Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 44
Noether's Theorem, Introduction to Second Variation Part 2

Now, we move onto a very crucial result regarding the reduction of our functional to reduce order Euler-Lagrange condition. We are going to talk about conservation laws, variational symmetry and the famous Noether's theorem.

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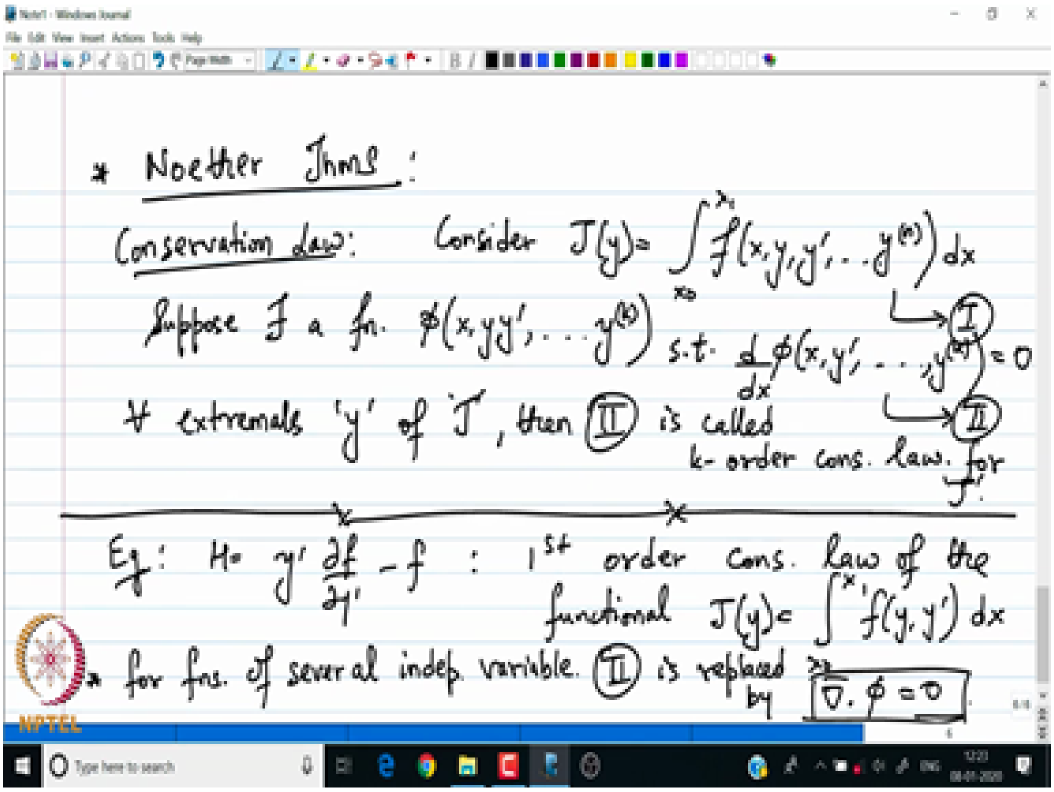

The topic that I am going to introduce in this half of discussion is all about Noether's theorem. So little bit of a trivia, Noether was a German mathematician whose major contribution in mathematical physics was Noether's theorem, which came in the early half of the 19^{th} century and which connects the various symmetries of the problem into a reduced order equation known as the conservation law, which we are going to see in a very simple form in this topic of discussion.

Conservation law : Let us consider a functional

$$J(y) = \int_{x_0}^{x_1} f\left(x, y, y', ..., y^{(n)}\right) dx \qquad \textbf{(1)}$$

This is a functional with derivatives up to n order. Now suppose, $\exists$ a function $\phi\left(x, y, y', ..., y^{(k)}\right)$ such that

$$\frac{d}{dx}\phi\left(x, y, y', ..., y^{(k)}\right) = 0 \qquad \textbf{(2)}$$

$\forall$ extremals y of J then (2) is called the k^{th} order conservation law for J. We have seen conservation laws in the past.I will just quickly recall notice this function

$$H = y' \frac{\partial f}{\partial y'} - f$$

This is the first order conservation law of the functional $J(y) = \int_{x_0}^{x_1} f(y, y')\, dx$. The relation (2) is the case when we have only one independent variable. Suppose we have multiple independent variables.For functions of several independent variables, several independent variables t_1, t_2, and so on then (2) is replaced by $\bar{\nabla}\phi = 0$.

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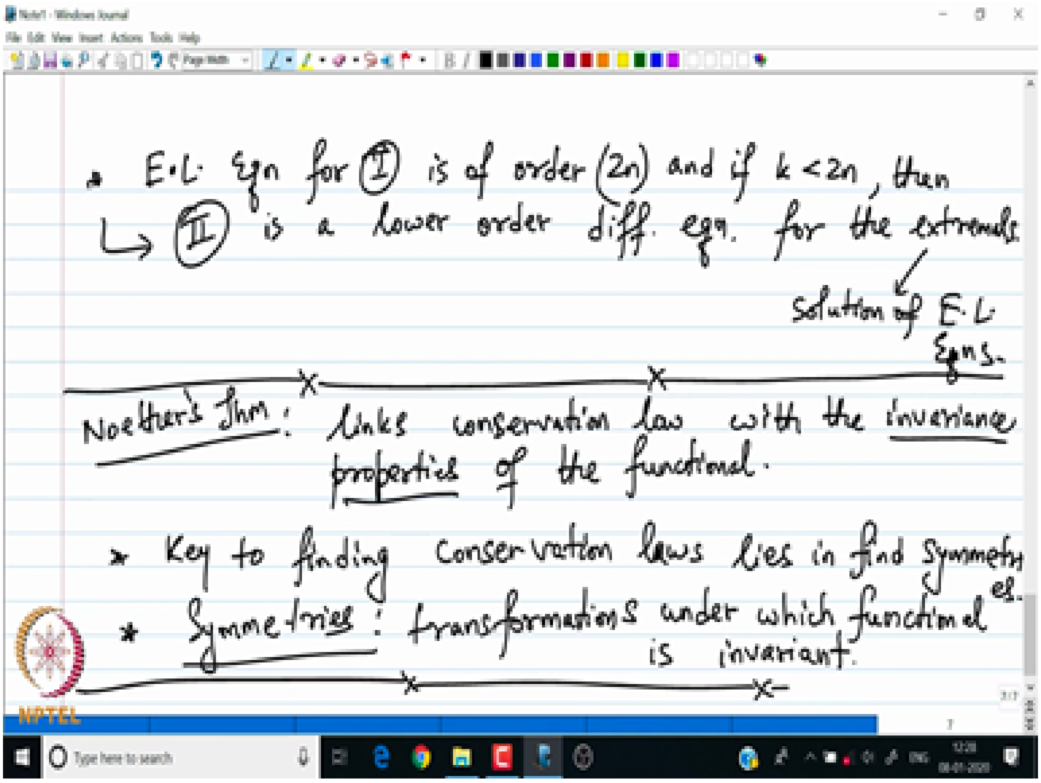

In general Euler-Langrange equation for (1) is of order $2n$ and if $k < 2n$ then (2) is a lower order differential equation for the extremals which were the solutions of Euler-Lagrange equation and this statement lies the power of conservation laws.

Noether's theorem : It links conservation law with the invariance properties of the functional. It basically tells that if you give me invariance properties, I can find out the conservation law. The key to finding conservation laws lies in finding symmetries or the invariance properties of the functional. In this topic of discussion in this lecture, we will assume that the symmetries exist or the invariance properties exist. We will look at only those examples for which we know the invariance properties and from there we are going to derive the conservation laws. In the next lecture we will we will highlight a method of finding these symmetries. Symmetries are transformations under which the functional is invariant.

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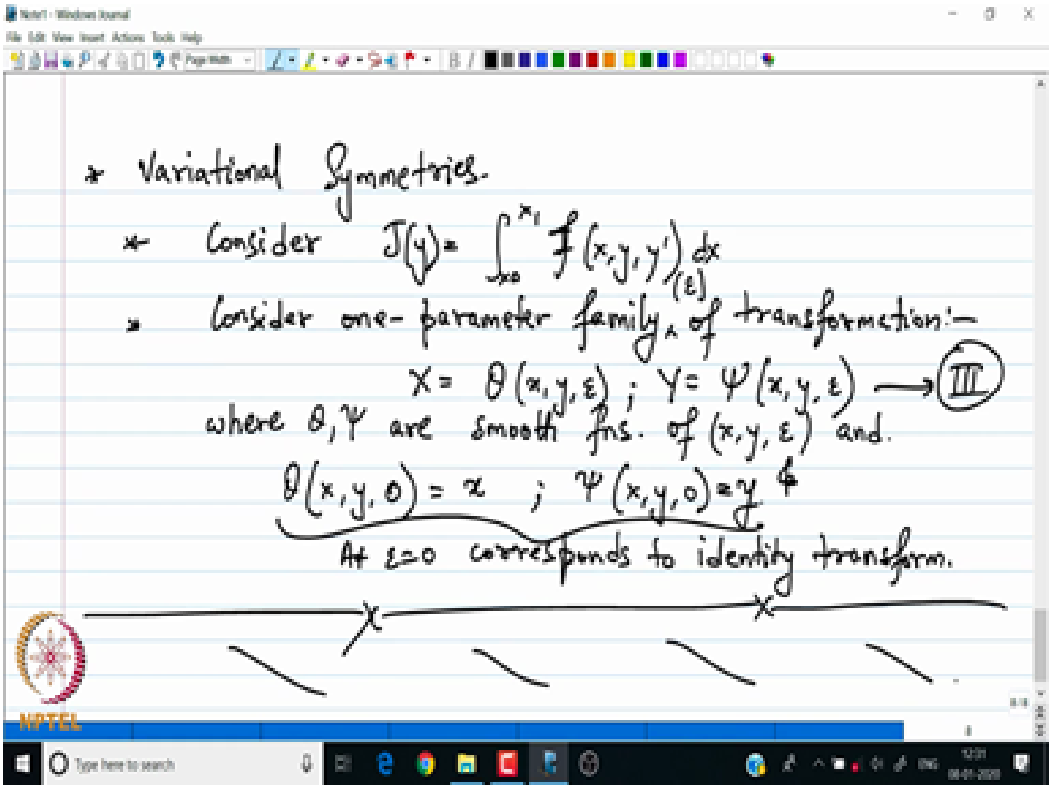

Let us look at the topic of variational symmetries. We consider the the simplest form of the functional to begin with

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

Now, let us consider a one-parameter family (ϵ) of transformation

$$X = \theta(x, y, \epsilon) \quad ; \quad Y = \psi(x, y, \epsilon) \qquad \textbf{(3)}$$

where θ and ψ are smooth functions of (x, y, ϵ) and we also require the following constraint:

$$\theta(x, y, 0) = x \quad ; \quad \psi(x, y, 0) = y$$

This constraint means that at $\epsilon = 0$ corresponds to the identity transformation. So let us look at few transformations and let us see how does the functional behave under those particular set of transformations.

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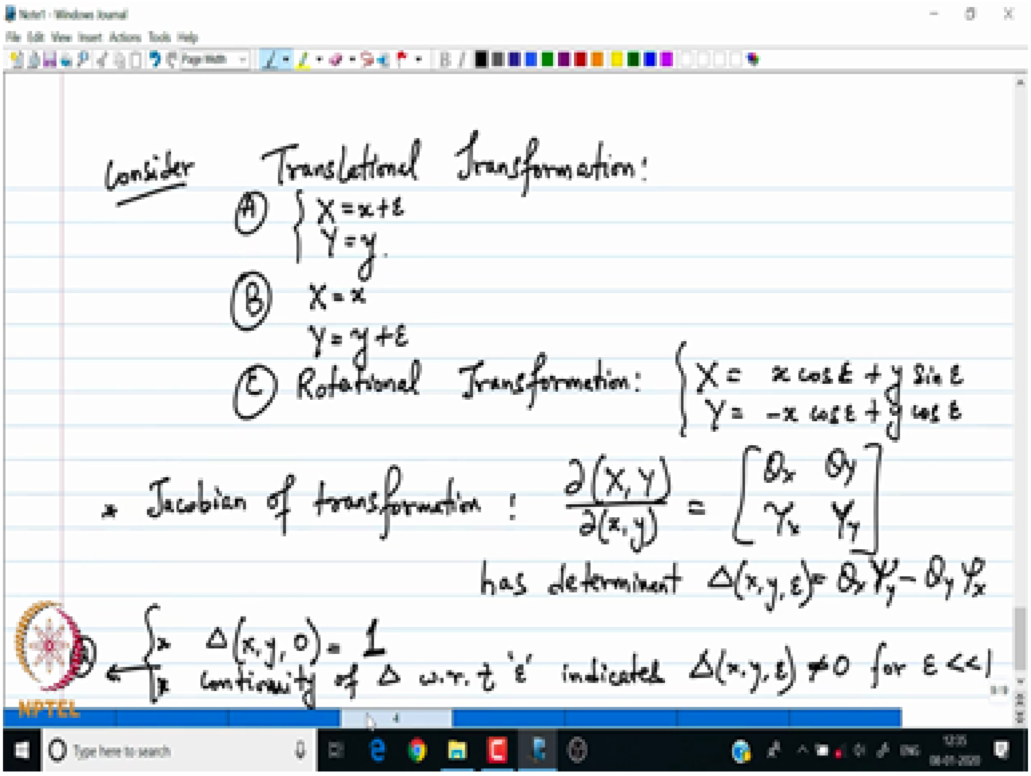

Consider the translational transformations. we consider the functions of the form

$$X = x + \epsilon \ \ ; \ \ Y = y \qquad (A)$$

And we can consider another set of transformation

$$X = x \ \ ; \ \ Y = y + \epsilon \qquad (B)$$

Notice that each of these transforms correspond to the cases where we are translating one of the coordinate axis by ϵ. We can also consider another set of transformation also known as the rotational transformation.

$$X = x\cos\epsilon + y\sin\epsilon \ \ ; \ \ Y = -x\cos\epsilon + y\cos\epsilon \qquad (C)$$

Let me also introduce the concept of Jacobian of transformation which is given by

$$\frac{\partial(X,Y)}{\partial(x,y)} = \begin{bmatrix} \theta_x & \theta_y \\ \psi_x & \psi_y \end{bmatrix}$$

Notice that this has determinant $\Delta(x, y, \epsilon) = \theta_x\psi_y - \theta_y\psi_x$. The way we have described θ and ψ we can see that $\Delta(x, y, 0) = 1$. Notice that at one value of ϵ the determinant is 1. So the determinant is non-singular for $\epsilon = 0$. And if functions θ and ψ they are continuous then we expect in the neighbourhood of $\epsilon = 0$ the determinant is non-zero or the transformation is non-singular. So the continuity of Δ with respect to ϵ indicates that $\Delta(x, y, \epsilon) \neq 0$ for $\epsilon << 1$. Let me call this set of arguments by (a). So I am going to refer these arguments later on when we will use in certain problems by (a). So (a) means that our transformations are uniquely determined in the neighbourhood of $\epsilon = 0$. They are invertible because the determinant of the Jacobian is non-zero which means that we can readily find the inverse of the transformation.

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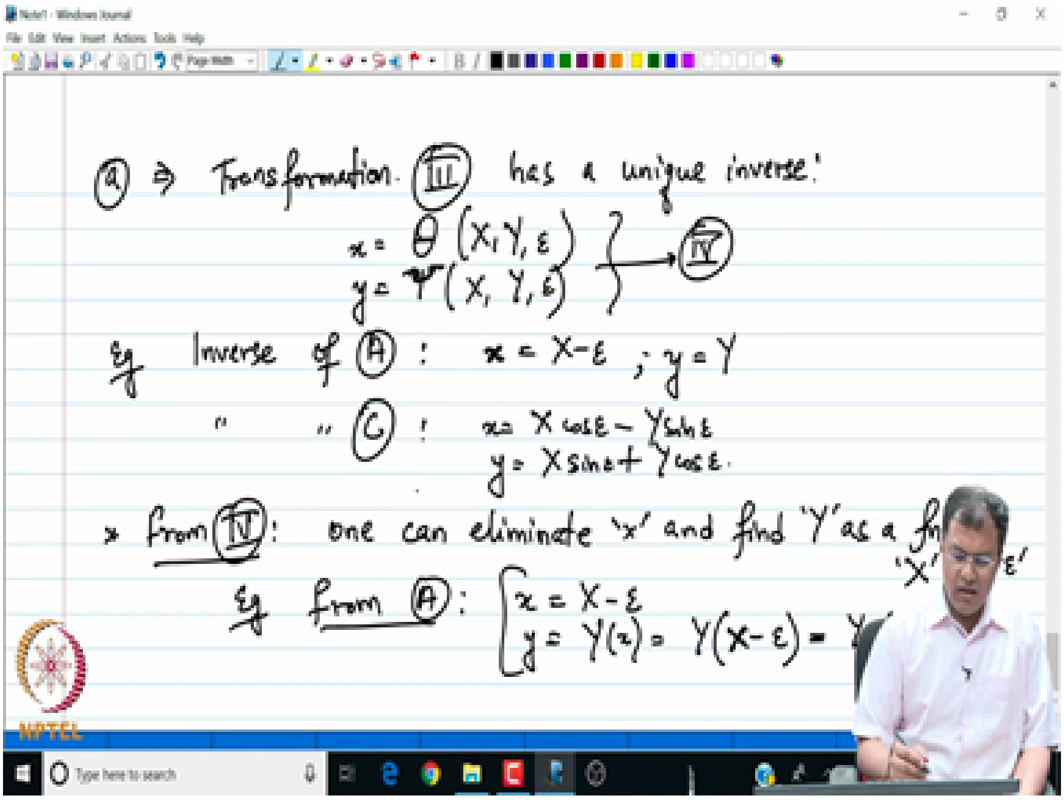

So (a) implies that the transformation given by (3) has a unique inverse and

$$x = \theta\left(X, Y, \epsilon\right) \quad ; \quad y = \psi\left(X, Y, \epsilon\right) \tag{4}$$

Let us recall our example of the translational transformation. The inverse of the transformation (A) is given by

$$x = X - \epsilon \;\; ; \;\; y = Y$$

And the inverse of (C) is given by

$$x = X\cos\epsilon - Y\sin\epsilon \;\; ; \;\; y = X\sin\epsilon + Y\cos\epsilon$$

Notice that we have two unknowns $'x'$ and $'y'$ in this inverse relation and we have two equations which means we can readily eliminate one of the unknowns . so, from (4) one can eliminate $'x'$ and find Y as a function of $'X'$ and ϵ.

. For example, from (A)

$$x = X - \epsilon$$

$$y = Y\left(x\right) = Y\left(X - \epsilon\right) = Y_\epsilon\left(X\right)$$

Now we can repeat this exercise in (C) to see how we can eliminate one variable.

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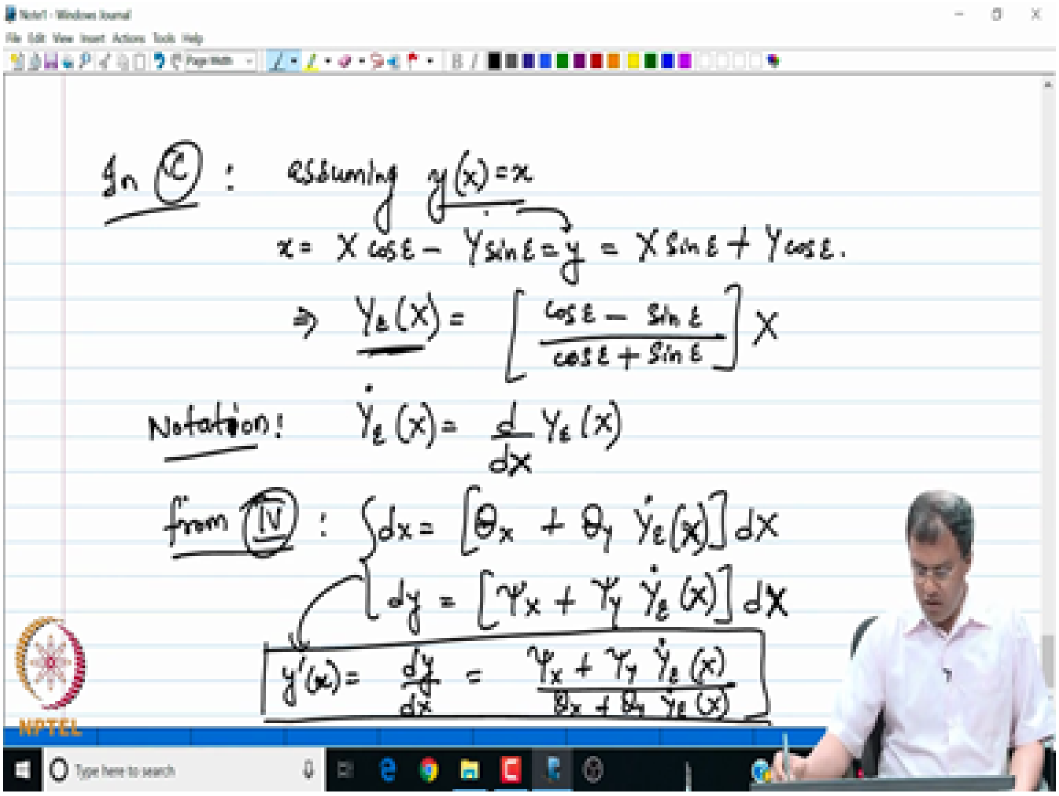

In (C) rotational transformation case, assuming $y\left(x\right)=x$ we see that

$$x = X\cos\epsilon - Y\sin\epsilon = y = X\sin\epsilon + Y\cos\epsilon$$

$$\Rightarrow Y_{\epsilon}\left(X\right) = \left[\frac{\cos\epsilon-\sin\epsilon}{\cos\epsilon+\sin\epsilon}\right]X$$

We also need to introduce some notation. So the notation that I have is

$$\dot{Y}_{\epsilon}\left(X\right) = \frac{\mathrm{d}}{\mathrm{d}X}Y_{\epsilon}\left(X\right)$$

From (4) we see that

$$dx = \left[\theta_x + \theta_y\dot{Y}_{\epsilon}\left(X\right)\right]dX$$

$$dy = \left[\psi_x + \psi_y\dot{Y}_{\epsilon}\left(X\right)\right]dX$$

From here we see that

$$y'\left(x\right) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\psi_x+\psi_y\dot{Y}_{\epsilon}(X)}{\theta_x+\theta_y\dot{Y}_{\epsilon}(X)}$$

Now, We are ready to describe the concept of variational symmetries after all this background.

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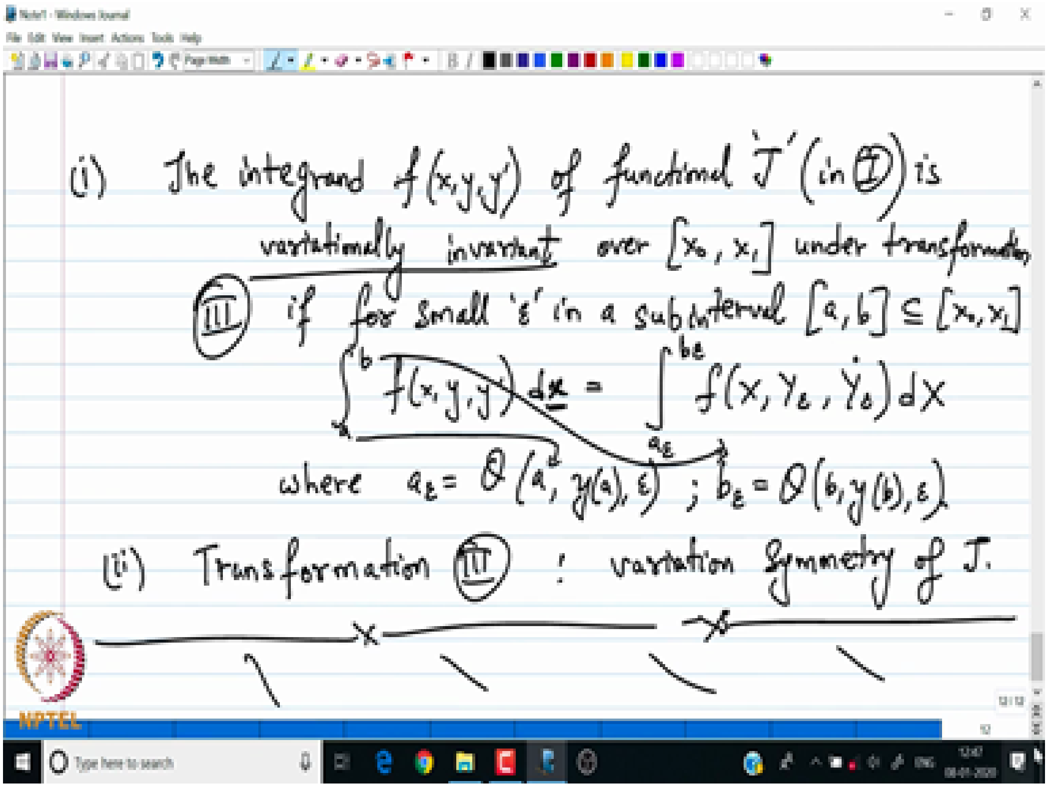

So what are variational symmetries? Before even that let me also introduce the concept of a functional being variationally invariant. The integrand $f\left(x,y,y'\right)$ of the functional J in (1) is called variationally invariant over $[x_0,x_1]$ under the transformation (3) if for small value of the parameter ϵ in a subinterval $[a,b]\subseteq[x_0,x_1]$

$$\int_a^b f\left(x,y,y'\right)dx=\int_{a_\epsilon}^{b_\epsilon} f\left(X,Y_\epsilon,\dot{Y}_\epsilon\right)dX \qquad \text{where}$$

$$a_\epsilon=\theta\left(a,y\left(a\right),\epsilon\right) \quad ; \quad b_\epsilon=\theta\left(b,y\left(b\right),\epsilon\right)$$

And then the second statement says that if the integrand is variationally invariant then the transformation (3) in that case is known as the variational symmetry of J.

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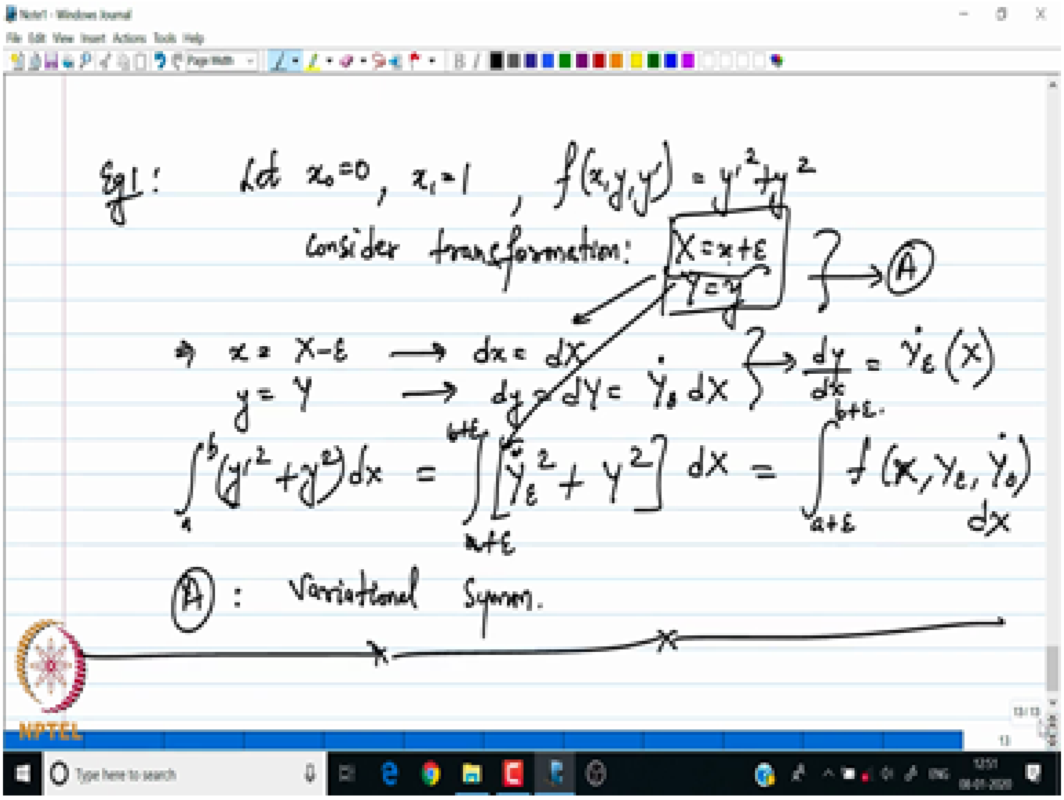

Let us look at an example. Let $x_0 = 0$, $x_1 = 1$, $f(x, y, y') = y'^2 + y^2$ and consider the transformation

$$X = x + \epsilon \;\; ; \;\; Y = y \qquad\qquad \textbf{(A)}$$

From here we can see that

$$x = X - \epsilon \quad \Rightarrow \quad dx = dX$$

$$y = Y \quad \Rightarrow \quad dy = dY = \dot{Y}_\epsilon dX$$

And we see that

$$\tfrac{dy}{dx} = \dot{Y}_\epsilon (X)$$

$$\int_a^b (y'^2 + y^2)\, dx = \int_{a+\epsilon}^{b+\epsilon} \left[\dot{Y}_\epsilon^2 + Y^2\right] dX = \int_{a+\epsilon}^{b+\epsilon} f\left(X, Y_\epsilon, \dot{Y}_\epsilon\right) dX$$

So what I have shown here is that for this sort of transformation or the translational transformation my integrand of this functional is variationally invariant or has the same form to begin with and which means that the transformation $(A)$is variational symmetry. Let us look at the same variational symmetry for another integrand.

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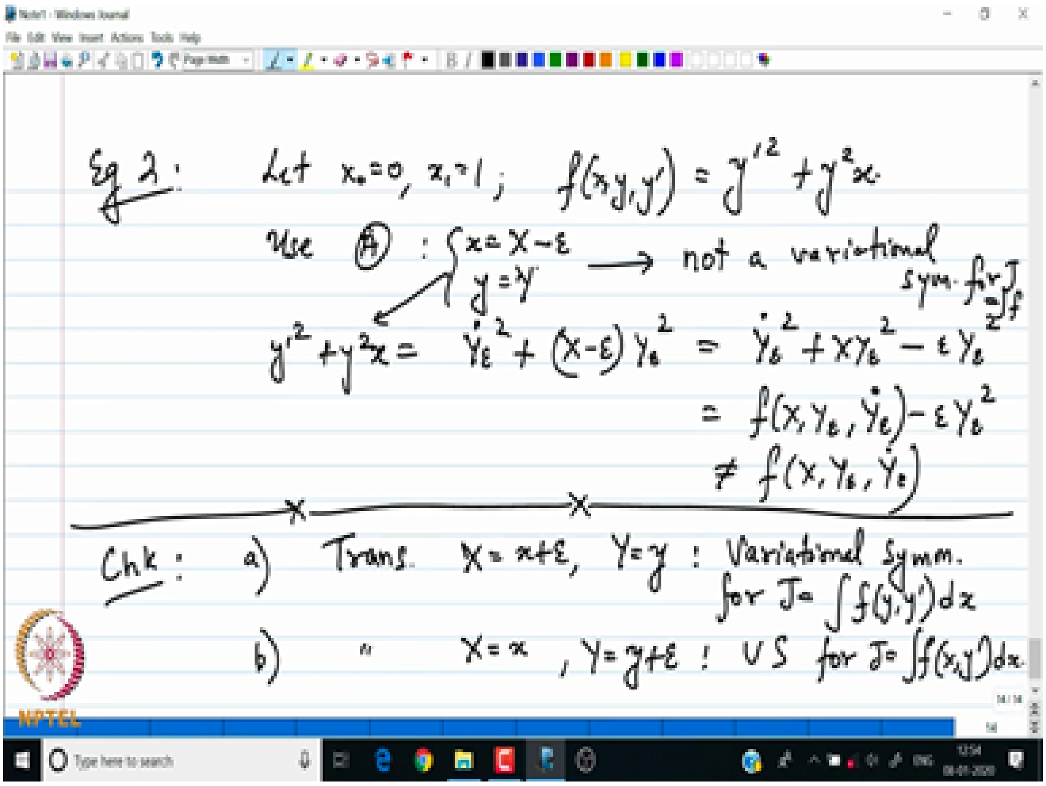

The example we have is the following: Let $x_0 = 0$, $x_1 = 1$, $f(x, y, y') = y'^2 + y^2 x$ and use (A) the same translational transformation

$$X = x - \epsilon \;\; ; \;\; Y = y$$

So from here we see that for this sort of transformation

$$y'^2 + y^2 x = \dot{Y}_\epsilon^2 + (X - \epsilon) Y_\epsilon^2 = \dot{Y}_\epsilon^2 + XY_\epsilon^2 - \epsilon Y_\epsilon^2 = f\left(X, Y_\epsilon, \dot{Y}_\epsilon\right) - \epsilon Y_\epsilon^2 \neq f\left(X, Y_\epsilon, \dot{Y}_\epsilon\right)$$

The conclusion is that the integrand is not variationally invariant for (A) So this is not a variational, variational symmetry for $J = \int f$. So this gives us some perspective as to how to check whether a transformation is a variational symmetry or not. Now in general, it should be mentioned that student should check that the transformation of the type $X = x + \epsilon$; $Y = y$, is a variational symmetry for $J = \int f(y, y')\, dx$ where we have a integrand which is independent of x. On the other hand, suppose we have a translational transformation of the form $X = x$; $Y = y + \epsilon$ is a variational symmetry for $J = \int f(x, y')\, dx$. So we can check, this is a general result for integrals of these two categories.