Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 45
Noether's Theorem, Introduction to Second Variation Part 3

So now we have, the next set of discussion involves the extension of these ideas of variational symmetry to functionals having integrants with several dependent variables.

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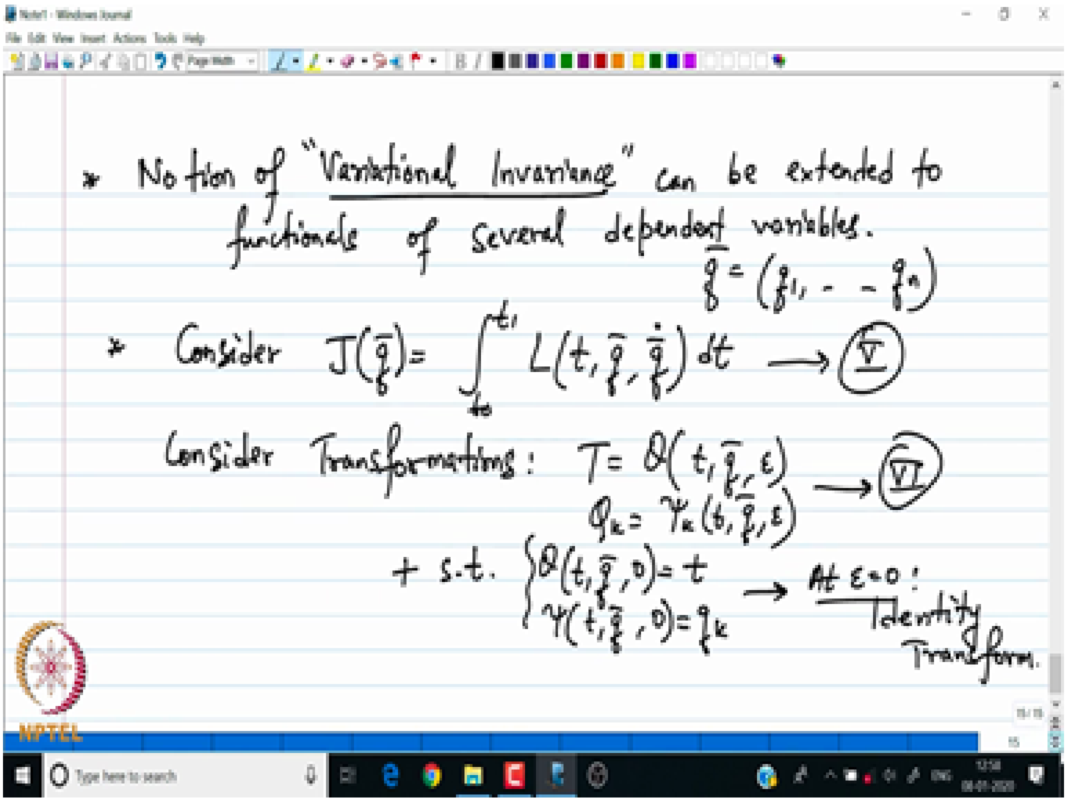

So what I said is that the notion of variational invariance can be extended to functional of several dependent variables given, let us say those variables are $\bar{q} = (q_1, q_2, ..., q_n)$. So we consider the functional

$$J(\bar{q}) = \int_{t_0}^{t_1} L(t, \bar{q}\dot{\bar{q}})\, dt \qquad \textbf{(5)}$$

We consider this general functional. Let me represent this equation by 5.

And also now we are going to transformations of this form. Consider transformation

$$T = \theta(t, \bar{q}, \epsilon)$$
$$Q_k = \psi_k(t, \bar{q}, \epsilon) \qquad \textbf{(6)}$$

So again, we need to satisfy certain consistency criterion, namely that the transformation for the parameter value $\epsilon = 0$ are nothing but the identity transformation.

We also assume the condition that

$$Q\left(t,\bar{q},0\right)=t$$
$$\psi\left(t,\bar{q},0\right)=q_k$$

So this is the value at $\epsilon=0$ I am getting back the identity transformation. So then I need to talk about the concept of variationally invariant functionals.

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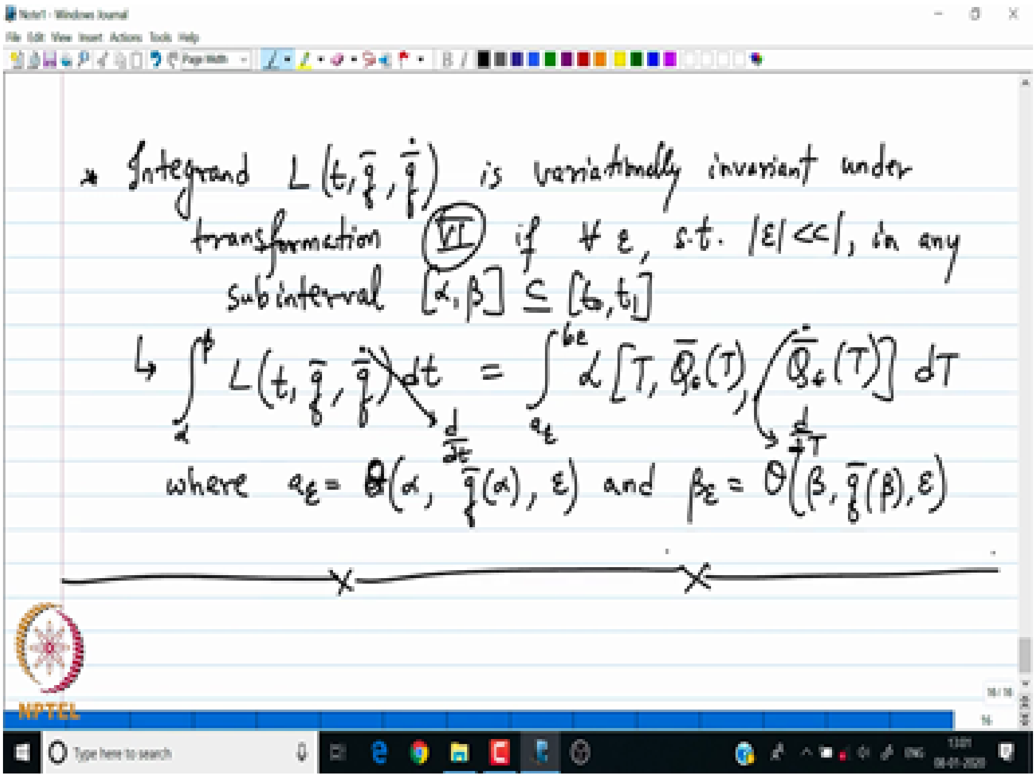

Similar to the case of one dependent variable I can extend my notion as follows:

$$L\left(\bar{q}\right)=\int_{t_0}^{t_1}L\left(t,\bar{q}\dot{\bar{q}}\right)dt$$

is variationally invariant under the transformation given by 6. Under the transformation 6 if for all ϵ such that ϵ is small in any subinterval $[\alpha,\beta]\subseteq[t_0,t_1]$ we get that

$$\int_{\alpha}^{\beta}L\left(t,\bar{q},\dot{\bar{q}}\right)dt=\int_{a_\epsilon}^{b_\epsilon}L\left(T,\bar{Q_\epsilon}(T),\dot{\bar{Q_\epsilon}}(T)\right)dT$$

So the integrant is variationally invariant if I can see that my integrant retains the same form when transforming from one coordinate to the other, where

$$a_\epsilon=\theta\left(\alpha,\bar{q}\left(\alpha\right),\epsilon\right)\quad,\quad b_\epsilon=\theta\left(\beta,\bar{q}\left(\beta\right),\epsilon\right)$$

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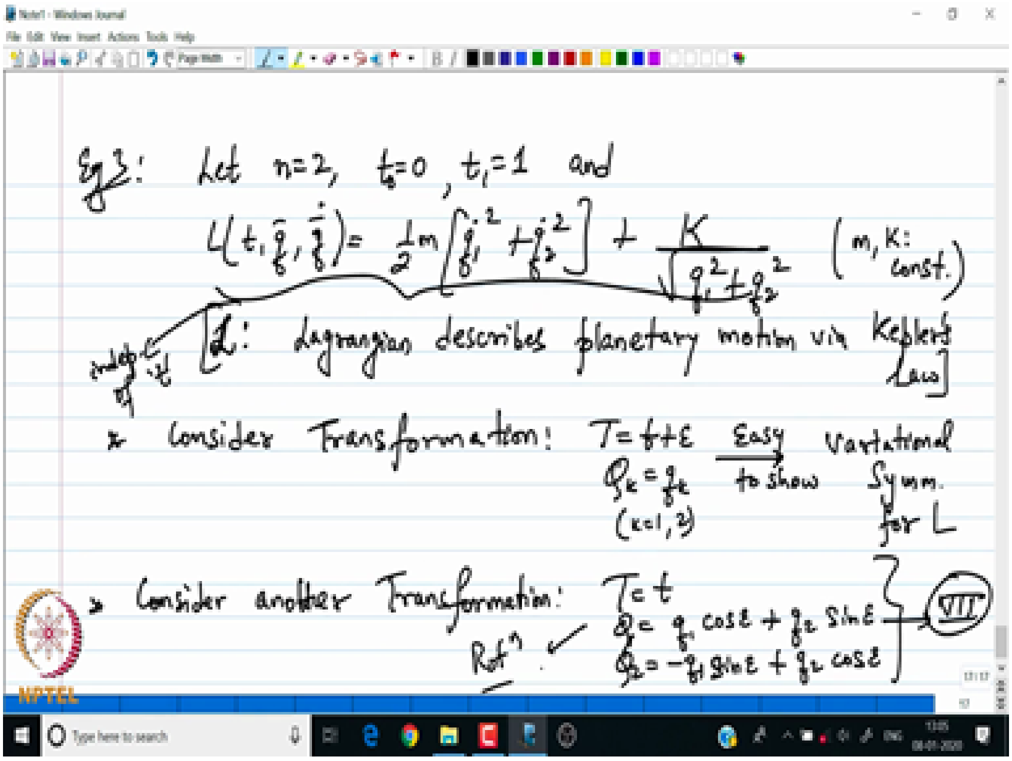

So let us look at an example for the multiply dependent variable case. See how the ideas of transformation work. So the example I have is as follows: Suppose

$$n = 2\ ,\ t_0 = 0\ ,\ t_1 = 1\ \ and$$
$$L\left(t, \bar{q}, \dot{\bar{q}}\right) = \frac{1}{2}m\left[\dot{q_1}^2 + \dot{q_2}^2\right] + \frac{K}{\sqrt{q_1^2+q_2^2}}$$

I am introducing my Lagrangian is as follows: Students who are looking at this Lagrangian will immediately recognize that this is the same example that we introduced in example 10 of our previous lecture, that is the central force problem.

And I want to highlight similar to the central force problem. Now this sort of Lagrangian frequently appears where we have to describe the planetary motion, also known as the Kepler's law. So my m and k are constants. So my Lagrangian, L here is the Lagrangian which describes my planetary motion via the Kepler's law. So let us consider a transformation to see how, what sort of transformation this Lagrangian is variationally invariant. So let us consider this transformation. The simplest one is the translational transformation. Consider the transformation of the form

$$T = t + \epsilon$$
$$Q_k = q_k$$

Notice that this Lagrangian is independent of the independent variable t. So we have shown, we have mentioned for integrants of one variable that for integrants which are independent of the independent variable will be invariant under this translational transformation. So what I am saying is that it is easy to show that this transformation is the variational symmetry for L. Student should check that by plugging in the respective variables. So where my k = 1 , 2. So we have three relations here.

So let us look at a more non-trivial case. So let us consider another transformation of the form

$$T = t$$
$$Q_1 = q_1 \cos\epsilon + q_2 \sin\epsilon$$

$$Q_2 = -q_1 \sin\epsilon + q_2 \cos\epsilon \tag{7}$$

And I see, let me call this transformation by 7 because so far we have introduced relations up to 6. So what we have done here. Is it? Yes, we have introduced relations up to 6. So 6 was my transformation, the general form. So we have called this is my rotational transformation. So let us look at what happens under this transformation. What is the form of my Lagrangian?

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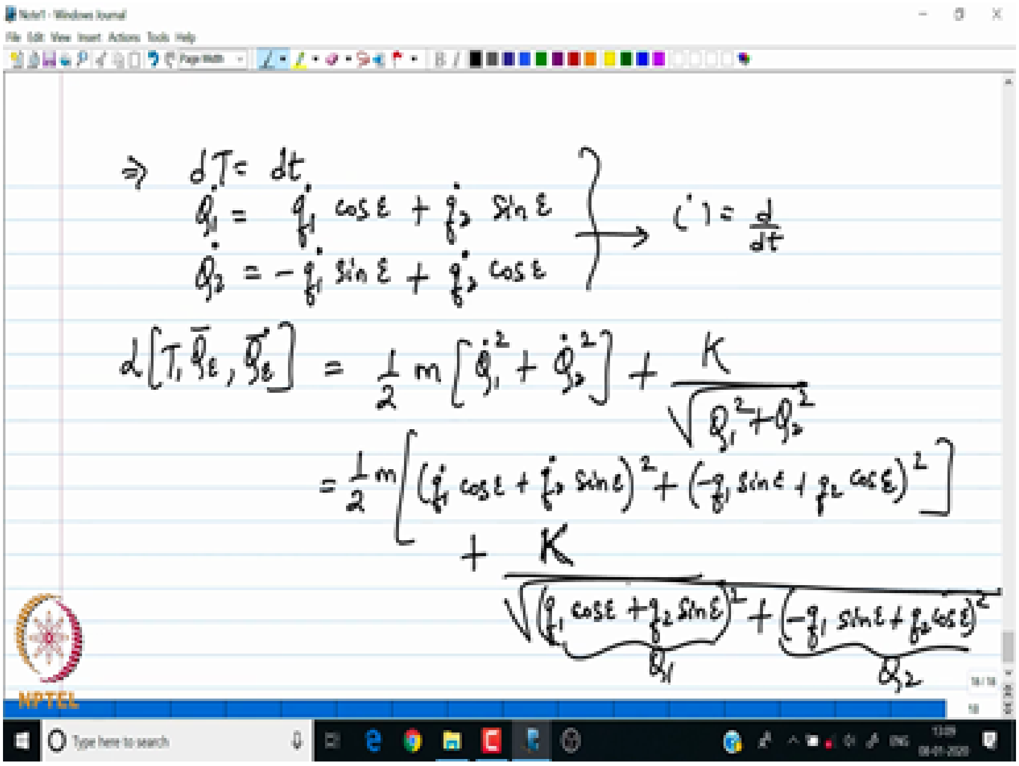

So which means for this transformation 7, I see that

$$dT = dt$$
$$\dot{Q}_1 = \dot{q}_1 \cos\epsilon + \dot{q}_2 \sin\epsilon$$
$$\dot{Q}_2 = -\dot{q}_1 \sin\epsilon + \dot{q}_2 \cos\epsilon$$

we are differentiating the variable q_1, q_2 because the parameter ϵ is fixed.

And so here it is quite clear that my dot is the derivative with respect to the original independent variable t. So then let us calculate

$$L\left[T, \bar{Q}_\epsilon, \dot{\bar{Q}}_\epsilon\right] = \frac{1}{2}m\left[\dot{Q_1}^2 + \dot{Q_2}^2\right] + \frac{k}{\sqrt{Q_1^2+Q_2^2}}$$

We plug in the values of $Q_1, Q_2, \dot{Q}_1 and\ \dot{Q}_2$

And we see that

$$L\left[T, \bar{Q}_\epsilon, \dot{\bar{Q}}_\epsilon\right] = \frac{1}{2}m\left[(\dot{q}_1\cos\epsilon + \dot{q}_2\sin\epsilon)^2 + (-\dot{q}_1\sin\epsilon + \dot{q}_2\cos\epsilon)^2\right] + \frac{k}{\sqrt{(q_1\cos\epsilon + q_2\sin\epsilon)^2 + (-q_1\sin\epsilon + q_2\cos\epsilon)^2}}$$

Note that after simplification let me write down the form of this function.

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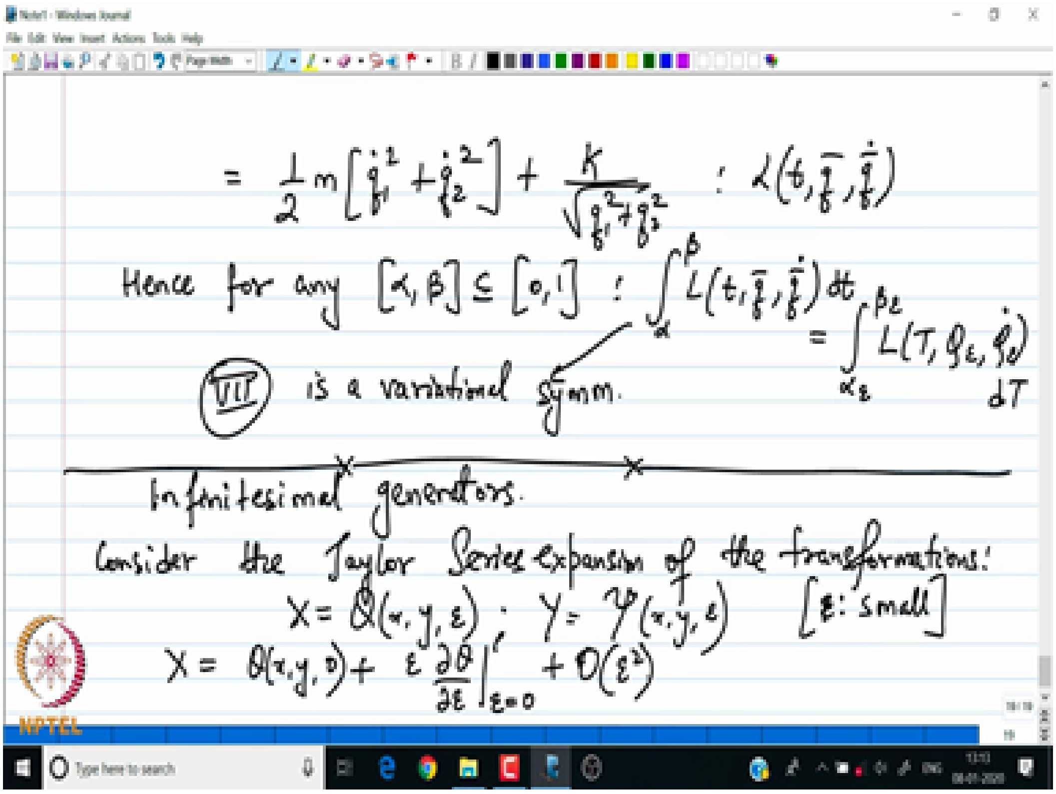

We see that I am going to get, after all the simplification the steps here are omitted. We are going to recover back our original original Lagrangian.

$$L\left(t, \bar{q}, \dot{\bar{q}}\right) = \frac{1}{2}m\left[\dot{q_1}^2 + \dot{q_2}^2\right] + \frac{K}{\sqrt{q_1^2+q_2^2}}$$

So what have we found? That, the transformation 7 which is a rotational transformation is the variational symmetry to this problem with the Lagrangian given as follows: for any subinterval, remember our original interval was from 0 to 1. For any subinterval $[\alpha, \beta]$ of the original interval $[0, 1]$ it turns out that

$$\int_{\alpha}^{\beta} L\left(t, \bar{q}, \dot{\bar{q}}\right) dt = \int_{\alpha_\epsilon}^{\beta_\epsilon} L\left(T, Q_\epsilon, \dot{Q}_\epsilon\right) dT$$

So what have we got? That we have shown this or the result is that our relation 7 is a variational symmetry. So that is what we have seen. We are almost at a point where we are ready to describe Noether's theorem but there is one more concept that I want to introduce at this point. So that is the concept of infinitesimal generators. So consider the Taylor series expansion of our transformation function θ *and* ψ. So consider the Taylor series expansion of the transformation

$$X = \theta\left(x, y, \epsilon\right) \quad , \quad Y = \psi\left(x, y, \epsilon\right)$$

my ϵ is small.

We see that we are going to do the Taylor series expansion around $\epsilon = 0$. So using Taylor series

$$X = \theta\left(x, y, 0\right) + \epsilon\left[\frac{\partial\theta}{\partial\epsilon}\right]_{\epsilon=0} + O\left(\epsilon^2\right)$$

Similarly, we are trying to represent our transformations up to linear order, we are ignoring higher order. When I talk about higher order, I am talking about higher order with respect to the parameter ϵ . So I am trying to ignore the effects of the higher order terms with respect to ϵ because ϵ is close to 0, it is small. So similarly, we can write the other expression.

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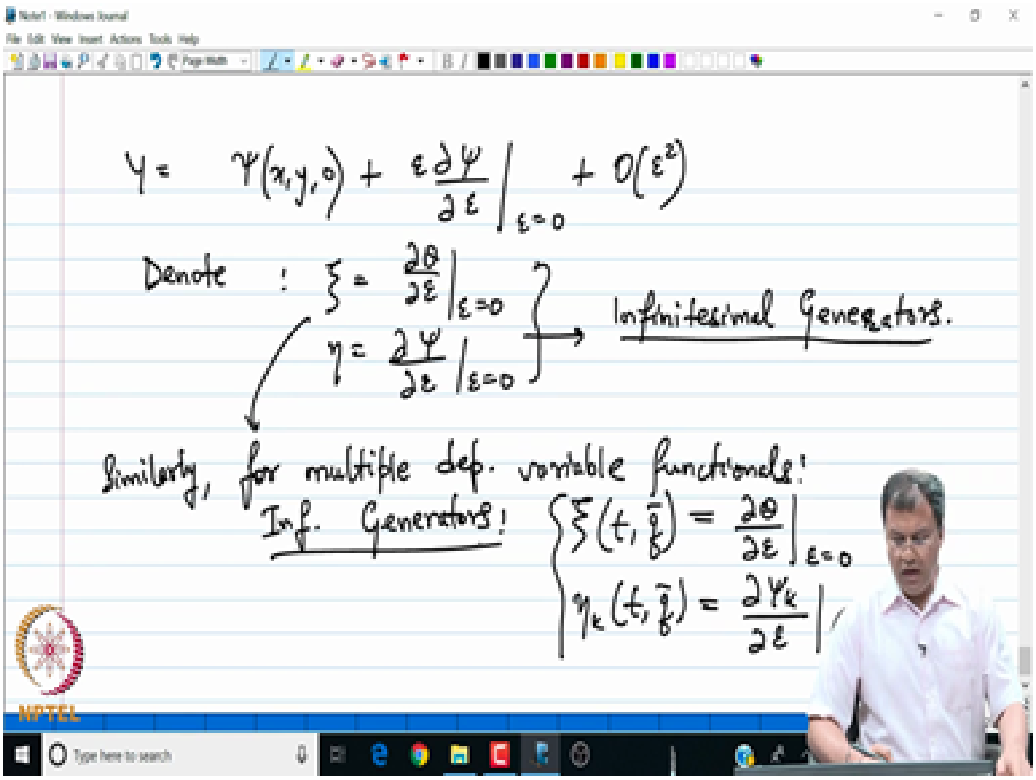

$$Y = \psi\left(x, y, 0\right) + \epsilon \left[\frac{\partial \psi}{\partial \epsilon}\right]_{\epsilon=0} + O\left(\epsilon^2\right)$$

we are differentiating with respect to the unknown parameter ϵ. We also denote

$$\xi = \left[\frac{\partial \theta}{\partial \epsilon}\right]_{\epsilon=0} \quad , \quad \eta = \left[\frac{\partial \psi}{\partial \epsilon}\right]_{\epsilon=0}$$

From here I get the infinitesimal generators.

So from here we can extend the concept of infinitesimal generator to functions of several dependent variables. So similarly for multiple dependent variables functionals I see that my infinitesimal generators are

$$\xi\left(t, \bar{q}\right) = \left[\frac{\partial \theta}{\partial \epsilon}\right]_{\epsilon=0}$$

$$\eta_k\left(t, \bar{q}\right) = \left[\frac{\partial \psi_k}{\partial \epsilon}\right]_{\epsilon=0}$$

So these are my infinitesimal generates in the general case with multiply dependent variables. So let us now state Noether's theorem so in the simplest possible form.

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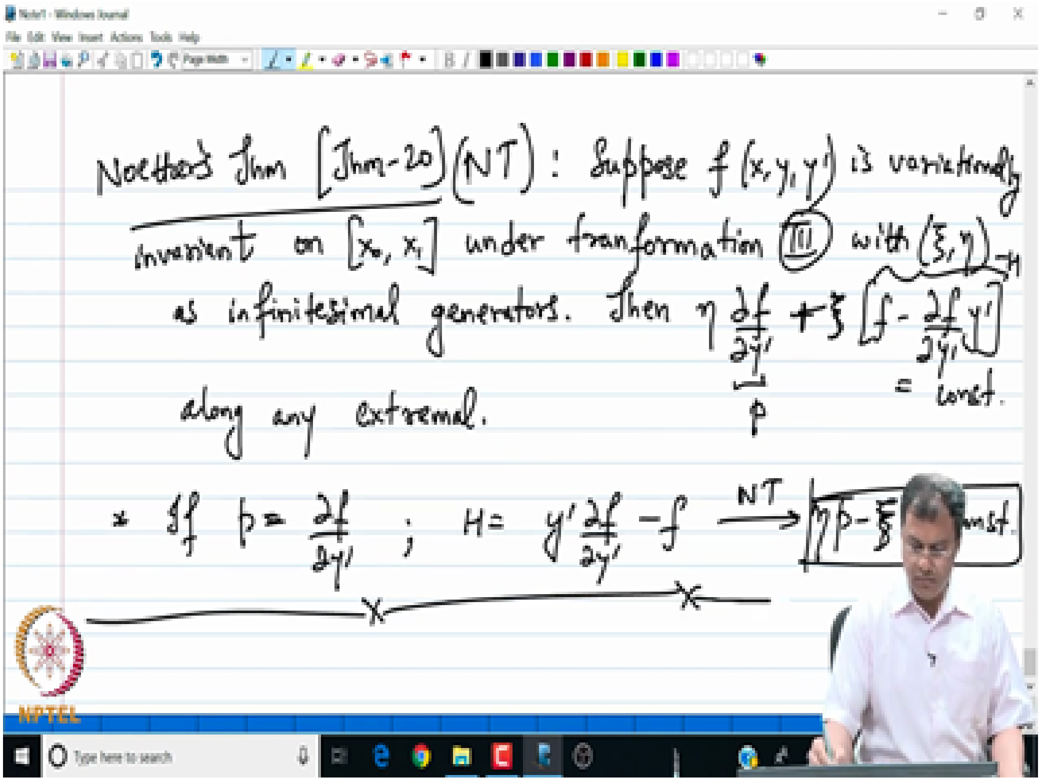

So Noether's theorem, I denoted by theorem 20 in the way we are numbering our theorems. I am going to denote it by the theorem NT in future reference. So Noether's theorem says suppose $f\,(x, y, y')$ is variationally invariant on the interval $[x_0, x_1]$ under the transformation 3 with (xi, η) as infinitesimal generators. Then what have I got?

Then the Noether's theorem tells us the conservation laws. So what it says is suppose you give me the infinitesimal generator, Noether gives you the conservation law. So,

$$\eta \frac{\partial f}{\partial y'} + \xi \left[f - \frac{\partial f}{\partial y'} y' \right] = constant$$

along any extremal. So that is in the simplest form the Noether's theorem. Now students can immediately recognize that thiss expression is not new. In the past we have denoted this first quantity by P and the second quantity here by $-H$.

And this expression arose in several places, especially when we were discussing broken extremal. So this expression is not new. And notice that $\eta \; and \; \xi$ are nothing but variation in $x \; or \; \delta x$, δy. So what I am saying is if

$$p = \frac{\partial f}{\partial y'} \quad , \quad H = y' \frac{\partial f}{\partial y'} - f$$

Then what I see is that the Noether's theorem states that

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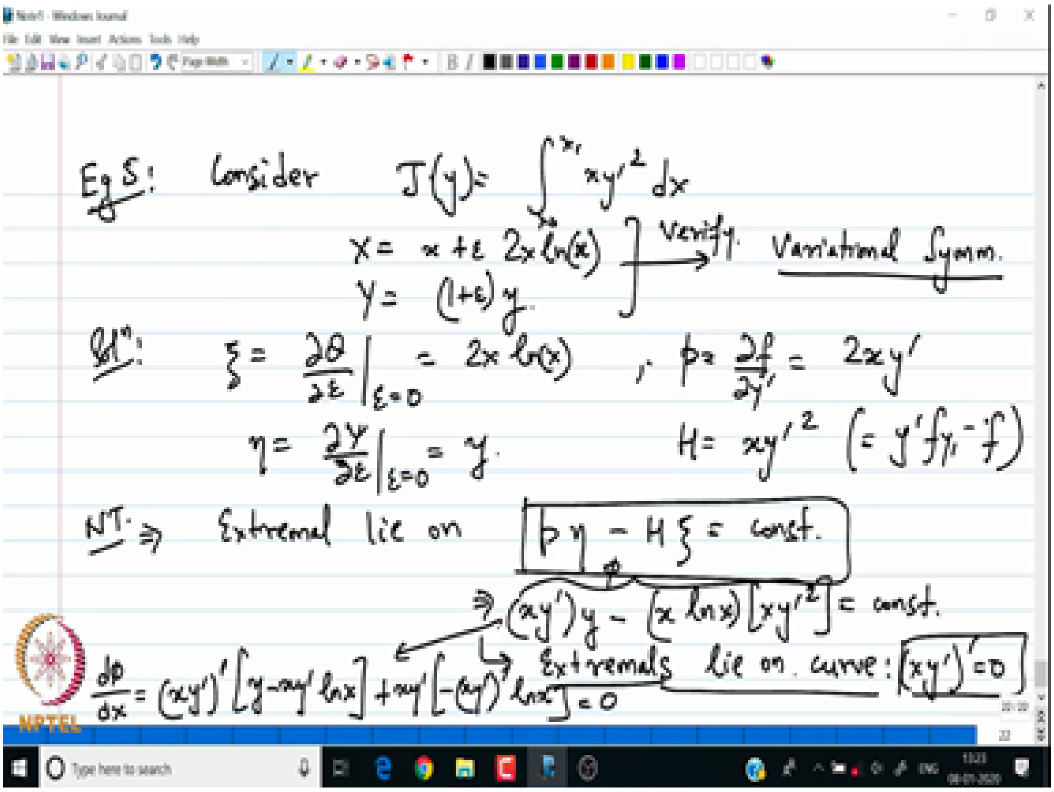

So let us look at some examples with the application of Noether's theorem. So the example I have in mind is, I denote it by example 5. So consider

$$J\left(y\right) = \int_{x_0}^{x_1} xy'^2 dx$$

$$X = x + \epsilon 2x \ln\left(x\right)$$

$$Y = \left(1+\epsilon\right) y$$

So which means my infinitesimal generators are

$$\xi = \left[\frac{\partial \theta}{\partial \epsilon}\right]_{\epsilon=0} = 2x\ln\left(x\right)$$

$$\eta = \left[\frac{\partial \psi}{\partial \epsilon}\right]_{\epsilon=0} = y$$

So we have our infinitesimal generators and further

$$p = \frac{\partial f}{\partial y'} = 2xy'$$

$$H = y'\frac{\partial f}{\partial y'} - f = xy'^2$$

You need to verify that this is a variational symmetry. So this transformation leads to a functional in the new variables which is variationally invariant. So that needs to be verified first before we can apply Noether's theorem. So the Noether's theorem says that the extremals lie on $p\eta - H\xi = constant$. We plug all these values and I am going to get the following relation

$$\left(xy'\right)y - \left(x\ln x\right)\left[xy'^2\right] = constant$$

So the conclusion is extremals lie on the curve given by $\left(xy'\right)' = 0$.

So let us now extend our Noether's theorem for functions of multiple variables and let us look at an example for that case.

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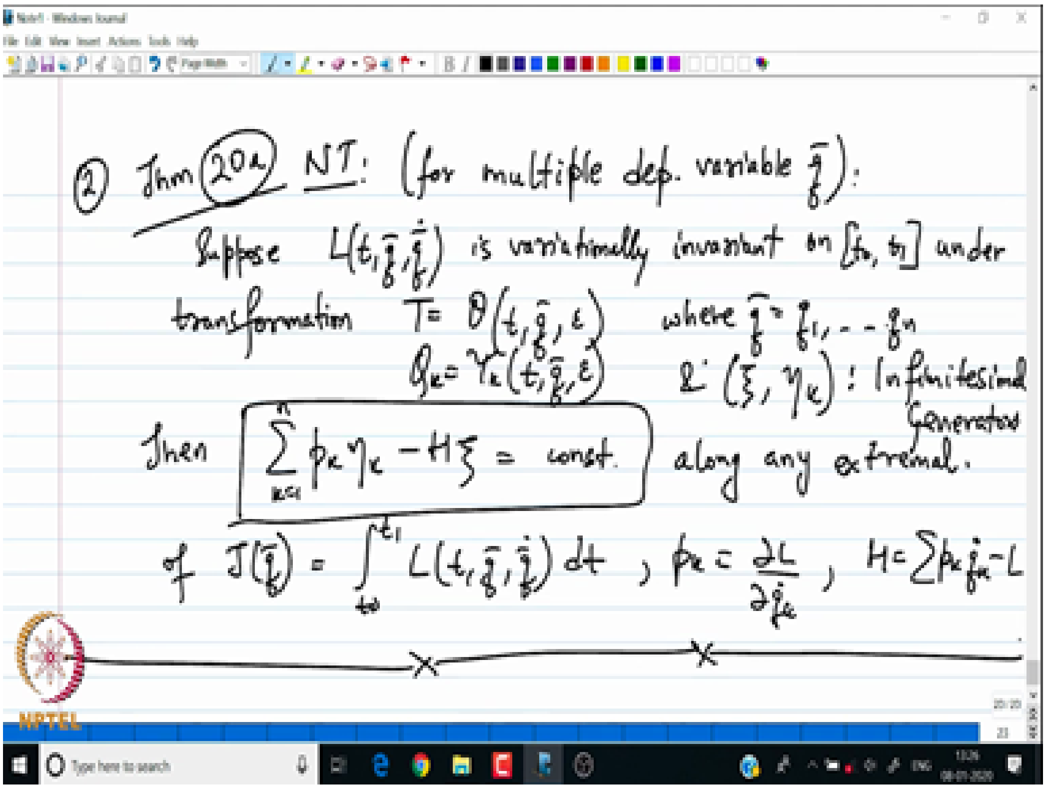

So the version, let me call this theorem 20 (a) because this is extension of the Noether's theorem. It tells us that for multiple dependent variables $\bar{q}$, suppose $L(t,\bar{q},\dot{\bar{q}})$ is variationally invariant on the interval $[t_0,t_1]$ under the transformation

$$T = \theta(t,\bar{q},\epsilon)$$

$$Q_k = \psi_k(t,\bar{q},\epsilon)$$

where $\bar{q} = (q_1, ..., q_n)$

And (ξ,η_k) the set of functions are my infinitesimal generators. So then the result says

$$\sum_{k=1}^{n} p_k\eta_k - H\xi = constant$$

So that is my Noether's theorem along any extremal of

$$J(\bar{q}) = \int_{t_0}^{t_1} L(t,\bar{q},\dot{\bar{q}})\,dt \quad , \quad p_k = \frac{\partial L}{\partial \dot{q}_k} \quad , \quad H = \sum p_k\dot{q}_k - L$$

let us look at an example in this situation.

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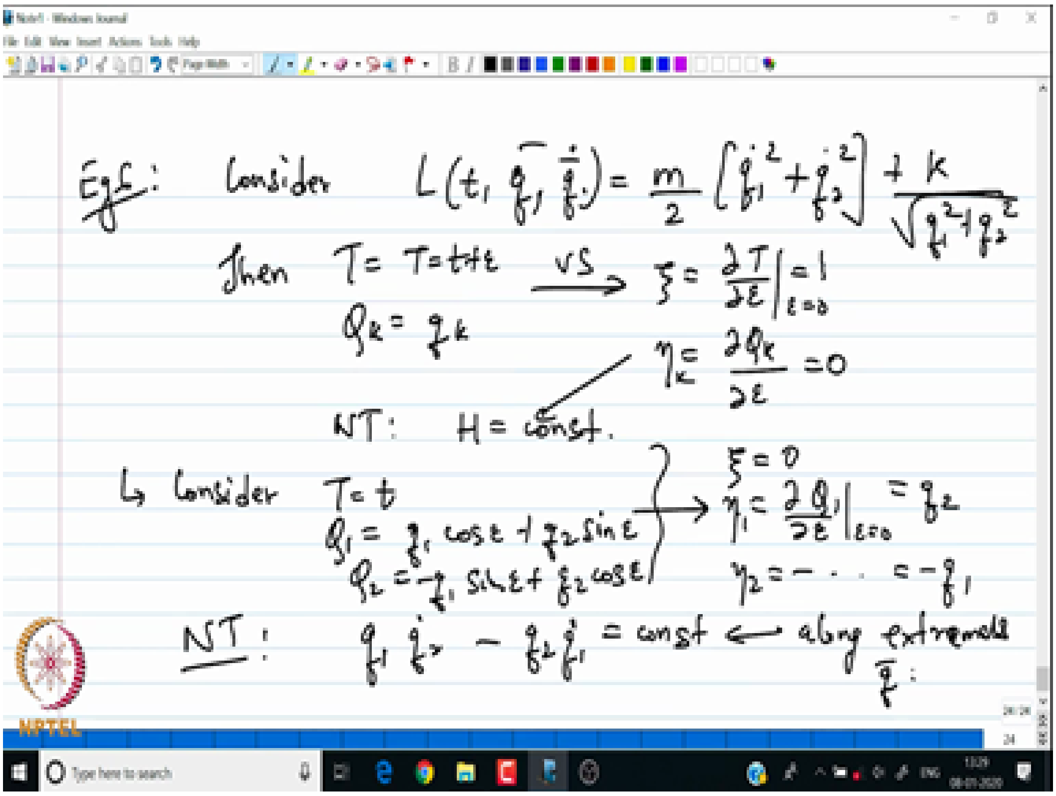

So the example I have is, this will be the final example of our discussion. So consider

$$L\left(t, \bar{q}, \dot{\bar{q}}\right) = \frac{m}{2}\left[\dot{q_1}^2 + \dot{q_2}^2\right] + \frac{K}{\sqrt{q_1^2+q_2^2}}$$

And we can see that if I take this transformation

$$T = t + \epsilon$$

$$Q_k = q_k$$

This is certainly a variational symmetry because in this case

$$\xi = \left[\frac{\partial T}{\partial \epsilon}\right]_{\epsilon=0} = 1$$

$$\eta_k = \frac{\partial Q_k}{\partial \epsilon} = 0$$

So my Noether's theorem says that I am going to get the conservation law $H = constant$. Now consider the transformation

$$T = t$$

$$Q_1 = q_1 \cos\epsilon + q_2 \sin\epsilon$$

$$Q_2 = -q_1 \sin\epsilon + q_2 \cos\epsilon$$

And from here I can see that via Noether's theorem that my infinitesimal generators

$$\xi = 0$$

$$\eta_1 = \left[\frac{\partial Q_1}{\partial \epsilon}\right]_{\epsilon=0} = q_2$$

$$\eta_2 = \left[\frac{\partial Q_2}{\partial \epsilon}\right]_{\epsilon=0} = -q_1$$

And from here Noether's theorem tells us that the conservation law is

$$q_1\dot{q_2} - q_2\dot{q_1} = constant$$

This is along the extremals $\bar{q}$. So let us look at an extension of this problem

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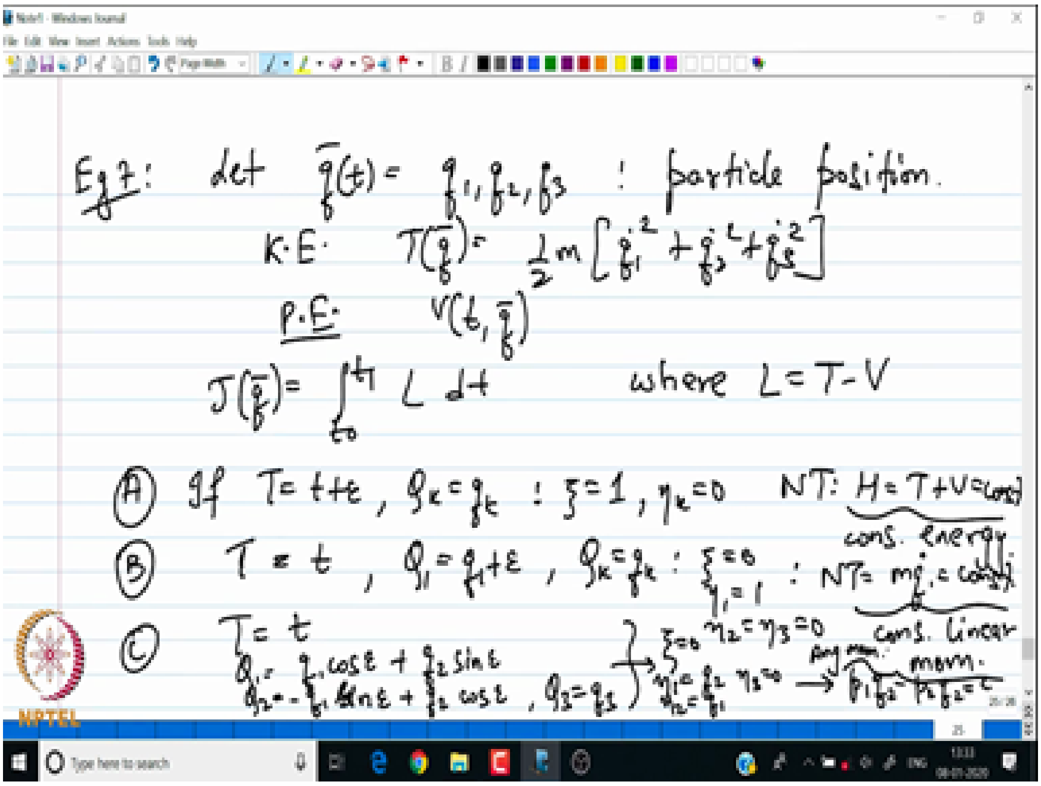

Let $\bar{q}\,(t) = (q_1, q_2, q_3)$. So I am talking about the problem in 3D, denote the particle position. So let us look at a problem in Newtonian mechanics. And kinetic energy for the system

$$T\,(\bar{q}) = \tfrac{1}{2}m\left[\dot{q_1}^2 + \dot{q_2}^2 + \dot{q_3}^2\right]$$

And my potential energy will be $V\,(t, \bar{q})$. Then functional is given by

$$J\,(\bar{q}) = \int_{t_0}^{t_1} Ldt$$

where my Lagrangian $L = T - V$

So now students can check that if we were to look at this sort of transformation, so if I take this transformation

$$T = t + \epsilon \ \ , \ \ Q_k = q_k \ \ , \ \ \xi = 1 \ \ , \ \ \eta = 0$$

And my Noether's theorem tells me that $H = T + V = constant$ Essentially, for this transformation I am satisfying the conservation of energy. On the other hand, if I take

$$T = t\ ,\ Q_1 = q_1 + \epsilon\ ,\ Q_k = q_k\ ,\ \xi = 0\ ,\ \eta_1 = 1\ ,\ \eta_2 = \eta_3 = 0$$

I see that my Noether's theorem gives me $m\dot{q_1} = constant$ this is nothing but the conservation of linear momentum. And finally, the last observation is the following: If I take rotational transformation

$$T = t$$

$$Q_1 = q_1 \cos\epsilon + q_2 \sin\epsilon$$

$$Q_2 = -q_1 \sin\epsilon + q_2 \cos\epsilon$$

$$Q_3 = q_3$$

$$\xi = 0 \;\;,\;\; \eta_1 = q_2 \;\;,\;\; \eta_2 = q_1 \;\;,\;\; \eta_3 = 0$$

From here I get the conservation laws $p_1 q_2 - p_2 q_1 = constant$ and we can see that this is the z component of the angular momentum and hence we are conserving angular momentum in this transformation. So thank you very much for listening and in the next lecture I am going to primarily talk about how to find these variational symmetries. It turns out that once we are able to find these variational symmetries, the Noether' theorem can easily take care of the conservation laws.