Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 51
Conjugate Points / Jacobi Accessory Equations / Introduction to Optimal Control Theory
Part 3

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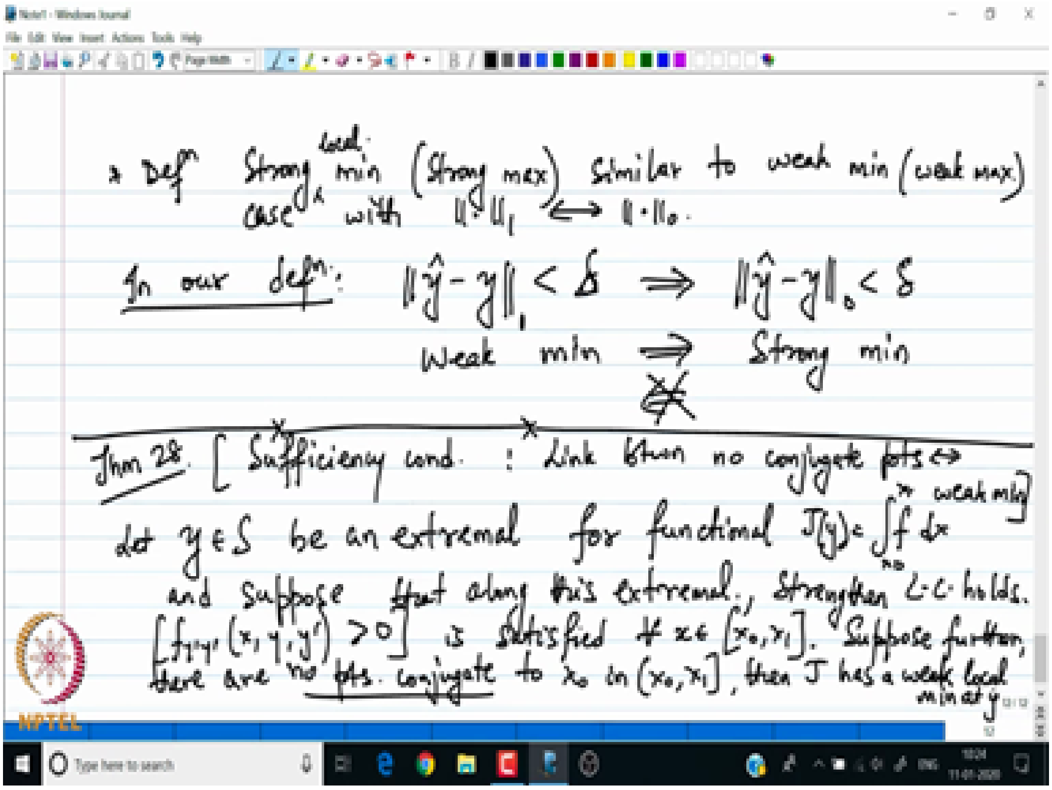

So, that brings us to a particular question, how to find these conjugate points? The moment we are able to find this conjugate points, we are guaranteed that the extremal that we have will be neither a minimum nor a maximum.

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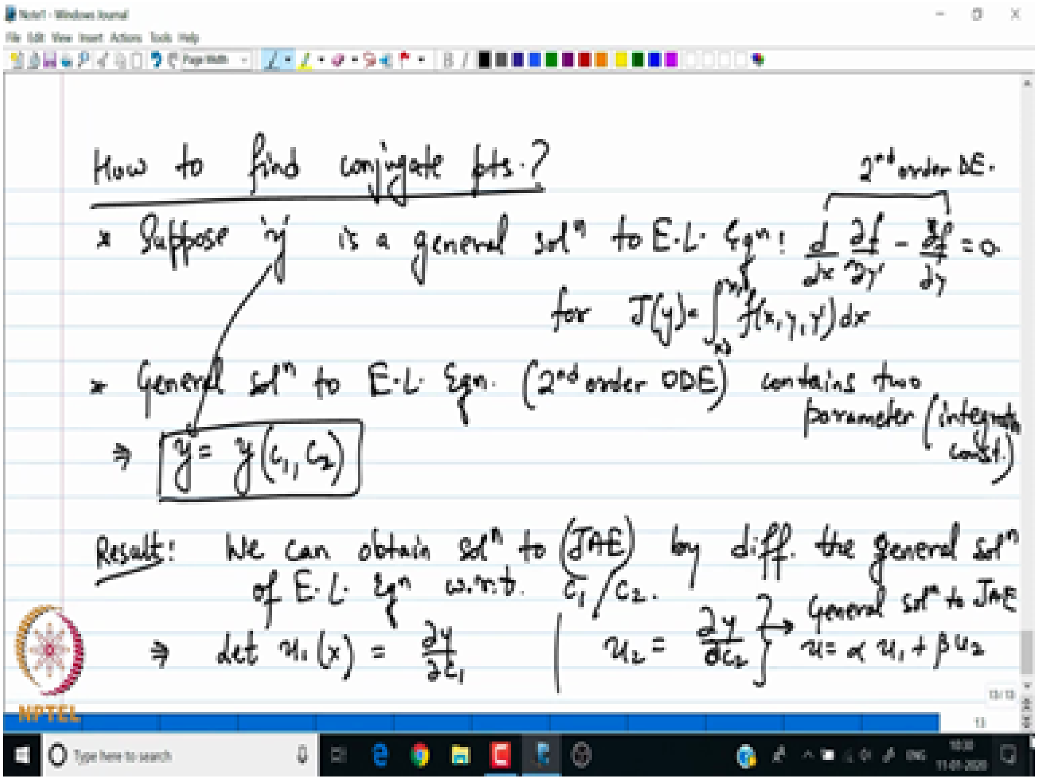

Thus question now reduces to how to find the conjugate points. We start with solution y being an extremal or the solution to the Euler Lagrange equation.Suppose 'y' is a general solution to Euler Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$ for the functional $J(y) = \int_{x_0}^{x_1} f(x, y, y')\, dx$. I know that the Euler Lagrange equation in general is a second order differential equation. So, we expect that we are going to get two linearly independent solutions in general. So, the general solution of Euler Lagrange equation contains two parameters or integration constants which means that my general solution can be written as follows: $y = y(c_1, c_2)$ .So, there is lots of arguments which students can follow from the text books that we are following. But the main result is that to find the conjugate points. All we need to do is to look at the solution to the Jacobi accessory equation and those are found by differentiating the extremal solution with respect to these constants. For different constants we are going to get different family of solutions. All we need to find is the variation of these family with respect to these constants respectively.

So, the result is as follows: We can obtain a solution to the Jacobi accessory equation by differentiating the general solution of Euler Lagrange equation with respect to c_1 and c_2. Let $u_1(x) = \frac{\partial y}{\partial c_1}$ *and* $u_2(x) = \frac{\partial y}{\partial c_2}$. Students can check that these are indeed the solution to the Jacobi accessory equation. From here I get that general solution to the Jacobi accessory equation is $u = \alpha u_1 + \beta u_2$. Now, suppose for these solutions to the Jacobi accessory equation, there are conjugate points.

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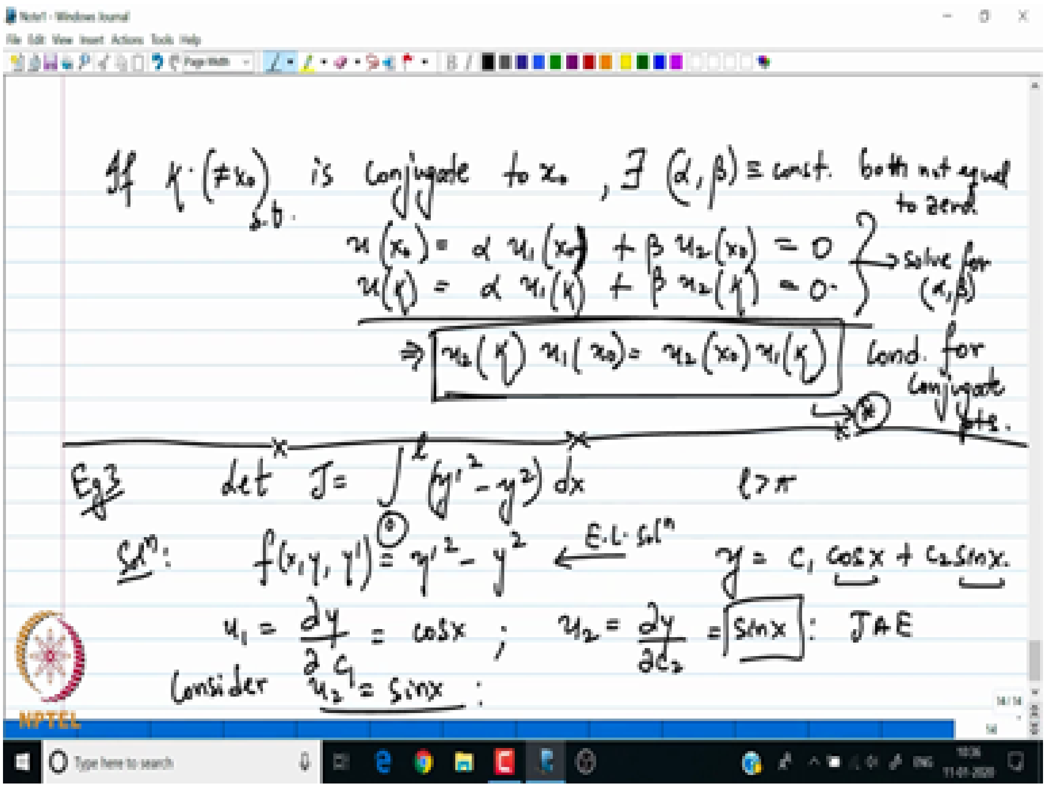

We are trying to find the condition for determining the conjugate points.Suppose if $\kappa\,(\neq x_0)$ is conjugate to x_0 then $\exists\,(\alpha, \beta) \equiv$ constants both not equal to 0 such that the solution to the JAE Jacobi accessory equation vanishes at these two points that is

$$u\,(x_0) = \alpha u_1\,(x_0) + \beta u_2\,(x_0) = 0$$
$$u\,(\kappa) = \alpha u_1\,(\kappa) + \beta u_2\,(\kappa) = 0$$

Certainly we have avoided the trivial equality by assuming that these constants are such that not both are 0. So, we have a unique solution to this problem. And the unique solution is not the trivial solution. So, we can solve for the unknowns (α, β) and we can get the following result:

$$u_2\,(\kappa)\,u_1\,(x_0) = u_2\,(x_0)\,u_1\,(\kappa) \qquad \text{(*)}$$

Now, suppose at one of the point the solution does not vanish. So, this is the condition for existence of conjugate points. Solving this will give us our conjugate points.

So, let us look at this idea with the help of an example. Let $J = \int_0^l \left(y'^2 - y^2\right) dx \quad ; \qquad l > \pi$

And for this functional, I see that $f\,(x, y, y') = y'^2 - y^2$. The Euler Lagrange solution will give me that $y = c_1 \cos x + c_2 \sin x$ .So, I can see that the function $\cos x$ and $\sin x$ they are the solution to the Jacobi accessory equation, by the discussion above that we have done. So, two solutions for the Jacobi accessory equation are following:

$$u_1 = \frac{\partial y}{\partial c_1} = \cos x \;\; ; \;\; u_2 = \frac{\partial y}{\partial c_2} = \sin x$$

Now, let us look at one of the solutions here, the $\sin x$. Note that the $\sin x$ certainly vanishes at one end point 0 and it also vanishes at another point π. Consider $u_2 = \sin x$. So, any point κ which is conjugate to 0 for u_2 will satisfy the relation $(*)$ above.Then we have the following:

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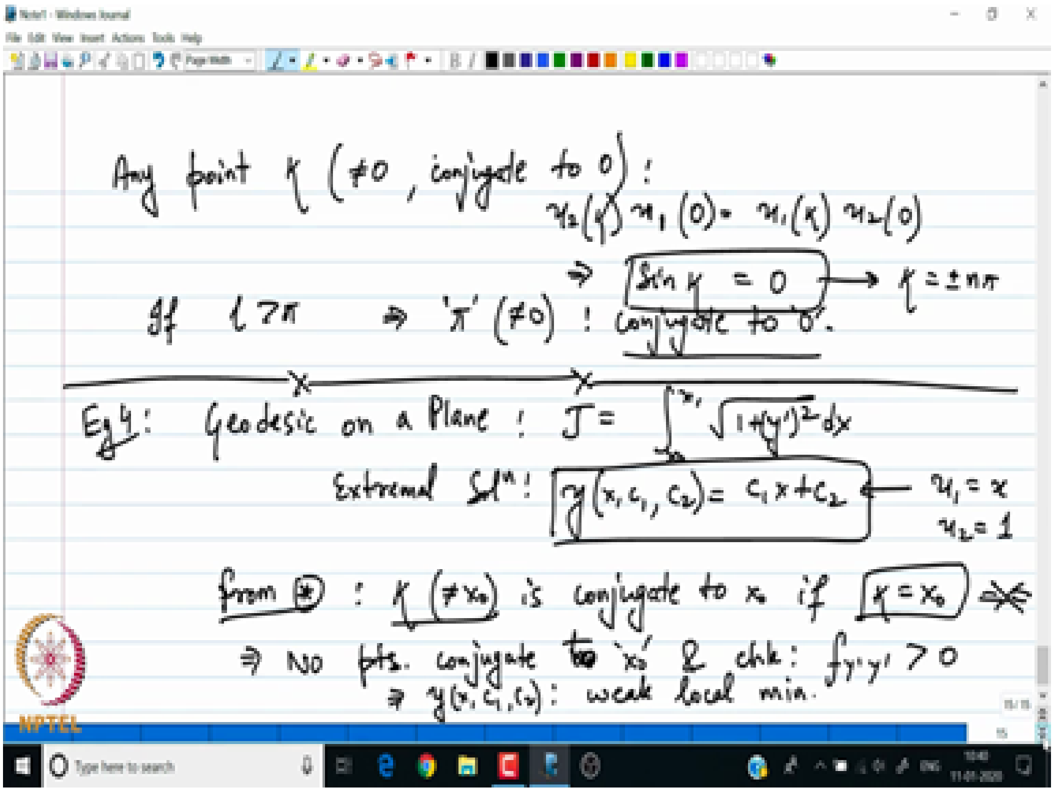

Any point $\kappa \neq 0$ but conjugate conjugate to 0 will satisfy $(*)$ condition $u_2(\kappa)\, u_1(0) = u_2(0)\, u_1(\kappa)$. From here I get $\sin\kappa = 0$ And the solution to this equation is $\kappa = \pm n\pi$. Now, if $l > \pi$ then it implies that $\pi\,(\neq 0)$ is point conjugate to 0. We will see that for the same example the moment we are able to find non-zero conjugate points, we are guaranteed that the extremal that we will get is neither the max nor the min. So, that concludes this example.

Let us also look at another example, namely the Geodesics on the plane. We know that the extremals are straight line. The functional is $J = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$ and general extremal solution is $y(x, c_1, c_2) = c_1 x + c_2$. . I know that these are straight lines and I can see that the solution to the Jacobi accessory equation here are $u_1 = x$ and u_2 is a constant, let us say $u_2 = 1$.

Again from $(*)$, $\kappa\,(\neq x_0)$ is conjugate to x_0 if I use $(*)$ condition, I will see that the conjugacy condition is$\kappa = x_0$. But that is itself a contradiction because we assumed that $\kappa \neq x_0$ which means that no points conjugate to x_0 and further check that strengthened Legendary condition $f_{y'y'} > 0$ is also satisfied which means that$y(x, c_1, c_2)$ are weak local minima. The moment we are able to find the extremal, we can determine under certain criteria we can determine whether the extremal we have got is a local minima or not. So, let us look at another example, the case of the Catenary.

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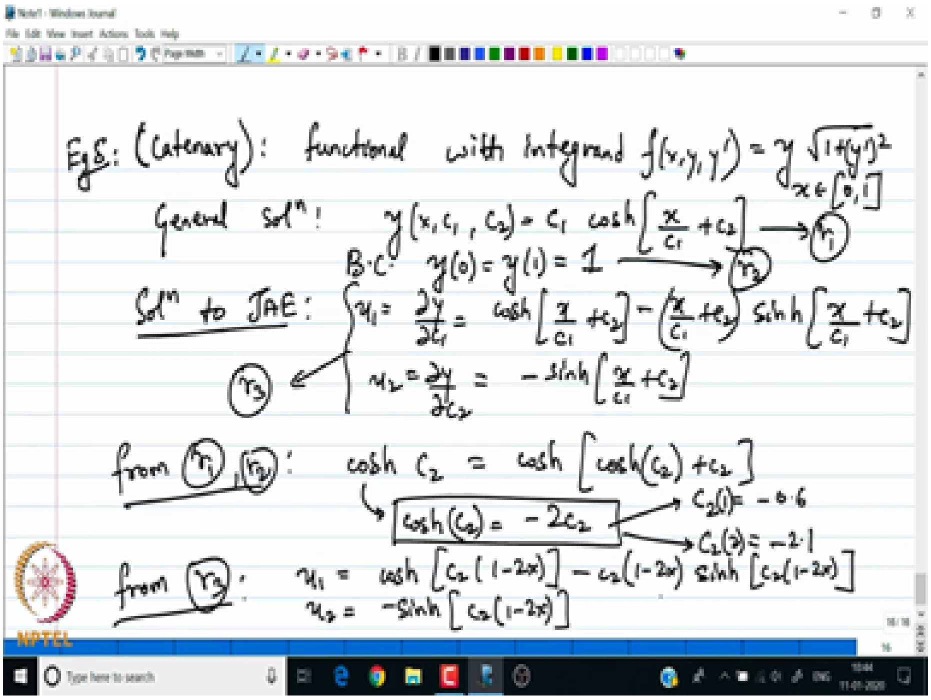

So, this is the case of the catenary. Let us consider the functional with the integrand $f(x, y, y') = y\sqrt{1+(y')^2}$ and $x \in [0,1]$. I know that the general solution to this catenary problem is :

$$y(x, c_1, c_2) = c_1 \cosh\left[\frac{x}{c_1} + c_2\right] \qquad (r_1)$$

Also we have boundary condition

$$y(0) = y(1) = 0 \qquad (r_2)$$

And the solution to the Jacobi accessory equation will be

$$u_1 = \frac{\partial y}{\partial c_1} = \cosh\left[\frac{x}{c_1} + c_2\right] - \left(\frac{x}{c_1} + c_2\right)\sinh\left[\frac{x}{c_1} + c_2\right]$$
$$u_2 = \frac{\partial y}{\partial c_2} = -\sinh\left[\frac{x}{c_1} + c_2\right] \qquad (r_3)$$

Then we can use the boundary condition to find these constants c_1 and c_2. So, from condition (r_1) and (r_2), I must have that the constants must satisfy the relation $\cosh c_2 = \cosh[\cosh(c_2) + c_2]$ or I can get

$$\cosh(c_2) = -2c_2 \qquad \textbf{(*)}.$$

I can directly solve. So, this is transcendental equation that I solved. I know that there are two solutions to this problem. One of the solution is given by $c_2(1) = -0.6$ and the other I get is $c_2(2) = -2.1$. Now, we can substitute these values of the constant into our Jacobi accessory equation and from (r_3) we have the following:

$$u_1 = \cosh[c_2(1-2x)] - c_2(1-2x)\sinh[c_2(1-2x)]$$
$$u_2 = -\sinh[c_2(1-2x)]$$

using (r_2) and $(*)$ i can see that

$$\frac{1}{c_1} = \cosh[c_2] = -2c_2 \qquad (r_4)$$

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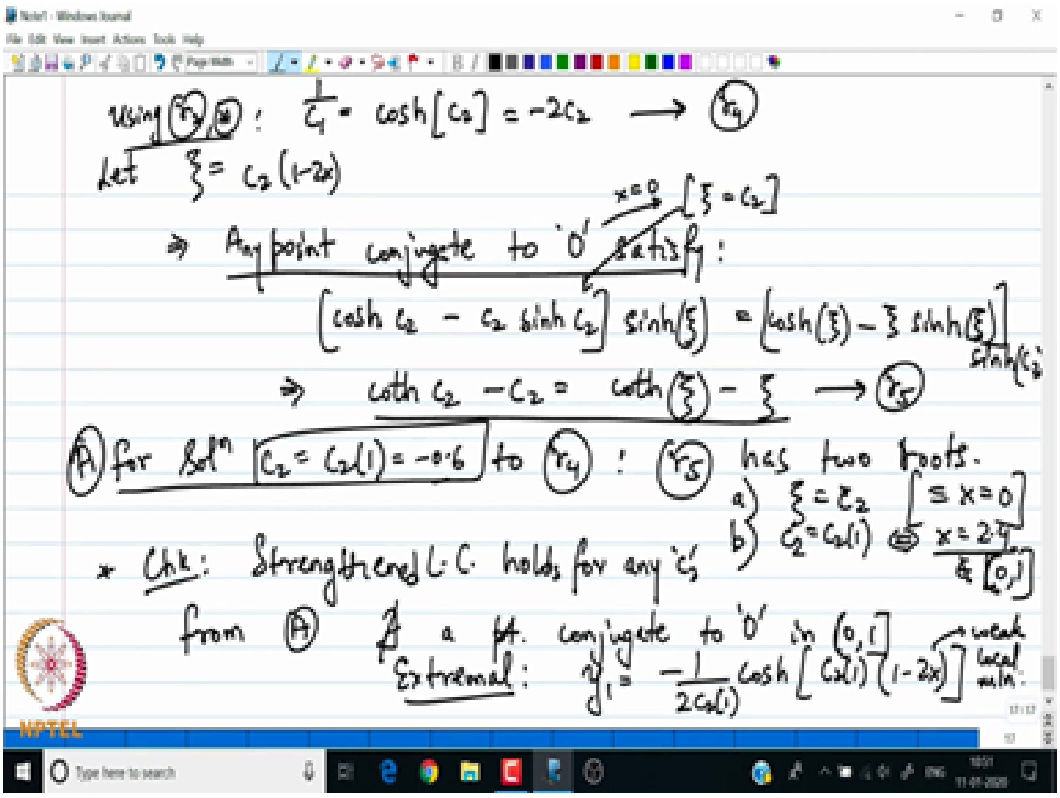

So, let $\xi = c_2\,(1-2x)$. And then it implies that any point conjugate to 0 will satisfy the following conjugacy condition

$$[\cosh c_2 - c_2 \sinh c_2] \sinh(\xi) = [\cosh(\xi) - \xi \sinh(\xi)] \sinh c_2$$

We can cross divide by $\sinh \xi$ and $\sinh c_2$. I see that the relation reduces to the following:

$$\coth c_2 - c_2 = \coth(\xi) - \xi \qquad (r_5)$$

Case (A): For solution $c_2 = c_2\,(1) = -0.6$ to (r_4) I can immediately see that (r_5) has two roots one of the root is $\xi = c2$ that is corresponding to $x = 0$.And the second root is $c_2 = c_2\,(1)$ when $x = 2.4$ but the most important part is that this does not lie in the interval $[0,1]$. we can see that there are no non-trivial roots which are conjugate to point 0 for the first case.Also students can check that the Strengthened Legendary condition holds for any constant c_2.

From case (A), $\nexists$ a point conjugate to 0 in $(0,1)$. The only point conjugate is outside the interval which means that the the extremal $y_1 = -\frac{1}{2c_2(1)} \cosh[c_2\,(1)\,(1-2x)]$ is a weak local minimum. We can do a similar exercise for the second case.

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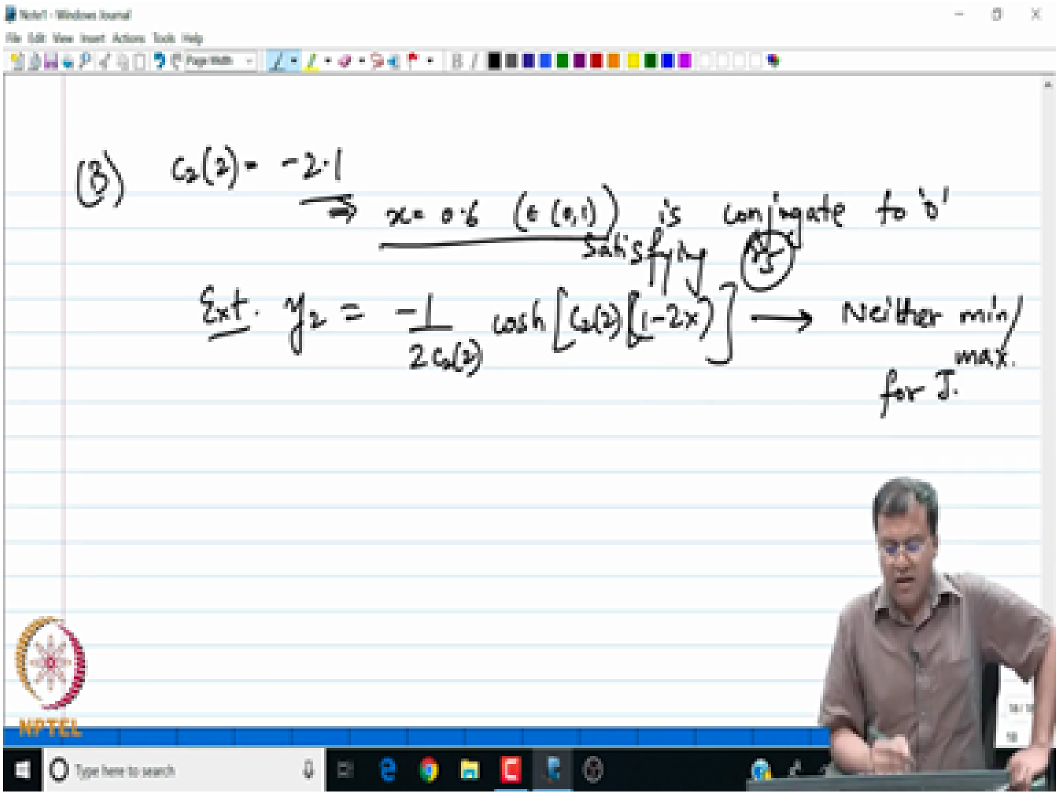

Case (B): For solution $c_2 = c_2\,(2) = -2.1$ to (r_4) we see that this leads to that $x = 0.6$ which lies in the interval$(0, 1)$ is conjugate to 0 satisfying condition (r_5).So, we have found a point which is conjugate to 0 and the conclusion is that the extremal $y_2 = -\frac{1}{2c_2(2)} \cosh\left[c_2\,(2)\,(1-2x)\right]$ is neither a minima nor maxima for J.

So, that concludes our discussion on the determination of the nature of the extrema using the existence and non-existence of conjugate points. In next topic, I am going to look at what is the geometric interpretation of conjugate points. And further we are going to look at a specific class of functional which contains convex integrands. And towards the later half of the next lecture, we will diversify all out theory into the development of the problems in optimal control theory.