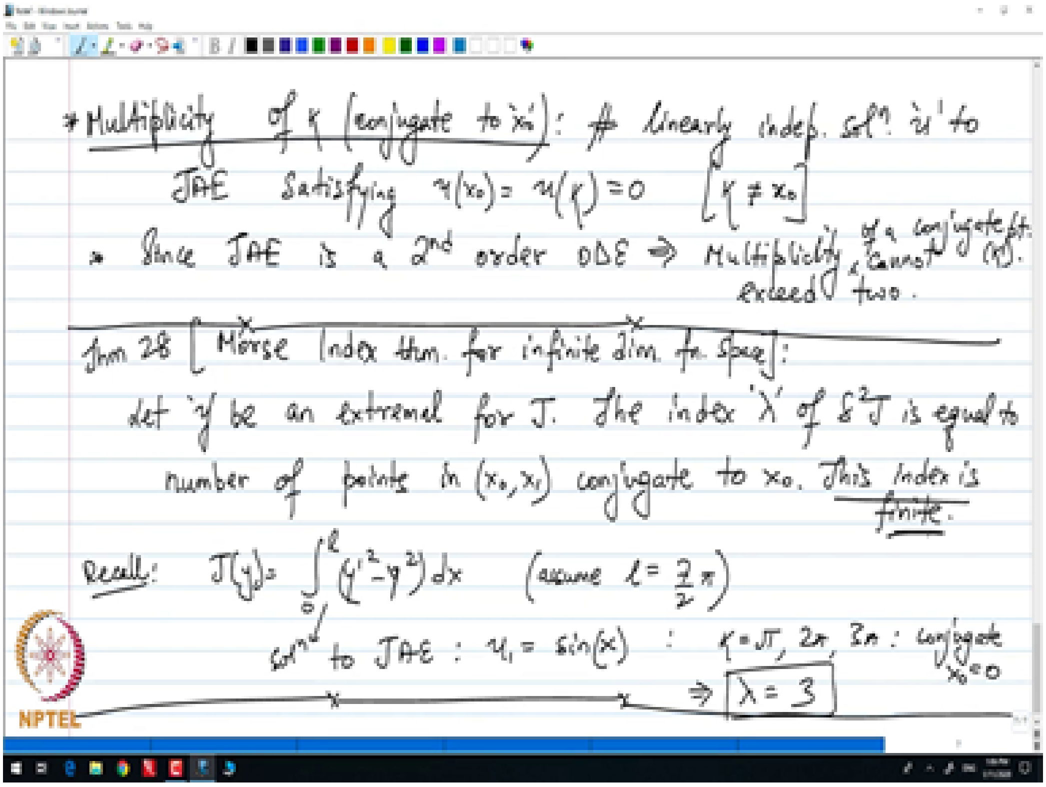

The multiplicity of κ, which is conjugate to x_0 is the number of linearly independent solution $'u'$ to the Jacobi accessory equation satisfying $u\left(x_0\right) = u\left(\kappa\right) = 0$ where $\kappa \neq x_0$.Since the Jacobi accessory equation is a second order ODE, so multiplicity of a conjugate point κ under consideration cannot exceed 2. So we are ready to discuss a major result by Morse for infinite dimensional case.

This is described in the form of a theorem or the Morse index theorem for the infinite dimensional dimension function space. It says that suppose $'y'$ be an extremal for the functional $'J'$ and the index λ of the second variation $\delta^2 J$ is equal to the number of points in (x_0, x_1) conjugate to x_0 and this index is finite.

Let me highlight the importance of this result with a example. Recall, our example $J\left(y\right) = \int_0^l \left(y'^2 - y^2\right) dx$ and assume that $l = \frac{7}{2}\pi$ so certainly it is bigger than π. Now note that one of the solution to the Jacobi accessory equation is $u_1 = \sin x$ and we saw that for this solution the point $\kappa = \pi, 2\pi, 3\pi$ are all conjugate to the point $x_0 = 0$ which means that Morse index $\lambda = 3$. So that is how we use theorem by Morse in the infinite dimensional function space. So certainly if the Morse index is 0, we are guaranteed to get a minima and if the Morse index is ∞ then we are guaranteed to get maxima. So the Morse index guarantees the existence of either saddle points or the local minima.Now, I am going to talk about a specific class of functionals, this is the last topic on the theory of calculus of variations namely the case where our integrands in the functional are convex or they are of certain characteristics. We will see that the extremals that we get for these class of functions are local minima or minima.

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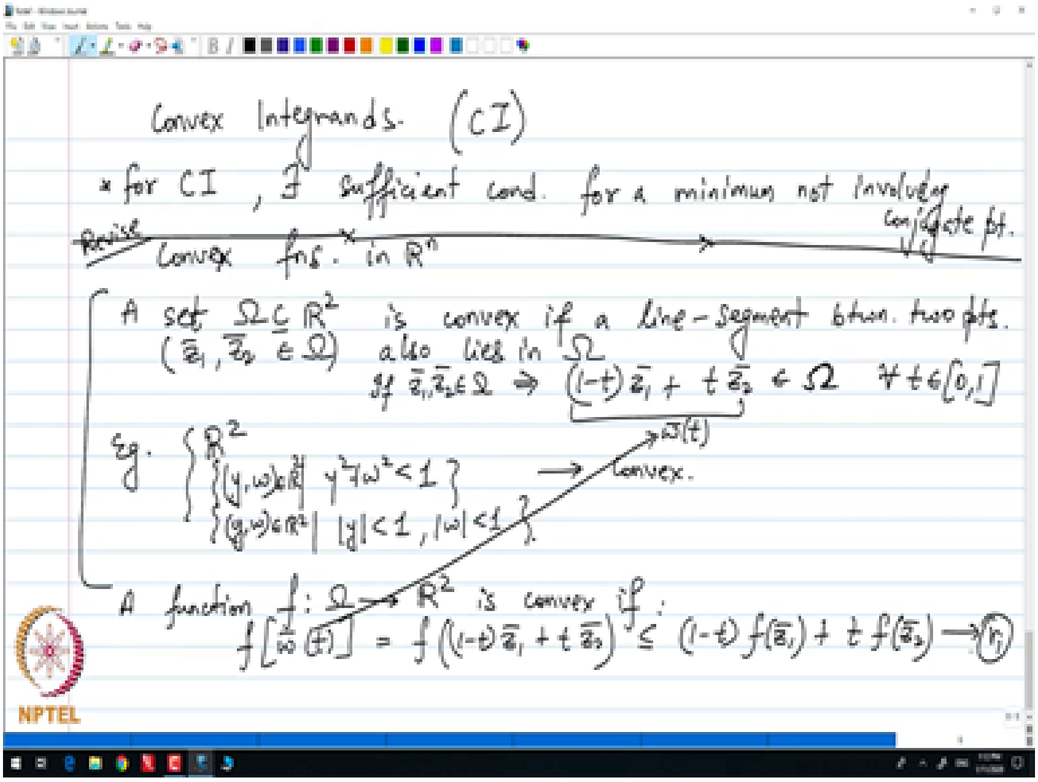

We are now going to focus our attention on convex integrands to the functional let me call this as CI. So as I just said for convex integrands there is a sufficient condition for a minimum not involving conjugate points. Essentially the extremal will be a minimum for convex integrands. So let me just revise the basics of convex functions or convex sets from our finite dimensional case. So, revise convex functions in $\mathbb{R}^n$.

We are going to restrict our attention only to $\mathbb{R}^2$ and similar ideas are extended to $\mathbb{R}^3$,$\mathbb{R}^4$ and so on, higher dimensions. A set $\Omega \subseteq \mathbb{R}^2$ is convex if a line segment between two points $\bar{z}_1, \bar{z}_2 \in \Omega$ also lies in Ω. So what I just said is the following, if the point $\bar{z}_1, \bar{z}_2 \in \Omega$, then

$$\omega(t) = (1-t)\,\bar{z}_1 + t\bar{z}_2 \in \Omega \;\; \forall \;\; t \in [0,1]$$

Examples of convex set are the set $\mathbb{R}^2$, $\left\{(y,\omega) \in \mathbb{R}^2 : y^2 + \omega^2 < 1\right\}$,$\left\{(y,\omega) \in \mathbb{R}^2 : |y| < 1, |\omega| < 1\right\}$. So these are some of the examples which are all convex. Now, I describe what are convex functions.

A function $f : \Omega \to \mathbb{R}^2$ is convex, if

$$f\left[\bar{\omega}(t)\right] = f\left[(1-t)\,z_1 \bar{+}\, t\bar{z}_2\right] \le (1-t)\,f(\bar{z}_1) + t f(\bar{z}_2) \qquad (r_1)$$

So then we can use certain results for example the mean value theorem to arrive at certain other relations.

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Use Mean-Value Thm: $\exists\ \tau \in (0,t)$ s.t.

$$f(\bar{w}(t)) = f(\bar{z}_1) + t(\bar{z}_2-\bar{z}_1)\cdot\bar{\nabla} f(\bar{w}(\tau)) \rightarrow (r_2)$$

Since $f, \bar{\nabla} f \leftarrow$ smooth & $0<\tau<t$ $\Rightarrow \lim\limits_{t\to 0} \bar{\nabla} f(\bar{w}(\tau)) = \bar{\nabla} f(\bar{z}_1) \rightarrow (r_3)$

[Recall $\bar{w}(t) = \bar{z}_1 + t(\bar{z}_2-\bar{z}_1)$]

Using (r_3) in (r_1): $\boxed{(\bar{z}_2-\bar{z}_1)\cdot\bar{\nabla} f(\bar{z}_1) \le f(\bar{z}_2)-f(\bar{z}_1)} \rightarrow (r_4)$

Suppose $\bar{z}_1 = $ C.P. or $\bar{\nabla} f(\bar{z}_1)=0 \Rightarrow f(\bar{z}_1) \le f(\bar{z}_2)$ [from (r_4)]

$\Rightarrow$ '$\bar{z}_1$' is the local minimum.

Consider $J(y) = \int_{x_0}^{x_1} f(x,y,y')\,dx$

Suppose $\Omega_x = \{(y,y') \in \mathbb{R}^2 \mid (x,y,y') \in D_f$: domain of $f\}$ is convex.

& further $f: \Omega_x \rightarrow \mathbb{R}$ is also convex

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Convex Integrands (CI)

* for CI, $\exists$ sufficient cond. for a minimum not involving conjugate pt.

~~Revise~~ Convex fns. in $\mathbb{R}^n$

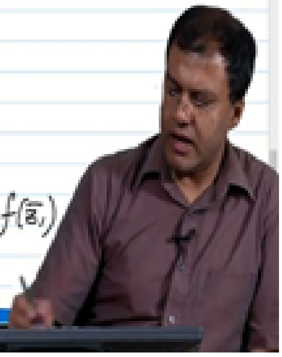
A set $\Omega \subseteq \mathbb{R}^2$ $(\bar{z}_1, \bar{z}_2 \in \Omega)$ is convex if a line-segment btwn. two pts. also lies in Ω

If $\bar{z}_1, \bar{z}_2 \in \Omega \Rightarrow \underbrace{(1-t)\bar{z}_1 + t\bar{z}_2}_{\bar{w}(t)} \in \Omega \quad \forall t \in [0,1]$

Eg. $\left\{ \begin{array}{l} \mathbb{R}^2 \\ \{(y,\omega)\in\mathbb{R}^2 \mid y^2+\omega^2<1\} \\ \{(y,\omega)\in\mathbb{R}^2 \mid |y|<1, |\omega|<1\} \end{array} \right. \rightarrow$ convex.

A function $f: \Omega \rightarrow \mathbb{R}^2$ is convex if:

$f[\bar{w}(t)] = f((1-t)\bar{z}_1 + t\bar{z}_2) \le (1-t) f(\bar{z}_1)$



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The mean value theorem for convex function says that $\exists$ a number $\tau \in (0, t)$ such that

$$f\left(\bar{\omega}\left(t\right)\right) = f\left(\bar{z}_1\right) + t\left(\bar{z}_2 - \bar{z}_1\right).\bar{\nabla} f\left(\bar{\omega}\left(\tau\right)\right) \qquad (r_2)$$

Since functions f and $\bar{\nabla} f$ are smooth, that is the derivatives are continuous and the function itself is continuous and $\tau \in (0, t)$, it implies that

$$\lim_{t \to 0} \bar{\nabla} f\left(\bar{\omega}\left(\tau\right)\right) = \bar{\nabla} f\left(\bar{z}_1\right) \qquad (r_3)$$

Because recall that $\bar{\omega}\left(\tau\right) = \bar{z}_1 + \tau\left(\bar{z}_2 - \bar{z}_1\right)$. Using (r_3) in (r_1) I arrive at the following relation

$$\left(\bar{z}_2 - \bar{z}_1\right).\bar{\nabla} f\left(\bar{z}_1\right) \leq f\left(\bar{z}_2\right) - f\left(\bar{z}_1\right) \qquad (r_4)$$

Further if we are talking about the functions in finite dimensional space in $\mathbb{R}^2$ and we know that a point z is a critical point provided the gradient vanishes. Suppose that z_1 is a critical point or I am saying that $\bar{\nabla} f\left(\bar{z}_1\right) = 0$, then $f\left(\bar{z}_1\right) \leq f\left(\bar{z}_2\right)$ from condition (r_4) or the point $\bar{z}_1$ is the local minima.

It is always going to be the minimum compared with the function values at other points in its neighborhood. So which means that any extremal for a convex function will be a local minimum. We can extend this same idea for functional optimization where our integrands of the functional are convex functions. So, let us continue the extension of this finite dimensional case to the functional case.

Consider $J\left(y\right) = \int_{x_0}^{x_1} f\left(x, y, y'\right) dx$ and suppose $\Omega_x = \left\{\left(y, y'\right) \in \mathbb{R}^2 : \left(x, y, y'\right) \in D_f : \; domain \;\; of \;\; f\right\}$ is convex set and further $f : \Omega_x \to \mathbb{R}$ is also convex.

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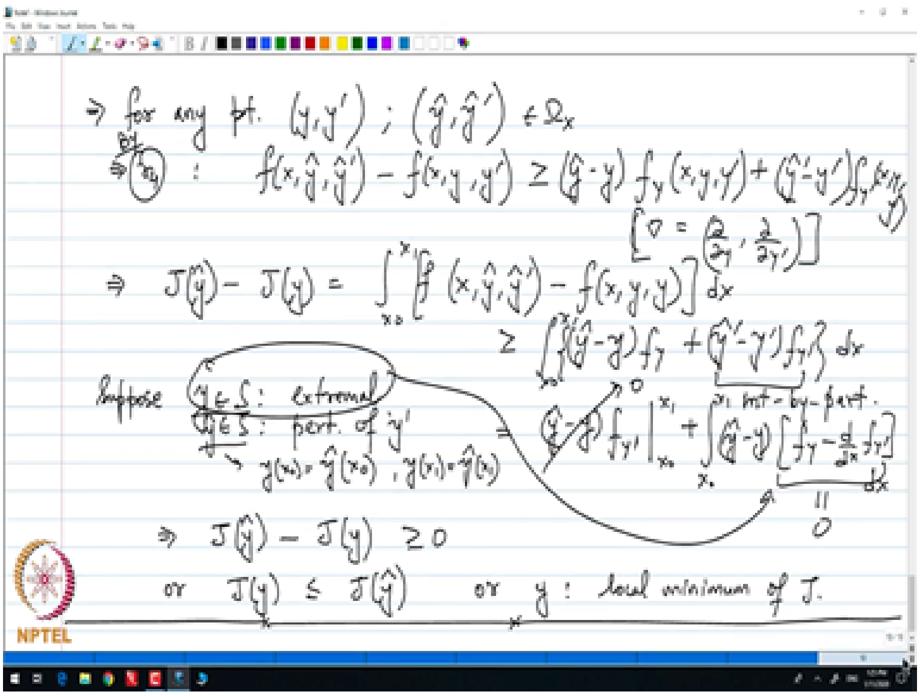

For any points $\left(y, y'\right); \left(\hat{y}, \hat{y}'\right) \in \Omega_x$, by (r_4) I have the following inequality

$$f\left(x, \hat{y}, \hat{y}'\right) - f\left(x, y, y'\right) \geq \left(\hat{y} - y\right) f_y\left(x, y, y'\right) + \left(\hat{y}' - y'\right) f_{y'}\left(x, y, y'\right)$$

The 2 dimensional gradient operator is $\nabla = \left(\frac{\partial}{\partial y} \, , \, \frac{\partial}{\partial y'}\right)$. So that is how we are getting this result. And from here,

$$J\left(\hat{y}\right)-J\left(y\right)=\int_{x_0}^{x_1}\left[f\left(x,\hat{y},\hat{y}'\right)-f\left(x,y,y'\right)\right]dx\geq\int_{x_0}^{x_1}\left[\left(\hat{y}-y\right)f_y+\left(\hat{y}'-y'\right)f_{y'}\right]dx$$

Suppose $y\in S$ is an extremal and $\hat{y}\in S$ is the perturbation of y. Now, we do integration by parts and we have the following

$$J\left(\hat{y}\right)-J\left(y\right)\geq\left(\hat{y}-y\right)f_{y'}\mid_{x_0}^{x_1}+\int_{x_0}^{x_1}\left(\hat{y}-y\right)\left[f_y-\frac{\mathrm{d}}{\mathrm{d}x}f_{y'}\right]dx$$

Since y and $\hat{y}$ both lie in S, so it must satisfy the endpoint conditions. So this means $y\left(x_0\right)=\hat{y}\left(x_0\right)$, $y\left(x_1\right)=\hat{y}\left(x_1\right)$. Since y is an extremal then it leads to the condition that the solution to this equation inside the bracket, which is the Euler Lagrange equation will be 0. So , we have found that

$$J\left(\hat{y}\right)-J\left(y\right)\geq0\quad or\quad J\left(y\right)\leq J\left(\hat{y}\right)$$

Or the extremal $'y'$ is a local minimum of J. So that is the conclusion that we can right away derive from our finite dimensional case. So the analysis becomes very easy when we are dealing with convex integrands. After few minutes, I am going to introduce a topic in optimal controls where we are exactly going to discuss convex integrands. So let us wrap up this section by providing some results and give some examples on how to solve optimization for convex integrands.

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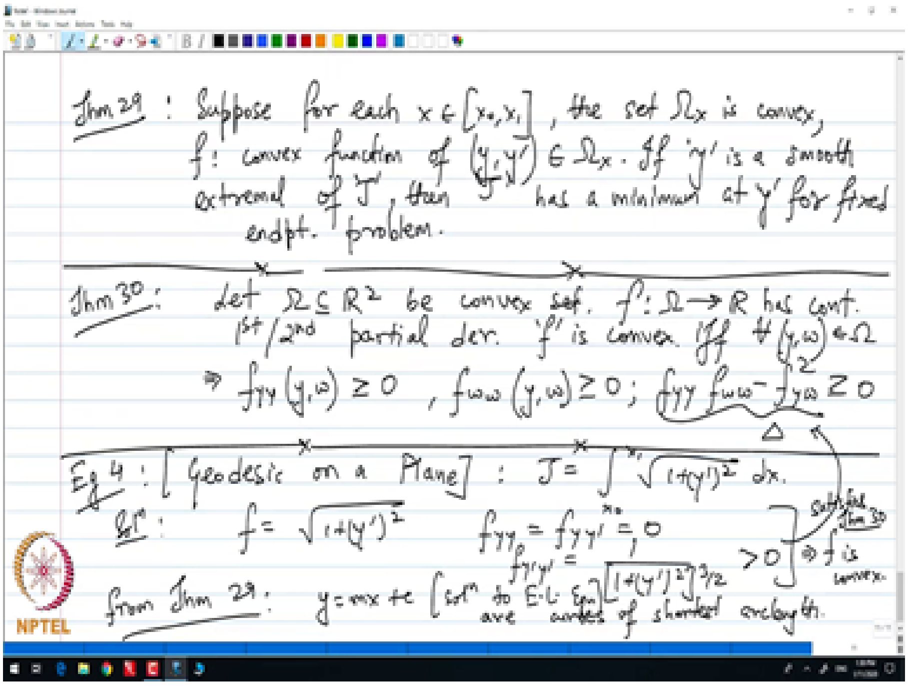

I have two theorems the first says that suppose for each point $x\in[x_0,x_1]$ the set Ω_x is convex and suppose f is also a convex function of $(y,y')\in\Omega_x$. If $'y'$ is a smooth is a smooth extremal of $'J'$ then $'J'$ j has a minimum at y for fixed endpoint problem.

The second result says that Let $\Omega\subseteq\mathbb{R}^2$ be a convex set and $f:\Omega\to\mathbb{R}$ has continuous first and second order partial derivatives then f is convex iff $\forall\ (y,\omega)\in\Omega$ we have

$$f_{yy}\left(y,\omega\right)\geq0\ \ ;\ \ f_{\omega\omega}\left(y,\omega\right)\geq0\ \ ;\ \ f_{yy}f_{\omega\omega}-f_{y\omega}^2\geq0$$

we call $f_{yy}f_{\omega\omega}-f_{y\omega}^2$ by Δ.So let me wrap up this discussion by looking at few examples, we are going to revisit our examples of the geodesics on a plane. Let us consider the geodesic example on a plane. Let $J = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$ and we have to check whether the integrand is convex or not, then only we can conclude something. So, $f = 1 + (y')^2$ and we see that $f_{yy} = f_{yy'} = 0$ and $f_{y'y'} = \frac{1}{\left[1+(y')^2\right]^{\frac{3}{2}}}$ and of course $f_{y'y'} > 0$ for any value of the derivative y'.

We can see that these condition satisfy convexity conditions, it implies that f is convex. The result from this statement that f is convex implies that the extremal that we find which is the straight lines will be local minima. So, from the previous theorem I can see that $y = mx + c$ which is the solution to the Euler Lagrange equation are curves of shortest arc length or they are the minimal extremal to the functional J.

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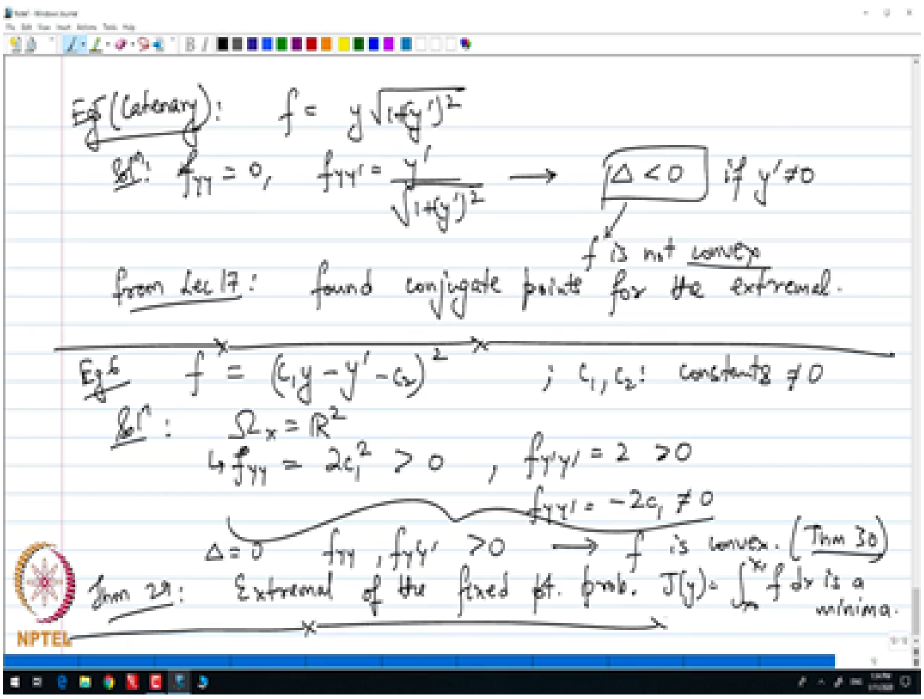

Let us look at one or two more examples. Another example that I have is that of the catenary. In this case integrand is of the form $f = y\sqrt{1 + (y')^2}$.Then $f_{yy} = 0$, $f_{yy'} = \frac{y'}{\sqrt{1+(y')^2}}$. And I see that in this case it does not matter whatever is $f_{y'y'}$ but discriminant $\Delta < 0$ if $y' \neq 0$ which means that f is not convex . We have seen that one of the extremals that we find for the catenary problem does not give either a minima or maxima.

So from our previous lecture, we found conjugate points guaranteeing the extremal not being a minima or maxima for the extremal. And then finally here is one more example. I consider the integrand of the form $f = (c_1 y - y' - c_2)^2$; c_1 and c_2 are non-zero constants. Let me $\Omega_x = \mathbb{R}^2$ so I know that the subset is convex, I have to check whether f is convex or not. Notice that

$$f_{yy} = 2c_1^2 > 0 \quad , \quad f_{y'y'} = 2 > 0 \quad , \quad f_{yy'} = -2c_1 \neq 0$$

From all these derivatives, I can conclude that discriminant $\Delta = 0$. But f_{yy} and $f_{y'y'}$ are positive implying that f is convex. And further f is convex or the extremal of fixed point problem given by $J(y) = \int_{x_0}^{x_1} f dx$ is a minima. So that is how we typically solve problems with convex integrands. That concludes my discussion on all about the theory of calculus of variations.