Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 50
Conjugate Points / Jacobi Accessory Equations / Introduction to Optimal Control Theory
Part 2

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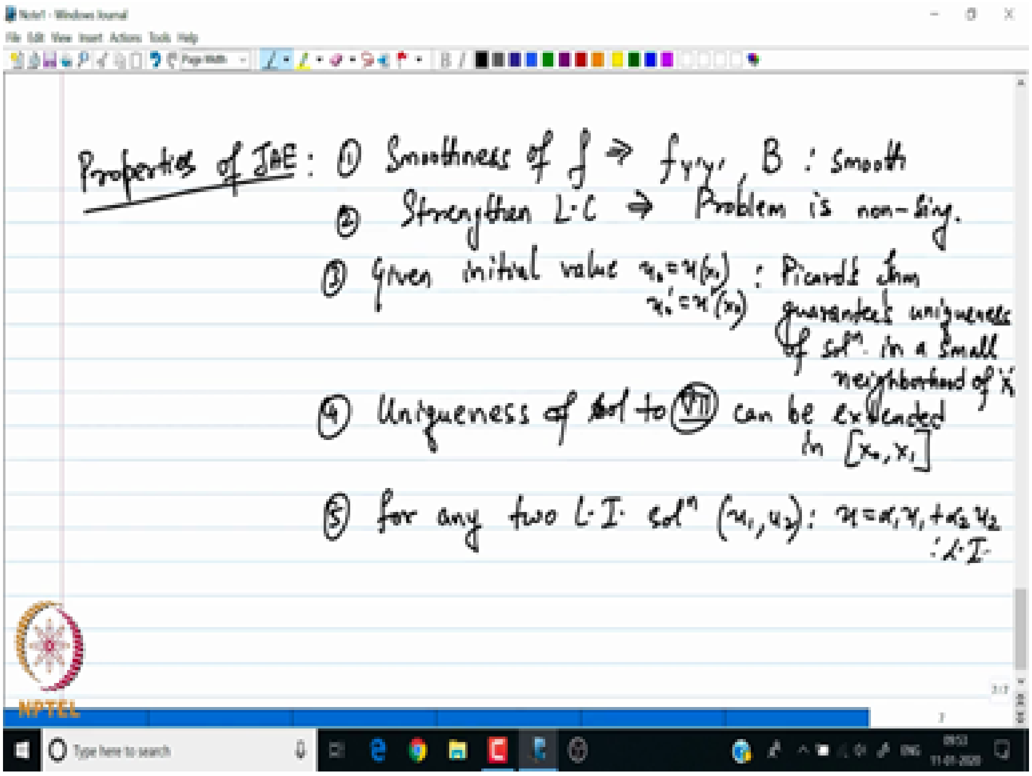

Next, I am going to introduce a very important concept known as the concept of conjugate points. It turns out that the solution to this Jacobi Accessory equation contains conjugate points, we will neither get a minima or a maxima. So, the extrema that we get will neither be a minima or a maxima if we are to find some conjugate points. And later on, we will see that the non-existence of these conjugate points will give us the necessary as well as the sufficient condition for the existence of minima. And the same result holds for maxima with the integrand taken with a minus sign. So,let us see what are these conjugate points.

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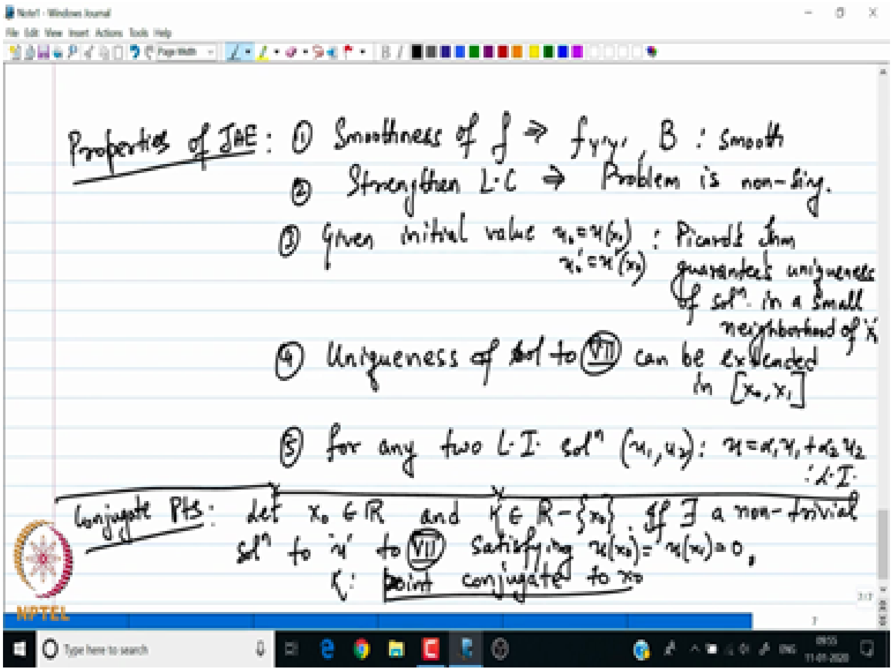

So,conjugate points are the roots to the solution to the Jacobi Accessory equation, such that they are the roots of the same solution to the Jacobi Accessory equation, such that these points are not equal to each other. So, what I said is the following. Let $x_0 \in \mathbb{R}$ and $\kappa \in \mathbb{R} - x_0$ .If there exists a non-trivial solution $'u'$ to equation (7) satisfying $u\left(x_0\right) = u\left(x_1\right) = 0$ then I see that κ is a point which is conjugate to x_0. So that is the definition. First of all, this new point κ should not be identically equal to x_0. And both must be the roots of the same solution to the Jacobi Accessory equation.

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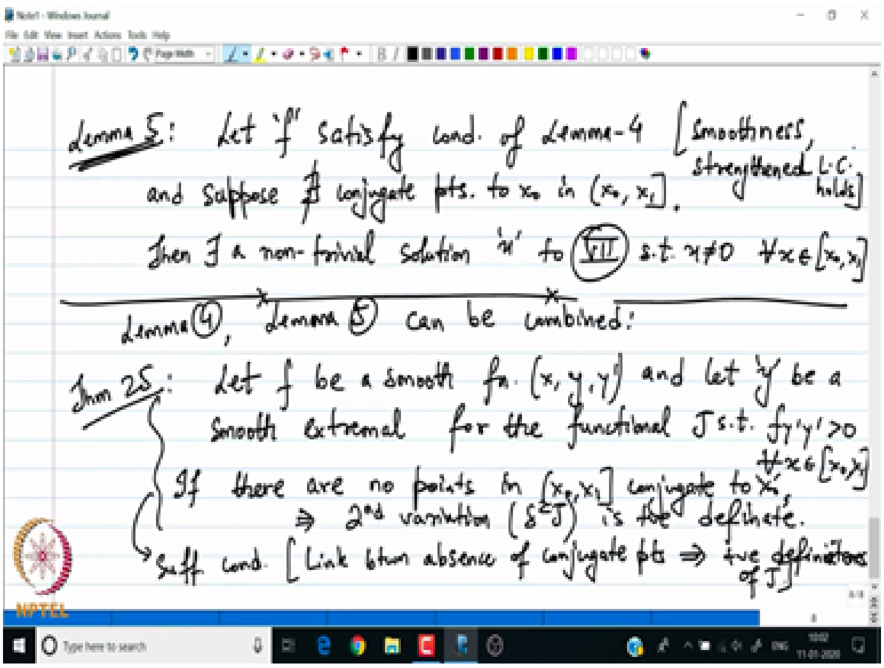

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I am going to highlight the importance of this conjugate points step by step starting from some small results, let us say in the form of a lemma. Let me term it as lemma 5. Let f satisfy conditions of my previous lemma, lemma 4, which is mainly that f is smoothness and the strengthened Legendre condition holds and suppose $\nexists$ conjugate points to x_0 in $(x_0, x_1]$ then $\exists$ a non-trivial solution $'u'$ to (7) such that $u \neq 0 \; \forall \; x \in [x_0, x_1]$. So, if we do not find conjugate points, it is guarantee that there is a non-trivial solution to the Jacobi accessory equation.

So then, so I can summarize results in lemma 5 and lemma 4. So again recall, lemma 4 is suppose Legendre condition holds then the secondary variation is positive definite.And then lemma 5 says that, suppose Legendre condition holds then there is a non-trivial solution to the Jacobi Accessory equation. Or in other words, we can find the link between the non-trivial solution to the Jacobi Accessory equation and the second variation being positive definite which is going to give the sufficient condition between the occurrence of conjugate points leading to the positive definiteness of the second variation. I am going to write this combined result in the form of a theorem, theorem 25 which says that let f be a smooth function of (x, y, y') and Let $'y'$ be a smooth extremal for the functional J such that $f_{y'y'} > 0 \; \forall \; x \in [x_0, x_1]$.If there are no points in $(x_0, x_1]$ conjugate to point x_0 then the second variation $\left(\delta^2 J\right)$ is positive definite. So, this is jus the combined result of the two lemmas we have just discussed.

So, as I said this result provides us with a sufficient condition or the link between the non-existence of conjugate points and the positive definiteness of the second variation. However, we still have to find the necessary condition, that is the other way round linking the positive definiteness to the non-existence of the conjugate points and also further we have to find the if and only if condition between the second variation being positive definite and $'y'$ the extremal being minimum.

So, this theorem provides us with the sufficient condition or it is the link between the absence of the conjugate points and positive definiteness of J.Then let me highlight this result with a small example.

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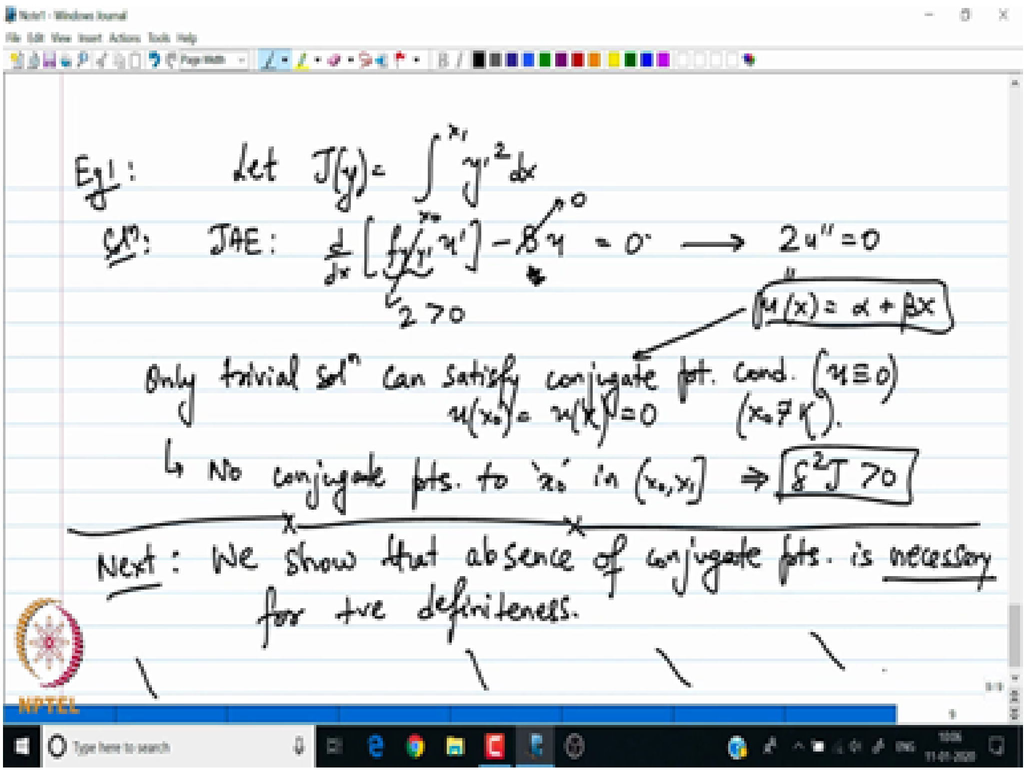

So, the example that I have is as follows. Let

$$J(y) = \int_{x_0}^{x_1} y'^2 dx$$

So Jacobi Accessory equation in this case will be

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f_{y'y'}u'\right] - Bu = 0$$

In this case $B = 0$ and $f_{y'y'} = 2$. Students can check this.

So, from here I see that the equation reduces to the following :

$$2u'' = 0 \Rightarrow u(x) = \alpha + \beta x \quad .$$

And from here I can see that only trivial solution can satisfy conjugate point condition given by $u(x_0) = u(\kappa) = 0$ for $\kappa \neq x_0$

So, this is only possible when $u \equiv 0$. But by our theorem, we cannot accept a trivial solution. We can also see that $f_{y'y'} = 2 > 0$. So, in this functional the strengthened Legendre condition is also satisfied. The integrand of the functional is smooth and also the Jacobi accessory equation tells us that there are no conjugate points.

The conclusion is that the second variation is positive definite. So, no conjugate points to x_0 in $(x_0, x_1]$ implies that $\delta^2 J > 0$. So that is how we are going to use our result described by theorem number 25. As I just said that the result that we have described in that previous theorem is all about the sufficient condition linking the non existence of conjugate points to the second variation being positive definiteness.

Next, we show that the absence of conjugate points is also a necessary condition for positive definiteness.

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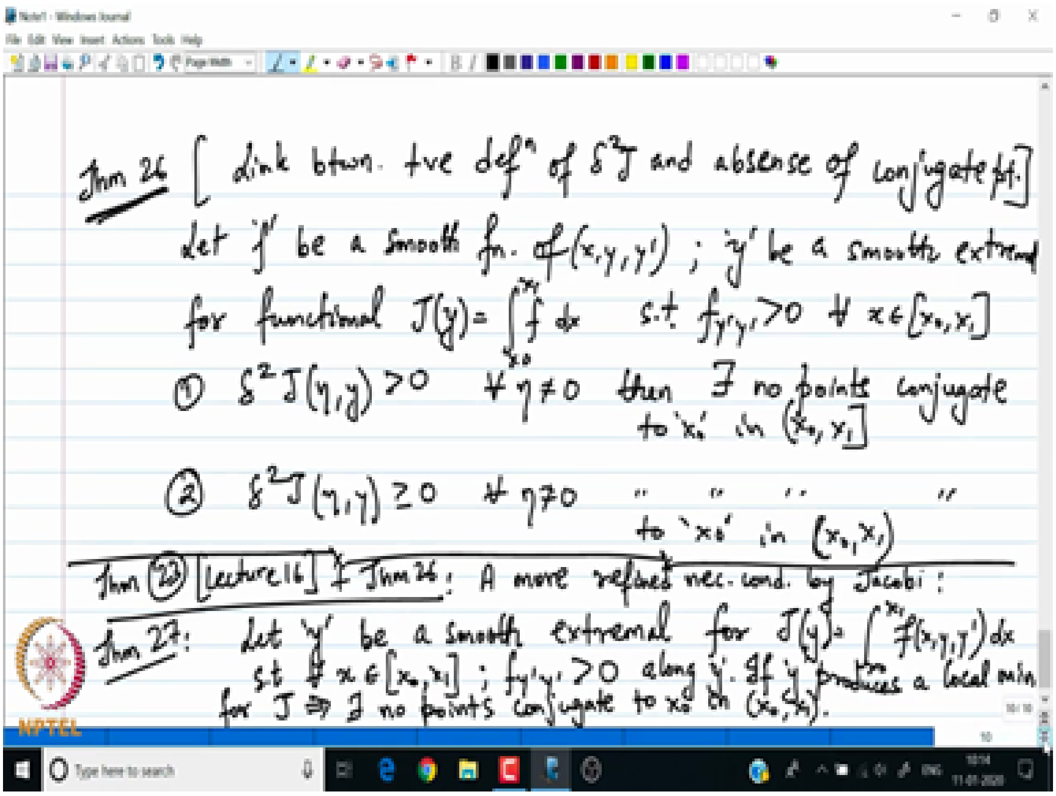

Next, we have a theorem ,i denote this theorem as theorem 26. So, this is going to provide us with a link between the positive definiteness of the second variation $\delta^2 J$ and the absence of conjugate points. The result is as follows:

Let f be a smooth function of (x, y, y') and y be a smooth extremal for the functional $J(y) = \int_{x_0}^{x_1} f dx$ such that the strengthened Legendre condition $f_{y'y'} > 0$ holds $\forall x \in [x_0, x_1]$. Then I have the following two results. The first result says that $\delta^2 J(\eta, y) > 0 \ \forall \ \eta \neq 0$ then there are no points conjugate to x_0 in $(x_0, x_1]$. And the second result says that $\delta^2 J(\eta, y) \geq 0 \ \forall \ \eta \neq 0$ then there are no points conjugate to x_0 in (x_0, x_1).

This result gives us the necessary condition or the link between the positive definiteness of the second variation to the non-existence of the conjugate points. But now we need another result which links or provides us the necessary condition as well as the extremal being the minimum to the second variation being positive definite. And that will complete the cycle.

Now, the result that I am going to write again in the form of a theorem will be the combination of theorem 26 and a previous theorem that we had described in my last lecture, namely the link between the positive definiteness of the second variation and y being the local minima that was theorem 23. So, theorem 23 and theorem 26 gives us the following result:

This is a more refined necessary condition by Jacobi. I denote it by next theorem, theorem 27. Let y be a smooth extremal for the functional $J(y) = \int_{x_0}^{x_1} f(x, y, y') dx$ such that $\forall x \in [x_0, x_1]$, $f_{y'y'} > 0$ along y. And if y produces a local minima for J,then there exists no points conjugate x_0 in (x_0, x_1) So, the necessary condition for a minimum is the non-existence of conjugate points, also known as the necessary conditions by Jacobi.

So, the non-existence of conjugate points is a necessary and sufficient condition for certainly positive definiteness of the second variation, but how about is it a necessary condition? We are certainly shown that the non-existence is a necessary condition for the existence of minima. But how about the non-existence of conjugate points being a sufficient condition for minima? To finally state that result, we have to define norm we are talking about.

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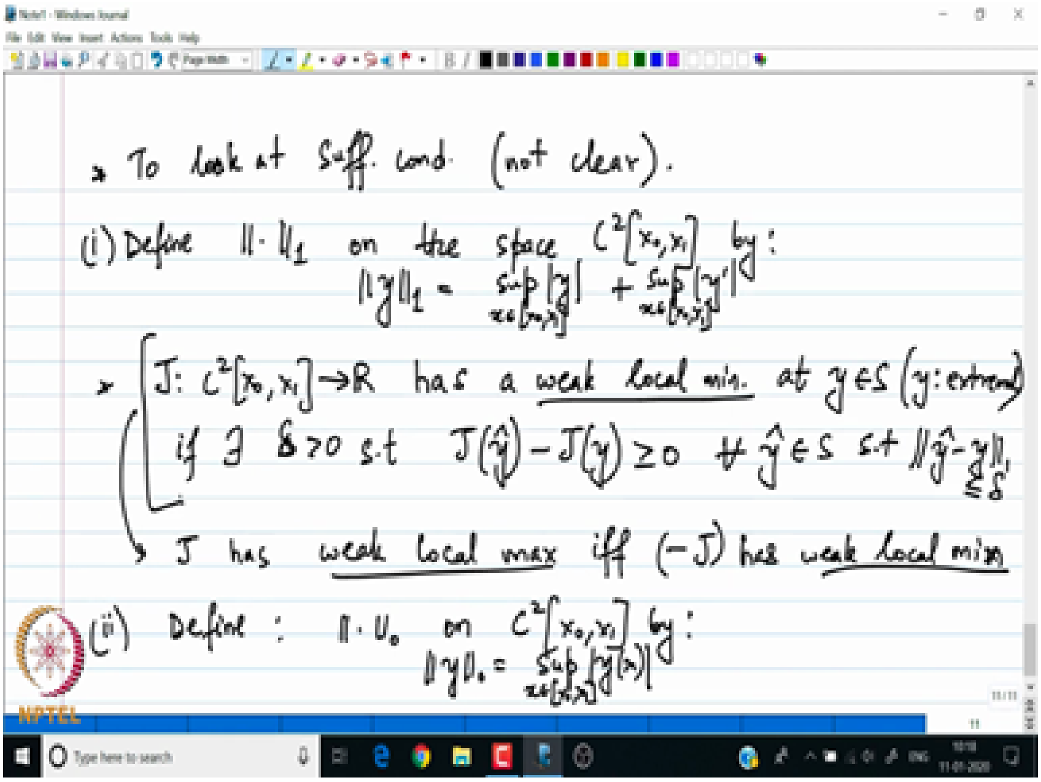

Let us define the function norm. Define $\|.\|_1$ on the space $\mathbb{C}^2[x_0, x_1]$ by the following:

$$\|y\|_1 = \sup_{x\in[x_0,x_1]} |y| + \sup_{x\in[x_0,x_1]} |y'|$$

I am going to talk about the concept of weak minima. Suppose $J : \mathbb{C}[x_0, x_1] \to \mathbb{R}$ has a weak local minima at $y \in S$, y is an extremal if there exists a $\delta > 0$ such that $J(\hat{y}) - J(y) \geq 0 \ \forall \ \hat{y} \in S$ such that $\|\hat{y} - y\|_1 < \delta$. So, if that is true, then I say that J has a weak local minima. Similarly I can say that J has a weak local maxima iff $(-J)$ has weak local minima. So, I don't bother about local maxima at all because the results with the minus sign will give me the results for the local maximum.I also describe another norm, the 0 norm as follows: $\|.\|_0$ on the space $\mathbb{C}^2[x_0, x_1]$ by the following:

$$\|y\|_0 = \sup_{x\in[x_0,x_1]} |y(x)|$$

So certainly, if the function is bounded above by the 1 norm, then it will certainly bounded above by the 0 norm. So, with the 0 norm, I can define another concept called the strong maxima or strong minima.

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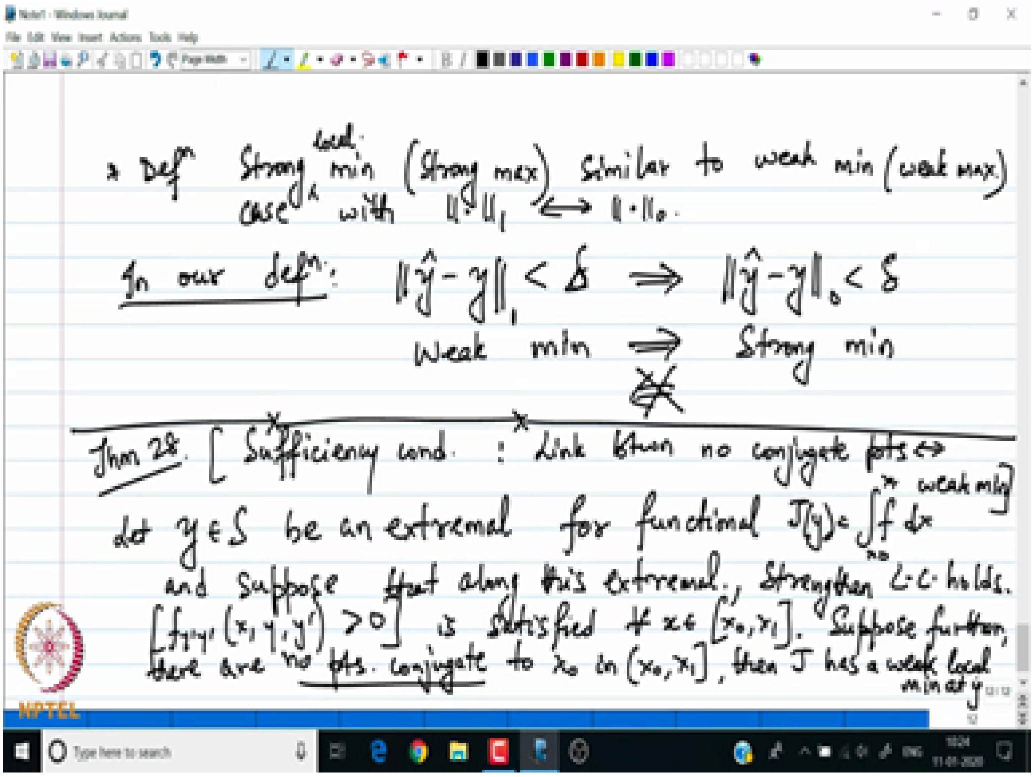

So, I can define, I can define my concept of strong local minima or even strong maxima similar to the weak local minima or the weak maxima case with my 1 norm replaced by 0 norm.So, in the same definition as in the previous slide, if we replace our 1 norm with the 0 norm, I am going to get the result for the existence of a strong local minima or local maxima. Certainly if within the definition, if the function satisfies the weakness condition then it will certainly satisfy the strong condition.In definition, $\|\hat{y} - y\|_1 < \delta \Rightarrow \|\hat{y} - y\|_0 < \delta$. So weak minima implies strong minima but the vice versa does not hold.

So, we are ready to give important result of the sufficiency of the local minima. Once we have described these norms, I am going to state the result, Theorem 28, namely the sufficiency condition which is the link between no conjugate pointsand weak minima. And that will complete our picture of if and only if criteria for non-existence of conjugate points linking to the existence of y being a local minimum. So, the result is as follows:

Let $y \in S$ be an extremal for functional $J(y) = \int_{x_0}^{x_1} f dx$ and suppose that along this extremal, the strengthened Legendre condition holds or $f_{y'y'}(x, y, y') > 0$ is satisfied $\forall x \in [x_0, x_1]$.

Suppose further there are no points conjugate to $x_0 \in (x_0, x_1]$ then J has a weak local minimum at extremal y. So, that is the result which completes our discussion or the link between the non-existence of conjugate point and the existence of local minimum.