Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture – 47

Noether's Theorem, Introduction to Second Variation Part 5

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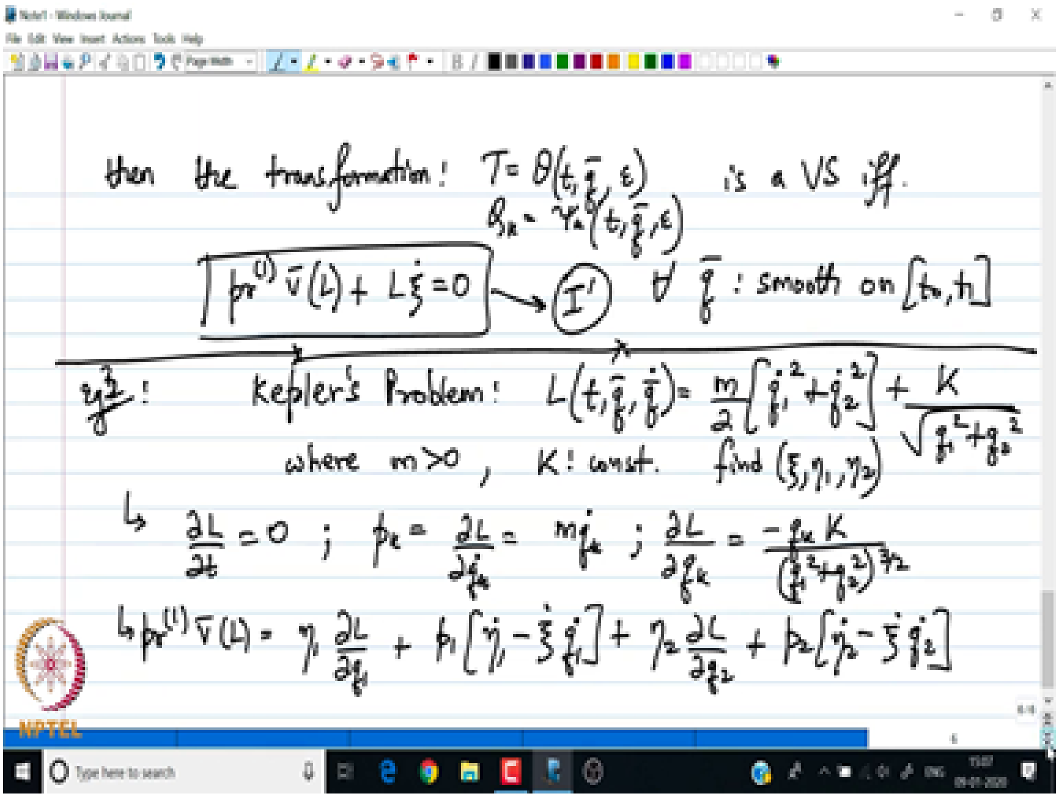

So, the example is number 3 here. So the example I have is that of the Lagrangian for the Kepler's problem. So for the Kepler's problem, my Lagrangian is as follows:

$$L\left(t, \bar{q}, \dot{\bar{q}}\right) = \frac{m}{2}\left[\dot{q_1}^2 + \dot{q_2}^2\right] + \frac{K}{\sqrt{q_1^2+q_2^2}}$$

where m is positive and k is a constant.

So, I have to set up my condition $(1')$ to find the infinitesimal generators. So, we have to find (ξ, η_1, η_2) infinitesimal generators for the variational symmetry. So again, to set up the prolongation operator, let us calculate some of these partial derivatives. So ,

$$\frac{\partial L}{\partial t} = 0 \quad ; \quad p_k = \frac{\partial L}{\partial \dot{q_k}} = m\dot{q}_k \quad ; \quad \frac{\partial L}{\partial q_k} = -\frac{q_k k}{\left(q_1^2+q_2^2\right)^{\frac{3}{2}}}$$

So, prolongation operator in this case is

$$pr^1\bar{v}\left(L\right) = \eta_1\frac{\partial L}{\partial q_1} + p_1\left[\dot{\eta_1} - \dot{\xi}\dot{q_1}\right] + \eta_2\frac{\partial L}{\partial q_2} + p_2\left[\dot{\eta_2} - \dot{\xi}\dot{q_2}\right]$$

So plug in all the quantities and write down the expression.

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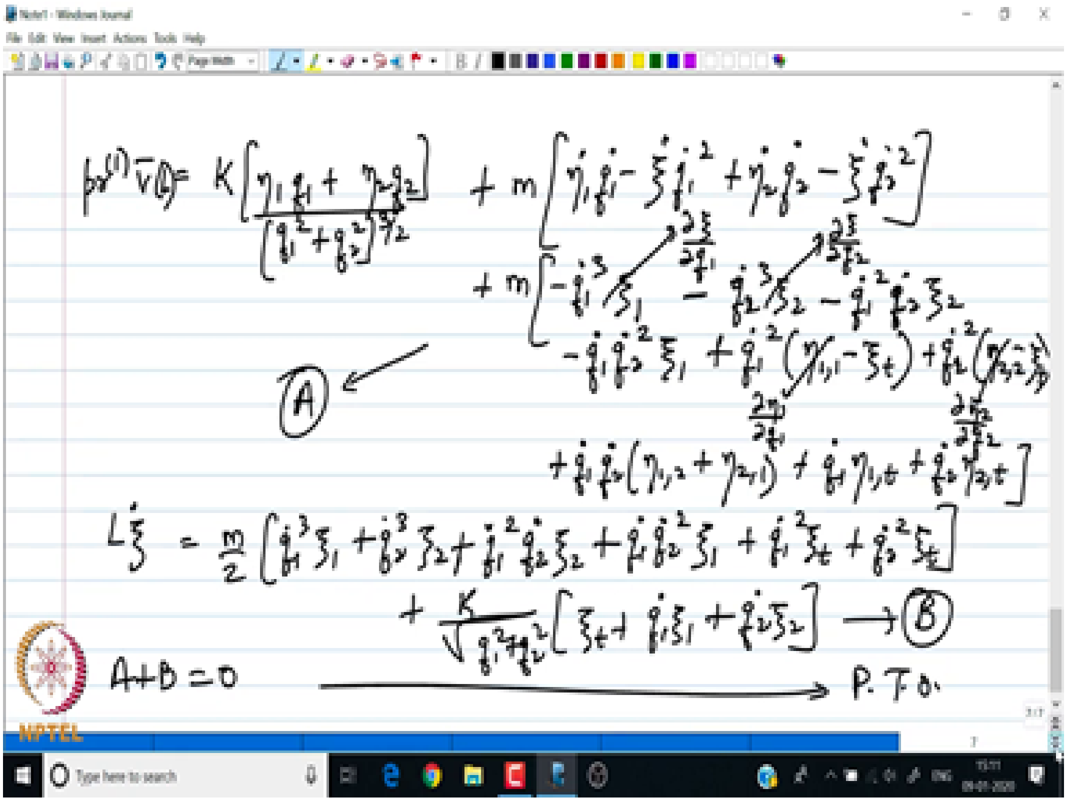

So, after plugging in all the quantities, I get the following expression:

$$pr^1\bar{v}\,(L) = k\frac{\eta_1 q_1+\eta_2 q_2}{\left(q_1^2+q_2^2\right)^{\frac{3}{2}}} + m\left[\dot{\eta}_1\dot{q}_1 - \dot{\xi}\dot{q_1}^2 + \dot{\eta}_2\dot{q}_2 - \dot{\xi}\dot{q_2}^2\right]$$

$$+m\left[-\dot{q_1}^3\xi_1 - \dot{q_2}^3\xi_2 - \dot{q_1}^2\dot{q}_2\xi_2 - \dot{q}_1\dot{q_2}^2\xi_1 + \dot{q_1}^2\left(\eta_{1,1}-\xi_t\right) + \dot{q_2}^2\left(\eta_{2,2}-\xi_t\right) + \dot{q}_1\dot{q}_2\left(\eta_{1,2}+\eta_{2,1}\right) + \dot{q}_1\eta_{1,t} + \dot{q}_2\eta_{2,t}\right] \quad \textbf{(A)}$$

Also we need to check

$$L\dot{\xi} = \frac{m}{2}\left[\dot{q_1}^3\xi_1 + \dot{q_2}^3\xi_2 + \dot{q_1}^2\dot{q}_2\xi_2 + \dot{q}_1\dot{q_2}^2\xi_1 + \dot{q_1}^2\xi_t + \dot{q_2}^2\xi_t\right] + \frac{k}{\sqrt{q_1^2+q_2^2}}\left[\xi_t + \dot{q}_1\xi_1 + \dot{q}_2\xi_2\right] \quad \textbf{(B)}$$

So I have to add A and B and equate the various powers of $\dot{q}_1$ *and* $\dot{q}_2$. I am going to get the following relations.

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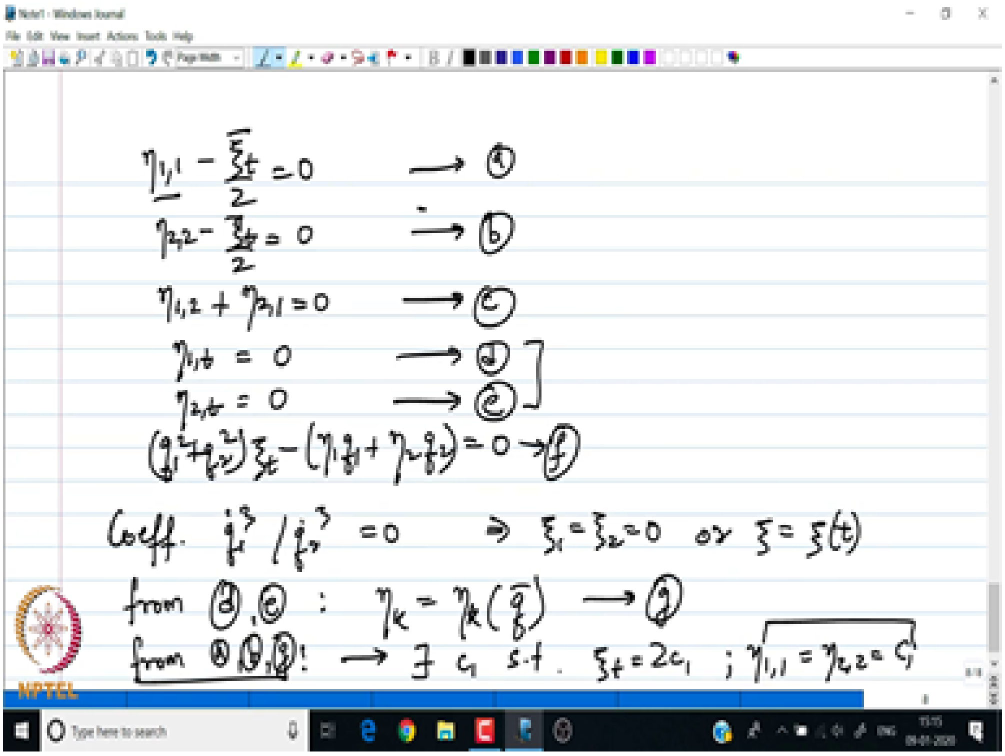

$$\eta_{1,1} - \frac{\xi_t}{2} = 0 \qquad \textbf{(a)}$$

$$\eta_{2,2} - \frac{\xi_t}{2} = 0 \qquad \textbf{(b)}$$

$$\eta_{1,2} + \eta_{2,1} = 0 \qquad \textbf{(c)}$$

$$\eta_{1,t} = 0 \qquad \textbf{(d)}$$

$$\eta_{2,t} = 0 \qquad \textbf{(e)}$$

$$\left(q_1^2 + q_2^2\right)\xi_t - \left(\eta_1 q_1 + \eta_2 q_2\right) = 0 \qquad \textbf{(f)}$$

From A, we see that, coefficients of ${\dot{q}_1}^3/{\dot{q}_2}^3$are 0 and what I get is $\xi_1 = \xi_2 = 0$ or I get that $\xi = \xi\left(t\right)$ is purely a function of t.

From,(d) and (e), I see that

$$\eta_k = \eta_k\left(\bar{q}\right) \qquad \textbf{(g)}$$

From (a) ,(b) and (g) we get there exixt a constant c_1 such that

$$\xi_t = 2c_1 \;\; ; \;\; \eta_{1,1} = \eta_{2,2} = c_1$$

So, from here we can conclude several things. First of all ξ must be a straight line. And we can continue, but most important notice this relation.

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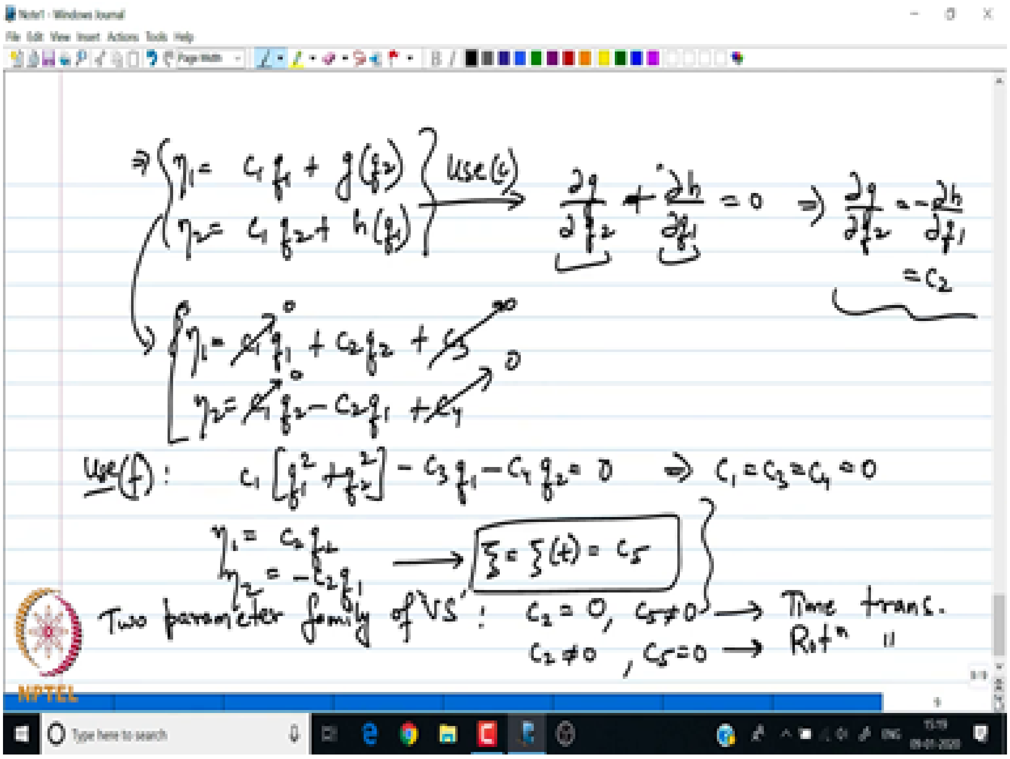

So, from that relation I can immediately get that

$$\eta_1 = c_1 q_1 + g\left(q_2\right) \;\; ; \;\; \eta_2 = c_1 q_2 + h\left(q_1\right)$$

use relation (c) , I see that

$$\frac{\partial g}{\partial q_2} + \frac{\partial h}{\partial q_1} = 0$$

one is a function of q2, the other is a function of q1 and both equated to 0, implies both are constants implies

$$\frac{\partial g}{\partial q_2} = -\frac{\partial h}{\partial q_1} = c_2$$

So, I can integrate this relation. And from here I get that

$$\eta_1 = c_1 q_1 + c_2 q_2 + c_3$$

$$\eta_2 = c_1 q_2 - c_2 q_1 + c_4$$

Finally, if I use my condition (f), I plug in the last relation and I get that

$$c_1\left[q_1^2 + q_2^2\right] - c_3 q_1 - c_4 q_2 = 0.$$

And if I were to equate this, I get that

$$c_1 = c_3 = c_4 = 0$$

So all I get is that is

$$\eta_1 = c_2 q_2 \;\; ; \;\; \eta_2 = -c_2 q_1$$

And finally, from all these observations, we can also get that ξ which was originally we found to be a function of t, comes out to be purely a constant say c_5 that is $\xi = \xi\left(t\right) = c_5$. So that comes from these set of relations that we have been using. So, students should check that. So what is the conclusion here? Notice that the way the infinitesimal generators are found, this is nothing but the rotational generators.

So η_1 depends on q_2, η_2 depends on q_1, provided constants c_1, c_2 are non-zero. So the conclusion is, we have two parameter family of variational symmetry. So, what have we got is that if $c_2 = 0$ and $c_5 \neq 0$ I can only get a time translational symmetry, there is no rotation. So this is time translation or translation in t. Now, on the other end if $c_2 \neq 0$ and $c_5 = 0$ then I have a pure rotational transformation. So, what have we found? That, in the Kepler's problem, the only variational symmetries, are linear combination of rotation and time translation.

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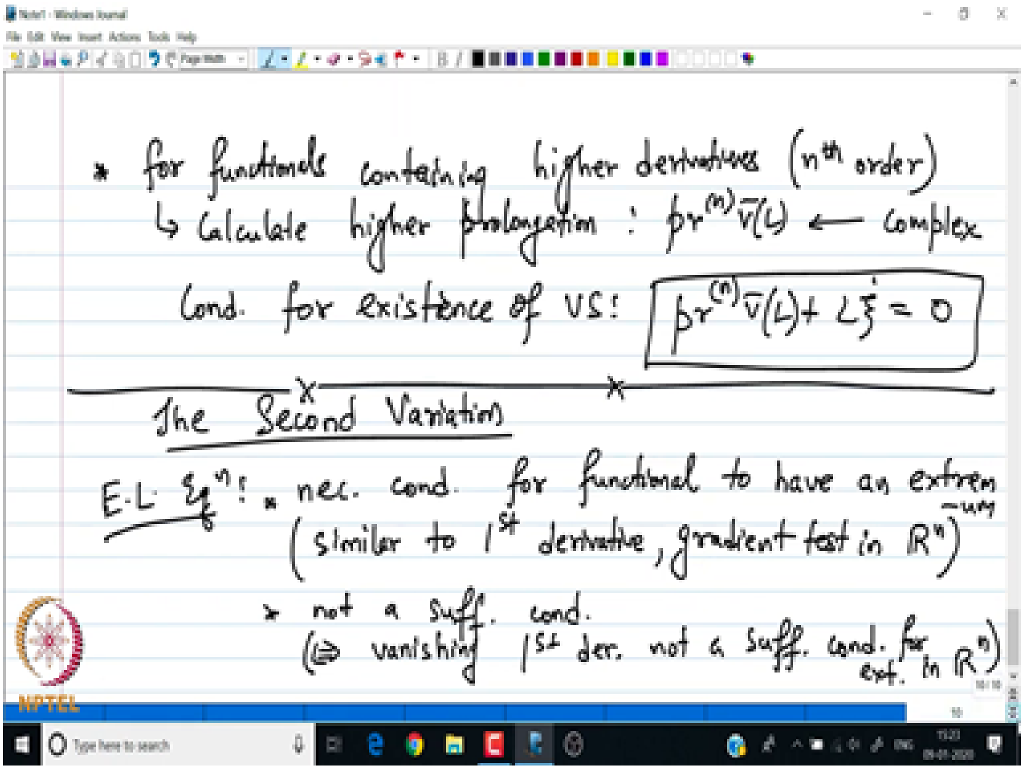

So we end our discussion by concluding that we can continue finding these variational symmetries for functionals containing integrand of higher order derivatives in a similar fashion. So let me write down the result. For functional containing higher derivatives let us say up to nth order we calculate higher prolongation.

We call this as $pr^n \bar{v}\,(L)$. As we increase this n, as n becomes larger and larger, this becomes quite complicated to find. So, I am not going to write the general form, but again, I will end the discussion by mentioning what is a condition for existence of variational symmetries is $pr^n \bar{v}\,(L) + L\dot{\xi} = 0$ differential equation needs to be satisfied.

So that is along the similar lines for functions with just first order derivatives. So, that completes our discussion on finding the necessary condition for the extrema of a functional. I am now going to shift my attention to calculate the second variation or to look at the sufficient condition for the existence of the extrema, namely, if the extrema that we found, whether it is a maxima or a minima. So we start our discussion on the second variation.

So,the idea is why is second variation topic is important, because so far we have looked at the various forms of Euler-Lagrange. But Euler-Lagrange only provides the necessary condition for the existence of the extrema. It does not provide the sufficient condition. So, it only provides a necessary condition for the functional to have an extremum. And this is similar to first derivative, gradient test in infinite dimensional space $\mathbb{R}^n$.

It is not a sufficient condition and specially to determine the nature of the extrema that we find. So this statement is equivalent to saying that a vanishing the first derivative in the finite dimensional calculus set equal to 0 is not a sufficient condition for the extrema. So, vanishing first derivative not a sufficient condition for the extrema in $\mathbb{R}^n$. So the moment the Euler-Lagrange does not provide us with the sufficient condition, the natural idea is to look for the higher order terms in our variation of the functional.

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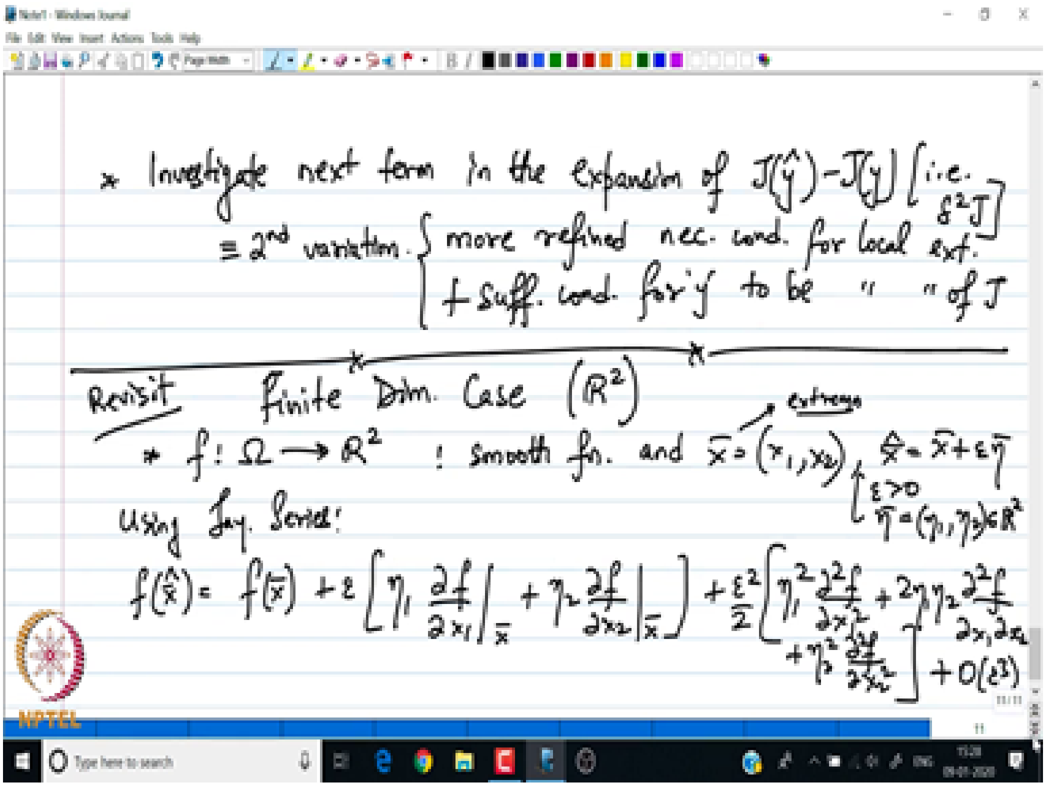

So we are going to look at the terms of the order of $\partial^2 J$. So the idea is we need to investigate next term in the expansion of $J\left(\hat{y}\right) - J\left(y\right)$ the variation, which will be our second variation of the functional. What is the importance of the second variation? Because we will investigating the second variation will not only provide us with a more refined necessary condition for the existence of extrema, it will also provide us with the sufficient condition for specific cases.

So, it provides us with more refined necessary condition for local extrema and it also provides the sufficient condition for y to be, to be a local extrema of J. So, all these statements that I am making for the functional, I am just writing down the equivalent statement in the finite dimensional calculus, so as if, our multivariate calculus is the guiding topic in investigating the second variation. So let us now briefly touch upon the topics in multivariate calculus related to the second derivative test.

So, I am going to revisit our concepts in finite dimensional calculus, related to the second derivative tests. So we will revisit finite dimensional case in$\mathbb{R}^2$. Let us consider a function $f : \Omega \to \mathbb{R}^2$ which is a smooth function and let $\bar{x} = (x_1, x_2)$. Let us say x is a extrema

Let us consider the perturbation in x ,$\hat{\bar{x}} = \bar{x} + \epsilon\bar{\eta}$ where $\epsilon > 0$ and $\bar{\eta} = (\eta_1, \eta_2) \in \mathbb{R}^2$. So, I can expand my smooth function using Taylor series about some known point $\bar{x} = (x_1, x_2)$. Using Taylor series, I see that

$$f\left(\hat{\bar{x}}\right) = f\left(\bar{x}\right) + \epsilon\left[\eta_1 \frac{\partial f}{\partial x_1}\Big|_{\bar{x}} + \eta_2 \frac{\partial f}{\partial x_2}\Big|_{\bar{x}}\right] + \frac{\epsilon^2}{2}\left[\eta_1^2 \frac{\partial^2 f}{\partial x_1^2} + 2\eta_1\eta_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + \eta_2^2 \frac{\partial^2 f}{\partial x_2^2}\right] + O\left(\epsilon^3\right)$$

So let me write down this expansion in a shorthand notation, assuming that $\bar{x} = (x_1, x_2)$ is an extrema.

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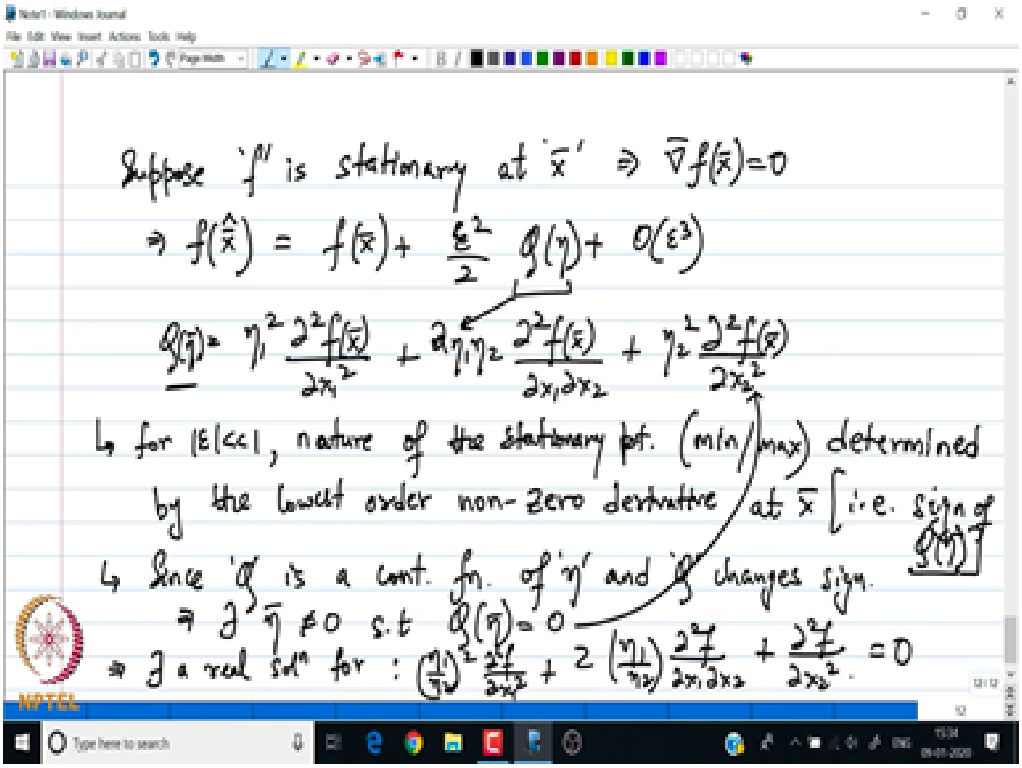

Suppose f is stationary or it has an extrema at the point $\bar{x} = (x_1, x_2)$. So , it means that $\bar{\nabla} f(\bar{x}) = 0$ or term in my Taylor series is set equal to 0 because this is nothing but $< \bar{\eta}, \bar{\nabla} f >$. When $\bar{\nabla} f = 0$, this term will be 0.

So, in that case

$$f(\hat{\bar{x}}) = f(\bar{x}) + \tfrac{\epsilon^2}{2} Q(\eta) + O(\epsilon^3)$$

$$where \qquad Q(\eta) = \eta_1^2 \tfrac{\partial^2 f(\bar{x})}{\partial x_1^2} + 2\eta_1\eta_2 \tfrac{\partial^2 f(\bar{x})}{\partial x_1 \partial x_2} + \eta_2^2 \tfrac{\partial^2 f(\bar{x})}{\partial x_2^2}$$

So we see that for $\epsilon << 1$, the nature of the extrema will be completely determined by the sign of function $Q(\eta)$. So what I said is, for $\epsilon << 1$, the nature of the stationary point say max or min, will be completely determined. The student should note that we are in finite dimensional calculus, whatever results we are saying, we are saying it without proof, because the assumption is, the students who are taking this course, have a background in vector calculus.

So they are requested to refer standard textbooks and we will see that all these results we state in vector calculus will have almost parallel in functional calculus. So the , max or min is determined by the sign by the lowest order non-zero derivative, at $\bar{x} = (x_1, x_2)$ [that is, that is the sign of $Q(\eta)$. So, it is all about finding the sign of Q and how does it change?

We need to check where the Q sign changes. Since Q is a function which is a continuous function of this vectorη and Q changes sign, which means that there will be a value of η where $Q(\eta) = 0$. So those are the points that we need to evaluate.

So what I said is, since Q is a continuous function of η and Q changes sign it implies that there exists a value or there exists a function $\eta \neq 0$ such that $Q(\eta) = 0$ which means there exists a real solution for

$$\left(\frac{\eta_1}{\eta_2}\right)^2 \frac{\partial^2 f}{\partial x_1^2} + 2\left(\frac{\eta_1}{\eta_2}\right) \frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial x_2^2} = 0$$

So this is a quadratic equation for the unknown $\left(\frac{\eta_1}{\eta_2}\right)$.Then we have to worry about the discriminant and see how the discriminant changes sign from here standard arguments in quadratic equation.

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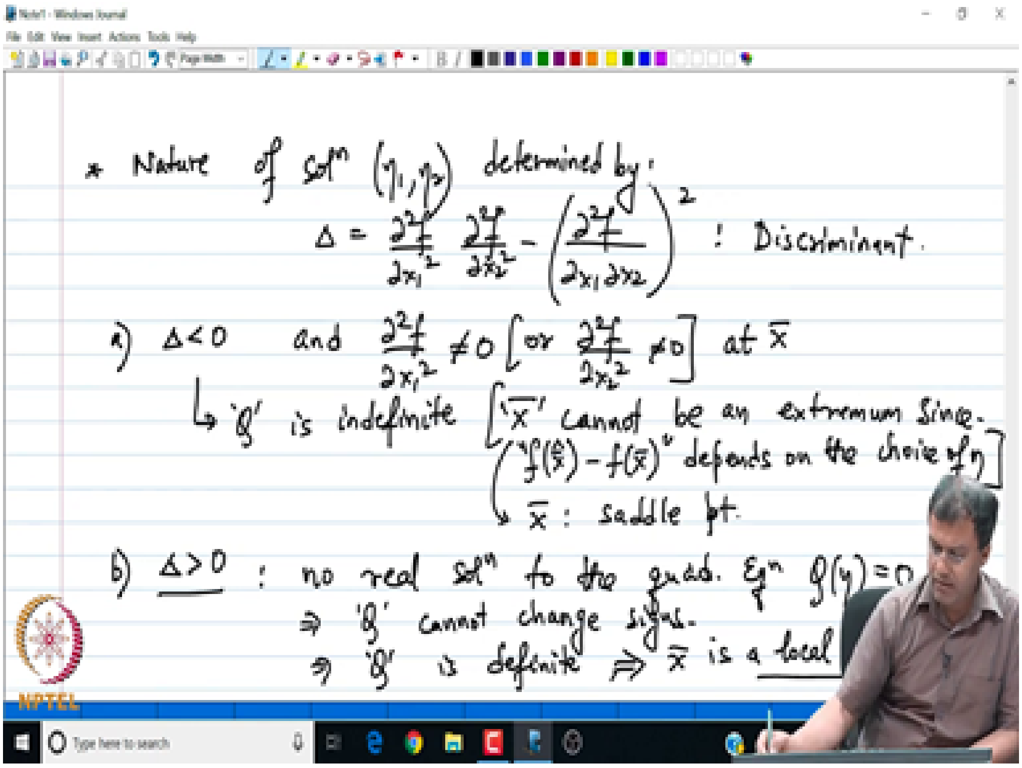

The nature of the solution (η_1, η_2) is determined by the discriminant

$$\Delta = \frac{\partial^2 f}{\partial x_1^2} \frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2$$

I call this delta to be discriminant. This is the standard quadratic equation solution theory. So the theory says if $\Delta < 0$, we will not have a real solution and, and further one of the derivatives either $\frac{\partial^2 f}{\partial x_1^2} \neq 0$ *or* $\frac{\partial^2 f}{\partial x_2^2} \neq 0$ *at* $\bar{x}$. So I see that this is a case where Q is indefinite. We would not get a real solution for (η_1, η_2) and the conclusion in the terms of the extremum is that in this situation $\bar{x}$ cannot be an extremum since $f\left(\hat{\bar{x}}\right) - f\left(\bar{x}\right)$ depends on the choice of the function η.

So, in this case $\bar{x}$ is also known as the saddle point. So then, the case that if $\Delta > 0$, we expect that there is no real solution to the quadratic equation given by $Q(\eta) = 0$.

Which means that Q cannot change signs. And in other words, I say that Q is definite and the conclusion from here is that $\bar{x}$ is a local extremum and further, we can conclude few more results from this, that this local extremum is maximum if one of the second derivatives of f with respect to either x_1 *or* x_2 are negative and it is a local minimum if those derivatives are positive.