Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 52
Conjugate Points / Jacobi Accessory Equations / Introduction to Optimal Control Theory
Part 4

In today's lecture, I am going to discuss the final topic related to the second variation namely what is the geometric meaning of the conjugate points . And then towards the later half of this course, I am also going to talk about an application of the calculus of variations namely the one arising in optimal control theory.

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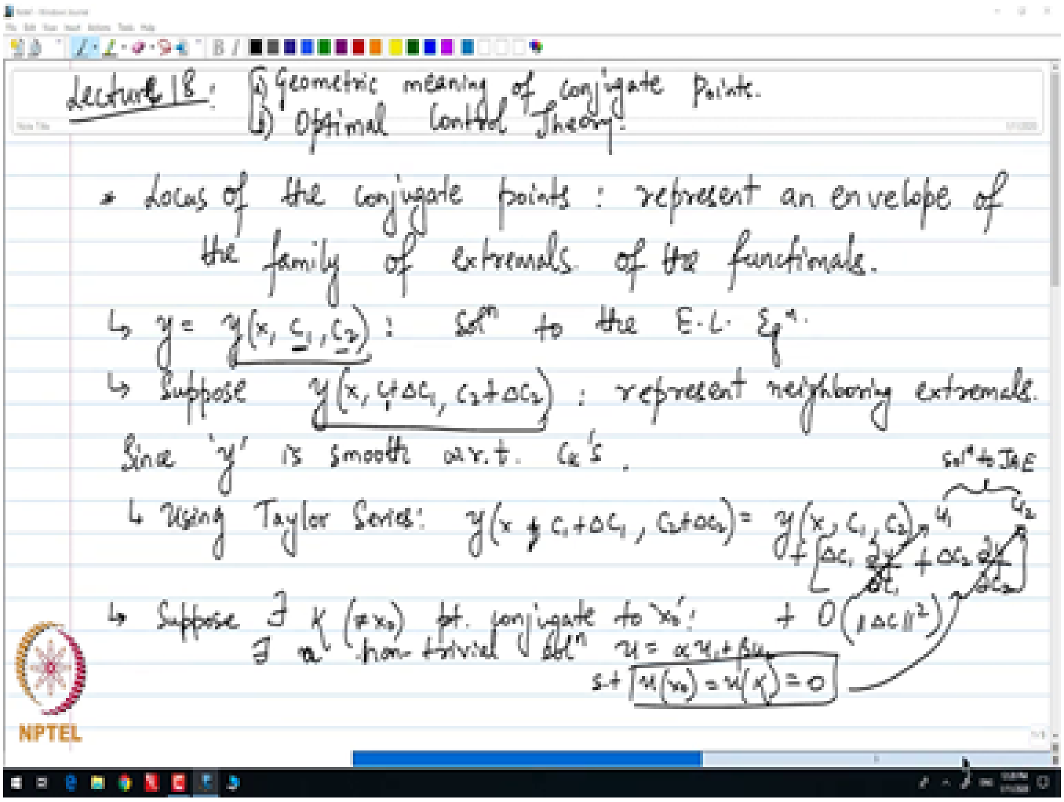

I am going to cover two topics. The first one will be mainly the continuation of our previous lectures discussion that is on the geometric meaning of conjugate points. And the second topic that I will be starting today will be applications of calculus of variations in optimal control theory.

Let us continue our discussion on the geometric meaning. We saw that the moment we have non-trivial conjugate points, we are guaranteed that the extrema that we find will either be a minima or maxima.Thus question is what are these conjugate points? I am going to right away give the result and I will show this result using some analysis. The result that I would like to show is that the locus of the conjugate points represent an envelope of the family of extremals of the functional under consideration. Let us choose a two parameter family of extremals $y = y(x, c_1, c_2)$ which is a solution to the Euler Lagrange equation.

Now, suppose $y\left(x, c_1+\Delta c_1, c_2+\Delta c_2\right)$ be the family which represents a neighboring extremal.Since $'y'$ is smooth with respect to c_k's.Then using Taylor series,

$$y\left(x, c_1+\Delta c_1, c_2+\Delta c_2\right)=y\left(x, c_1, c_2\right)+\left[\Delta c_1 \frac{\partial y}{\partial c_1}+\Delta c_2 \frac{\partial y}{\partial c_2}\right]+O\left(\|\Delta c\|^2\right)$$

Notice that from our discussion in the previous lecture, we know that $\frac{\partial y}{\partial c_1}=u_1$; $\frac{\partial y}{\partial c_2}=u_2$ where u_1 and u_2 are the solution to the Jacobi accessory equation. Now, I introduce a conjugate point suppose $\exists \kappa\left(\neq x_0\right)$ is a point conjugate to x_0 which means that $\exists$ a non-trivial solution given by $u=\alpha u_1+\beta u_2$ such that $u\left(x_0\right)=u(\kappa)=0$ .So we can use this information in this Taylor series expansion. Let us evaluate the Taylor series expansion at the conjugate points x_0 and κ.

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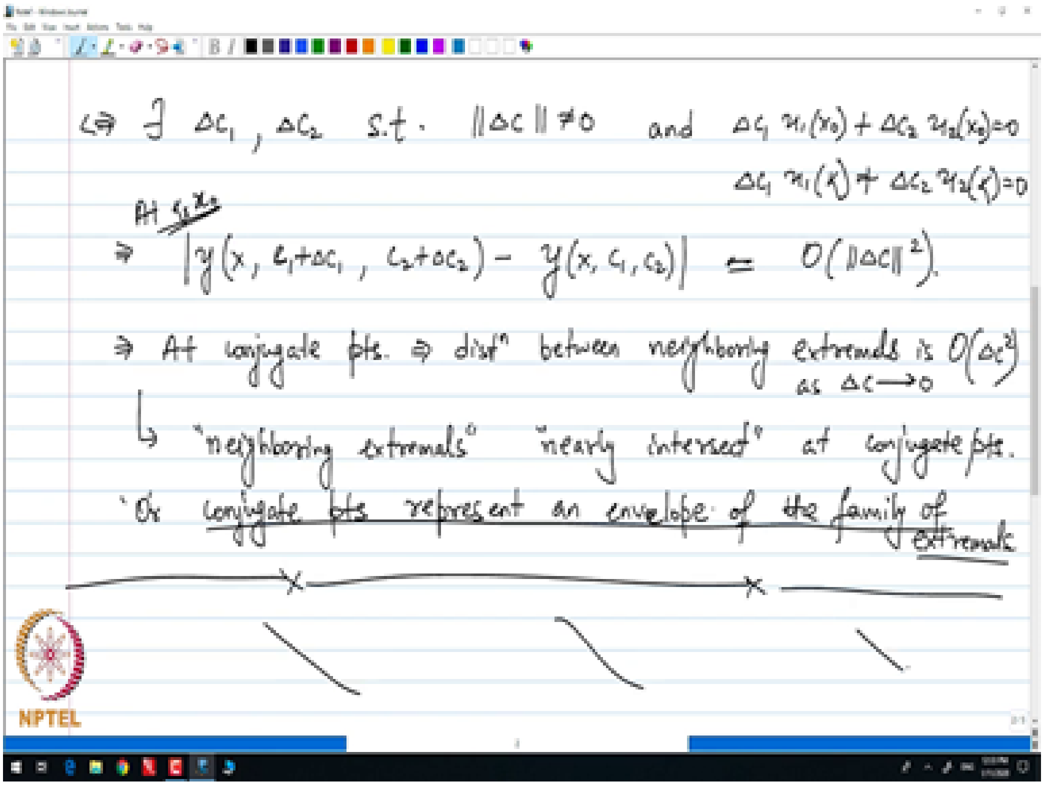

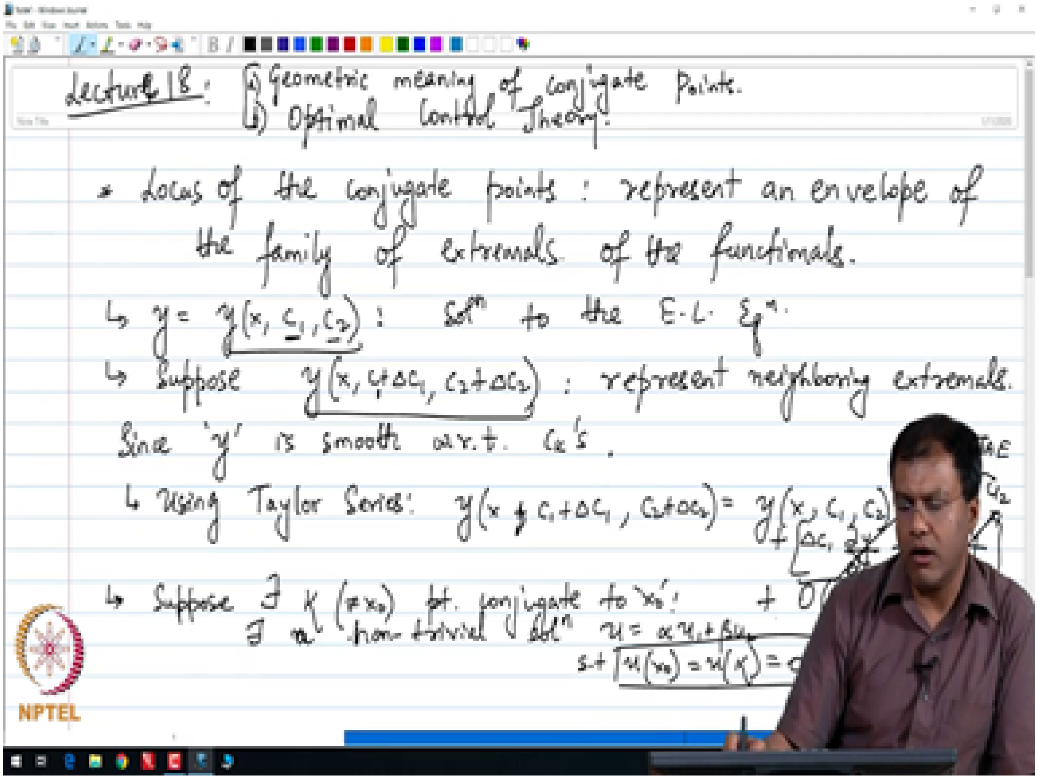

So, the previous statement is equivalent to saying that $\exists \Delta c_1, \Delta c_2$ such that $\|\Delta c\| \neq 0$ and

$$\Delta c_1 u_1 (x_0) + \Delta c_2 u_2 (x_0) = 0$$

$$\Delta c_1 u_1 (\kappa) + \Delta c_2 u_2 (\kappa) = 0$$

That comes from the definition of points x_0 and κ being conjugate to each other. Now at point x_0 and κ

$$|y (x, c_1 + \Delta c_1, c_2 + \Delta c_2) - y (x, c_1, c_2)| = O \left(\|\Delta c\|^2 \right)$$

So, the conclusion is at the conjugate points, the distance between the neighboring extremals is $O \left(\Delta c^2 \right)$ as $\Delta c \to 0$.Further the final conclusion out of all these exercises that the neighboring extremals are nearly intersect at the conjugate points or the conjugate points represent an envelope of the family of extremals. So, at least we know what do we mean by the locus of these conjugate points they are the envelope of the family of extremals. So, that completes the geometric meaning of these conjugate points.

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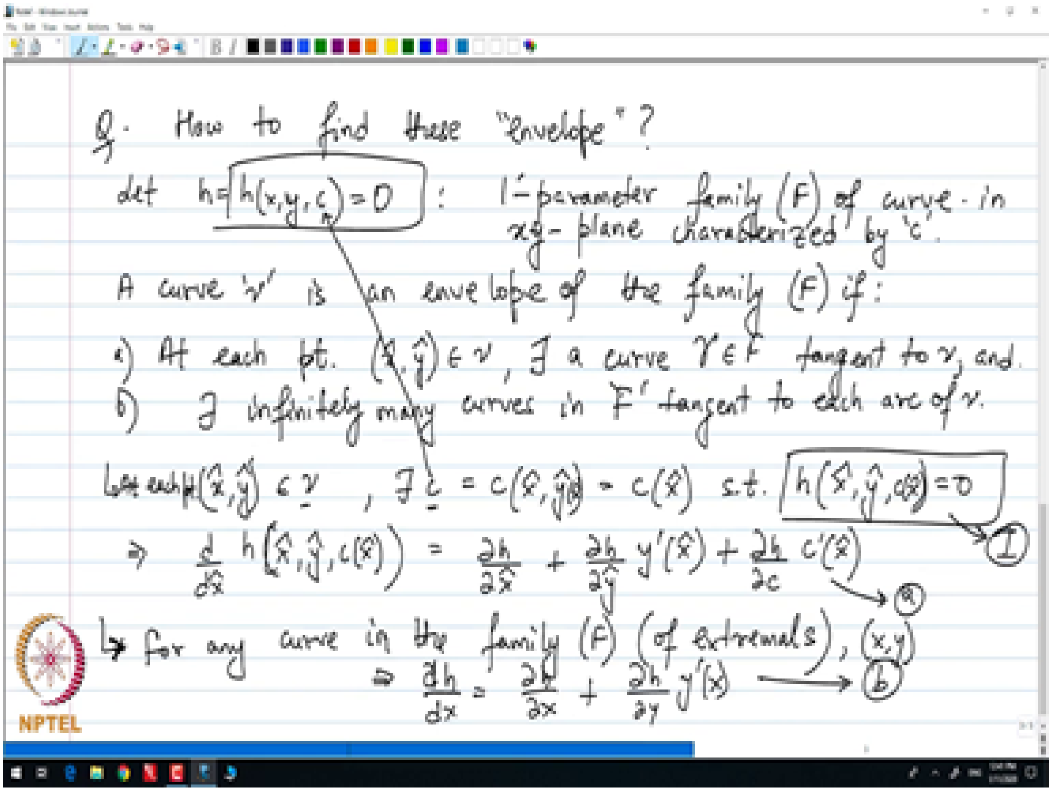

Thus question is how to find these “envelope”? Let $h = h\left(x, y, c\right) = 0$ be one parameter family (F) of curves in the xy- plane characterized by c.Then for this family what is an envelope? A curve $'\nu'$ is an envelope of the family (F) if the following sets of two condition holds :

a) At each point $(\hat{x}, \hat{y}) \in \nu$, $\exists$ a curve $\gamma \in F$ tangent to ν ,and

b) $\exists$ infinitely many curves in F tangent to each arc of ν. So, we are going to use this definition of the envelope to derive the condition for the envelope of the extremals.

So, let us start analysis . Let at each point $(\hat{x}, \hat{y}) \in \nu$, $\exists$ $c = c\left(\hat{x}, \hat{y}\left(\hat{x}\right)\right) = c\left(\hat{x}\right)$ (Notice that this constant c is a constant with respect to the family of extermals h, it is not a constant with respect to the points lying on the curve ν.) such that

$$h\left(\hat{x}, \hat{y}, c\left(\hat{x}\right)\right) = 0 \qquad \textbf{(1)}$$

Now If I differentiate relation (1) with respect to $\hat{x}$ a point on the envelope we get the following :

$$\frac{\mathrm{d}}{\mathrm{d}\hat{x}} h\left(\hat{x}, \hat{y}, c\left(\hat{x}\right)\right) = \frac{\partial h}{\partial \hat{x}} + \frac{\partial h}{\partial \hat{y}} y'\left(\hat{x}\right) + \frac{\partial h}{\partial c} c'\left(\hat{x}\right) \qquad (a)$$

For any curve in the family F of extremals, at point (x, y) the following relation holds

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} y'\left(x\right) \qquad (b)$$

Now, at the point of tangency we must have that the point (x, y) which lies on the family of extremals must also lie on envelope ν. So, they must satisfy condition a and b simultaneously. So, we have the following:

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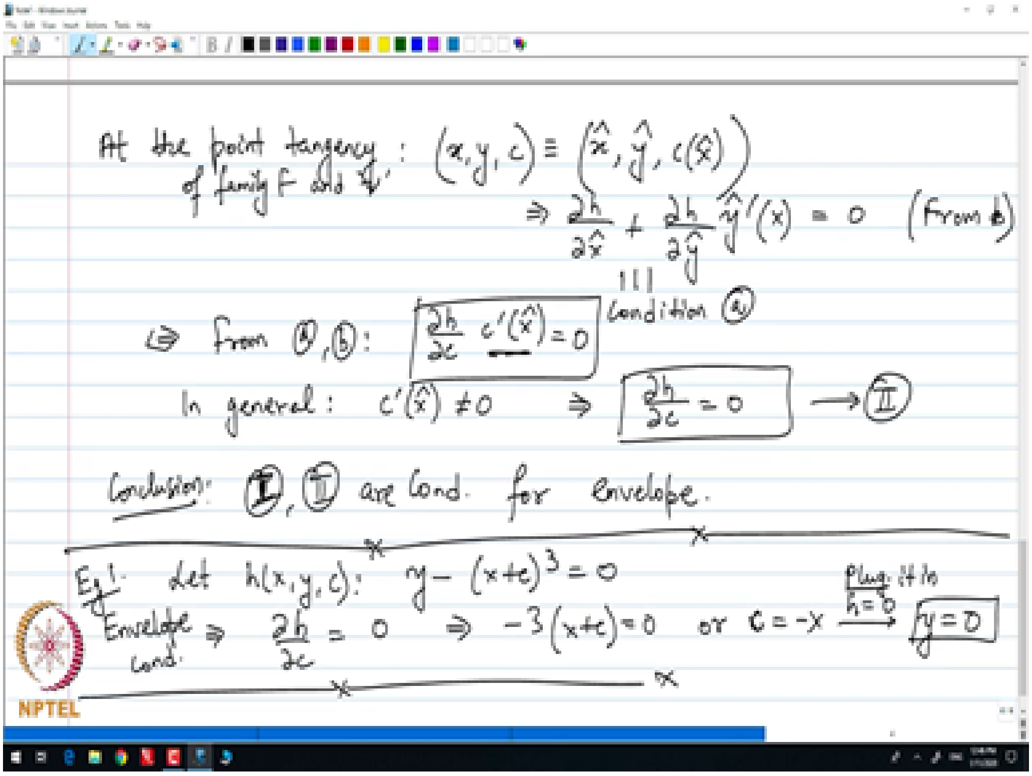
At the point tangency of family F and ν: $(x, y, c) \equiv (\hat{x}, \hat{y}, c(\hat{x}))$
$\Rightarrow \frac{\partial h}{\partial \hat{x}} + \frac{\partial h}{\partial \hat{y}} \hat{y}'(x) = 0$ (From b)
Condition (a)
From (a), (b): $\frac{\partial h}{\partial c} c'(\hat{x}) = 0$
In general: $c'(\hat{x}) \neq 0 \Rightarrow \frac{\partial h}{\partial c} = 0 \longrightarrow$ (II)
Conclusion: (I), (II) are Cond. for Envelope.
Eg 1. Let $h(x, y, c)$: $y - (x+c)^3 = 0$
Envelope Cond. $\Rightarrow \frac{\partial h}{\partial c} = 0 \Rightarrow -3(x+c) = 0$ or $c = -x$ $\xrightarrow{h=0}$ $y = 0$
Plug it in
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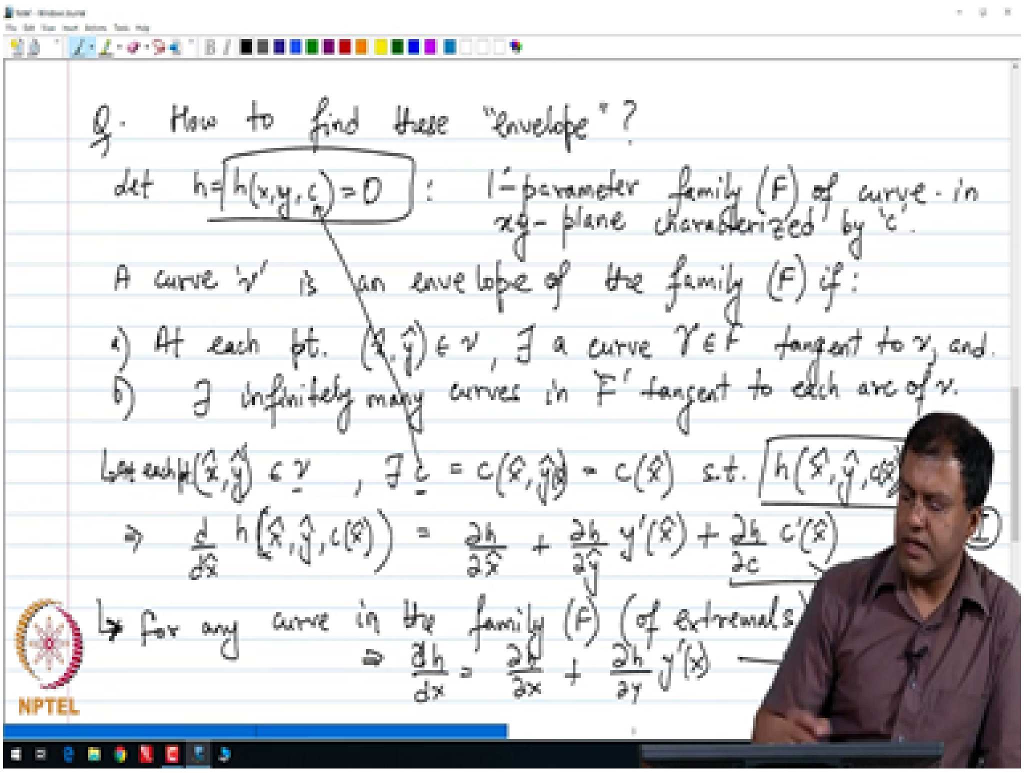
Q. How to find these "envelope"?
Let $h = h(x, y, c) = 0$: 1-parameter family (F) of curve in xy-plane characterized by 'c'.
A curve 'ν' is an envelope of the family (F) if:
a) At each pt. $(\hat{x}, \hat{y}) \in \nu$, ∃ a curve $\gamma \in F$ tangent to ν, and
b) ∃ infinitely many curves in F tangent to each arc of ν.
Let each pt. $(\hat{x}, \hat{y}) \in \nu$, ∃ $c = c(\hat{x}, \hat{y}) = c(\hat{x})$ s.t. $h(\hat{x}, \hat{y}, c(\hat{x}))$
$\Rightarrow \frac{d}{d\hat{x}} h(\hat{x}, \hat{y}, c(\hat{x})) = \frac{\partial h}{\partial \hat{x}} + \frac{\partial h}{\partial \hat{y}} y'(\hat{x}) + \frac{\partial h}{\partial c} c'(\hat{x})$
For any curve in the family (F) (of extremals)
$\Rightarrow \frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} y'(x)$
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So, at the point of tangency of the family F and the curve ν, I must have that $(x, y, c) \equiv (\hat{x}, \hat{y}, c(\hat{x}))$ and we see that $\frac{\partial h}{\partial \hat{x}} + \frac{\partial h}{\partial \hat{y}} \hat{y}'(\hat{x}) = 0$. we get that from our condition b.

From (a) and (b), we see that $\frac{\partial h}{\partial c} c'(\hat{x}) = 0$ .In general,$c'(\hat{x}) \neq 0$ that gives me the conclusion that

$$\frac{\partial h}{\partial c} = 0 \qquad \textbf{(2)}$$

So, the conclusion is condition (1) and condition (1) are the condition for finding the envelope.

Let us look at some examples on how to find this envelope. Let $h(x, y, c)$ be the following function $y - (x + c)^3 = 0$. And I want to find the family of envelopes for this family of curves. So, to find the family of envelope, envelope condition is

$$\frac{\partial h}{\partial c} = 0 \;\; \Rightarrow \;\; -3(x + c) = 0 \;\; or \;\; c = -x$$

Now, we plug it in the condition for the family of extremals $h = 0$ we get that envelope are $y = 0$,all the points lying on the x axis.

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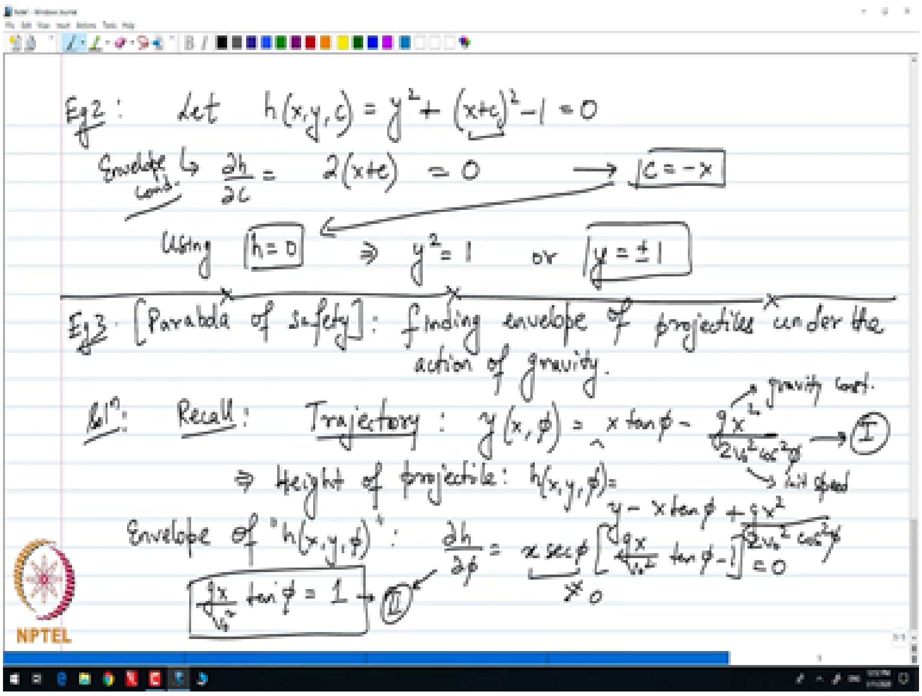

Let us look at the second example. Let $h(x, y, c) = y^2 + (x + c)^2 - 1 = 0$ and I want to find the envelope for this class of functions.Now, differentiate h with respect to c and we get

$$\frac{\partial h}{\partial c} = 2(x + c) = 0 \;\; \Rightarrow \;\; c = -x$$

Now, using $h = 0$ the family of extremals and the envelope condition, I see that $y^2 = 1 \;\; or \;\; y = \pm 1$ and I get two straight lines at $y = 1$ and $y = -1$ as the family of envelope for the curves represented by h.

Let us quickly wrap up our discussion in this topic by giving one more example. This example is related to the parabola of safety. So, students who have done science measures in high school they must have been taught about the motion of a projectile under the action of gravity.And when we throw a projectile at a particular angle it is going to follow a path which represents a parabola. The following exercise will tell us what should be a typical height of an aeroplane or an object to fly so that it avoids a projectile path which is shot at a given velocity and a given angle also known as the parabola of safety.

So, find the envelope of projectiles under the action of gravity. Recall that the trajectory by the projectile follows the following path :

$$y(x,\phi) = x\tan\phi - \frac{gx^2}{2v_0^2\cos^2\phi} \qquad \textbf{(1)}$$

where v_0 is initial speed and g is gravity constant.So, the height of the projectile is given by

$$h(x,y,\phi) = y - x\tan\phi + \frac{gx^2}{2v_0^2\cos^2\phi}$$

So, to find the envelope of the height function we need to differentiate h with respect to ϕ and we get the following expression :

$$\frac{\partial h}{\partial \phi} = x\sec\phi\left[\frac{gx}{v_0^2}\tan\phi - 1\right] = 0$$

And we can see that the function $x\sec\phi \neq 0$ because that means $\cos\phi = \infty$ that is not possible.So, we are getting that

$$\frac{gx}{v_0^2}\tan\phi = 1 \qquad \textbf{(2)}.$$

Then the condition (1) also satisfies the equation of the trajectory.

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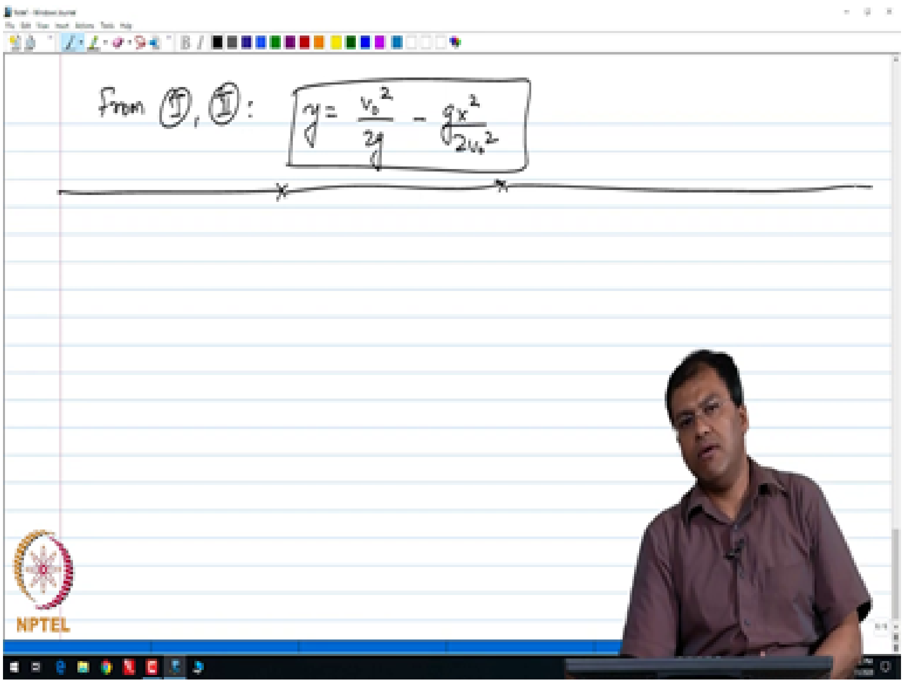

So, from (1) and (2) , I see that

$$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

So, parabola of safety or the parabola over which any object can fly without getting hit has the following curve. So, that completes the discussion on the importance of these conjugate points.