Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 43
Noether's Theorem, Introduction to Second Variation Part 1

So good morning everyone! In today's lecture I am going to talk about two topics namely, the continuation of our discussion on the conditions for separation of variables. But most important we are going to introduce famous Noether's theorem, which talks about the result or the relation between finding variational symmetries to the existence of the conservation laws.

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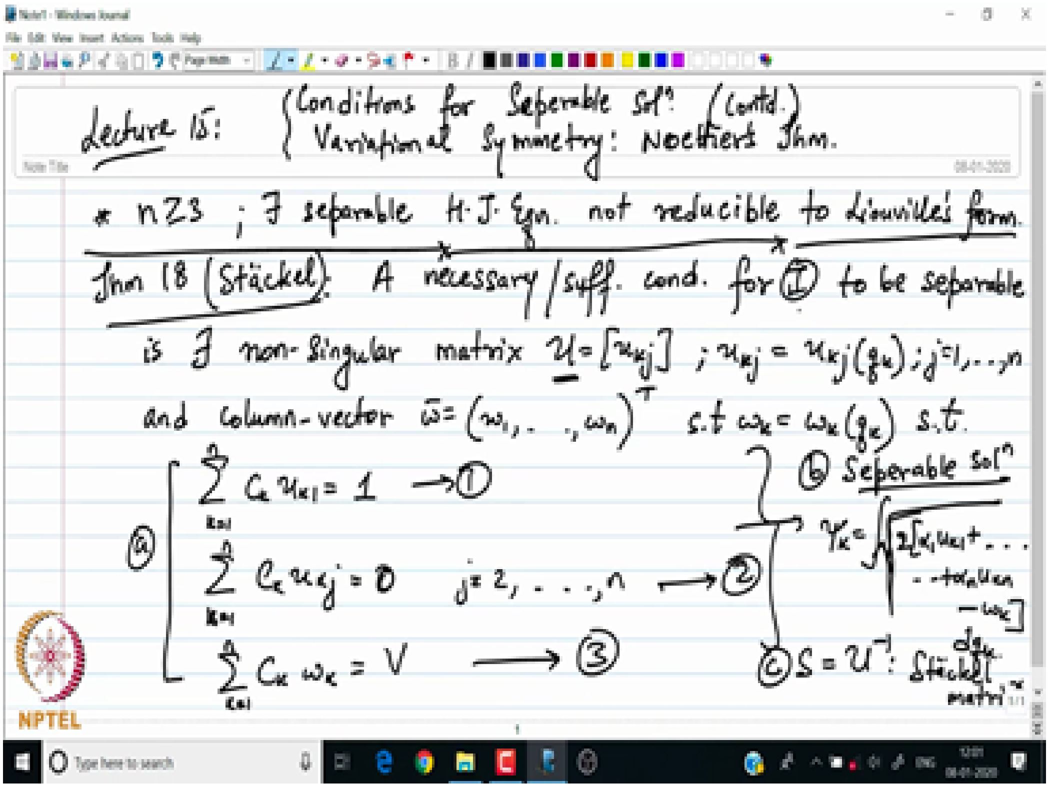

we will wrap our discussion on the conditions for separable solutions. So this is our continuation of our previous lecture topic but most of the time we are going to spend on the topic of variational symmetry leading to the famous results of Noether's theorem.

In the previous lecture, we had ended our discussion on finding the necessary and sufficient condition for two-dimensional problem where the Hamilton Jacobi equation can be variable separated. Now for $n \geq 3$ it can be readily shown that there are separable Hamilton Jacobi equations which cannot be reduced to the Liouville's form. For that we have to look at another result given by Stackel.

So for $n \geq 3$, there exists a separable Hamilton Jacobi equation not reducible to the Liouville's form. And for that scenario we have to look at another result by Stackel. I call this my theorem number 18, the theorem by Stackel. So what it says is the following: A necessary and sufficient for 1 which is our reduced Hamilton Jacobi equation.

So, for 1 to be separable is there exists a non-singular matrix $u = [u_k j]$; $u_k j = u_k j\,(q_k)$; $j = 1, ..., n$

And, the column vector $\bar{w} = (w_1, ..., w_n)^T$ such that $w_k = w_k\,(q_k)$ and these elements are such that I have the following setup:

$$\sum_{k=1}^{n} C_k u_{k1} = 1 \qquad \textbf{(1)}$$

$$\sum_{k=1}^{n} C_k u_{kj} = 0 \ ; \ j = 2, ..., n \qquad \textbf{(2)}$$

$$\sum_{k=1}^{n} C_k w_k = V \qquad \textbf{(3)}$$

And finally, once these requirements are satisfied, the Stackel's result also tells us that the separable solution, so this is, so let me call this entire block of requirement as a, and then once these requirements are satisfied the Stackel's result tells that there is a separable solution which is the following:

So the separable solution is given by

$$\psi_k = \int \sqrt{2\,[\alpha_1 u_{k1} + ... + \alpha_n u_{kn} - w_k]}dq_k$$

So this is my separable solution.Finally, the third block of the result also says that the matrix S which is my u^{-1} , also known as the Stackel matrix. This has special significance in this result, the Stackel matrix. So before I end the statement of this result, let me just briefly say what is 1. So 1 is the Hamilton Jacobi equation for the conservative system.

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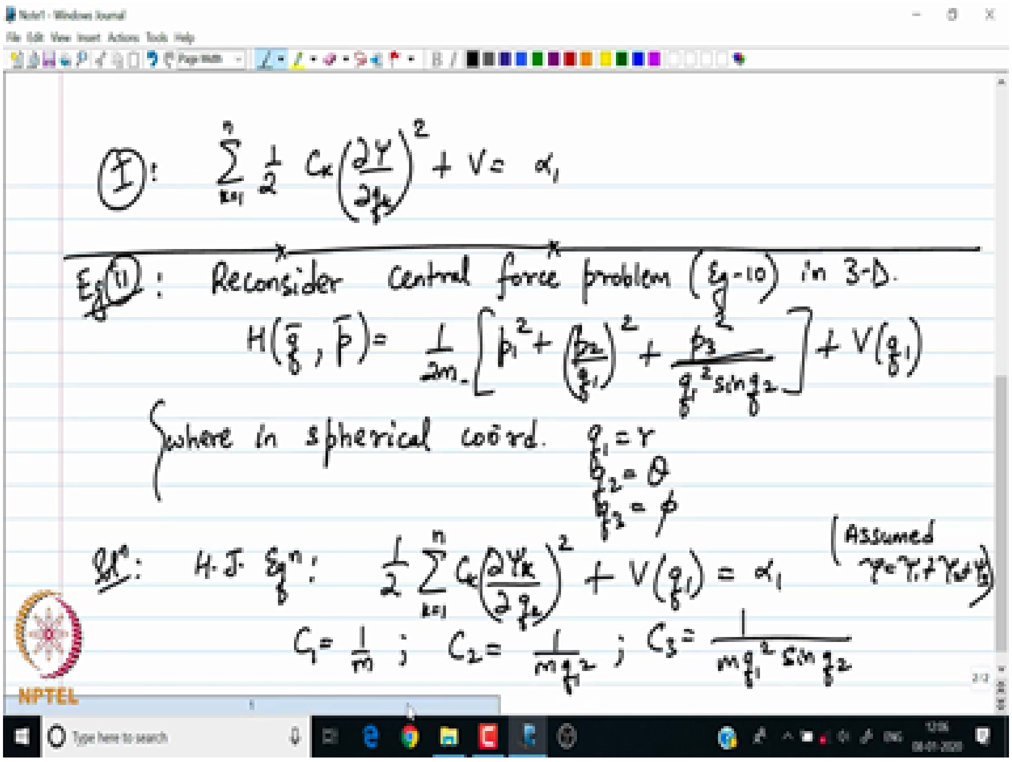

So my expression 1 is

$$\sum_{k=1}^{n} \frac{1}{2} C_k \left(\frac{\partial \psi}{\partial q_k}\right)^2 + V = \alpha_1$$

So this is my expression.Note this which means the result by Stackel tells us that the necessary and sufficient condition for the system to be variable separable is that we are able to find this set of functions u_{kj} and the column vector w_k is such that 1, 2, 3, these three relations are satisfied, that gives us u and

w. And finally, once these unknowns are found, we can write our solution in the form of these unknowns as follows:

So let us quickly look at an example. I am going to continue our discussion by the same numbering that we started with from two lectures ago. So this time my example number is 11 and we are going to reconsider our previous case of the central force problem in 3D. So reconsider the central force problem. This was first introduced in example number 10 and this time in 3D.

So this time our Hamiltonian is of this form

$$H(\bar{q},\bar{p}) = \frac{1}{2m}\left[p_1^2 + \left(\frac{p_2}{q_1}\right)^2 + \frac{p_3^2}{q_1^2 \sin q_2}\right] + V(q_1)$$

Where in spherical coordinates $q_1 = r$. This is just one example. So if you are in spherical coordinates, my coordinate q_1 would have represented the radial component r and my coordinate q_2 would have represented the angular component theta and q_3 would have represented the Azimuthal component ϕ that is angle with respect to the z axis.

So then for this Hamiltonian I can quickly write the Hamilton Jacobi equation. So my Hamilton Jacobi equation is

$$\frac{1}{2}\sum_{k=1}^{n} C_k \left(\frac{\partial \psi_k}{\partial q_k}\right)^2 + V(q_1) = \alpha_1$$

So I am assuming separable solution. So we have assumed, that $\psi = \psi_1 + \psi_2 + \psi_3$ We have already assumed separable solutions and hence ψ_k's.

So then I also need to mention what are my C_k's. So

$$C_1 = \frac{1}{m} \quad ; \quad C_2 = \frac{1}{mq_1^2} \quad ; \quad C_3 = \frac{1}{mq_1^2 \sin q_2}$$

So ,once we have all these constants, we are ready to set up our Stackel's equation.

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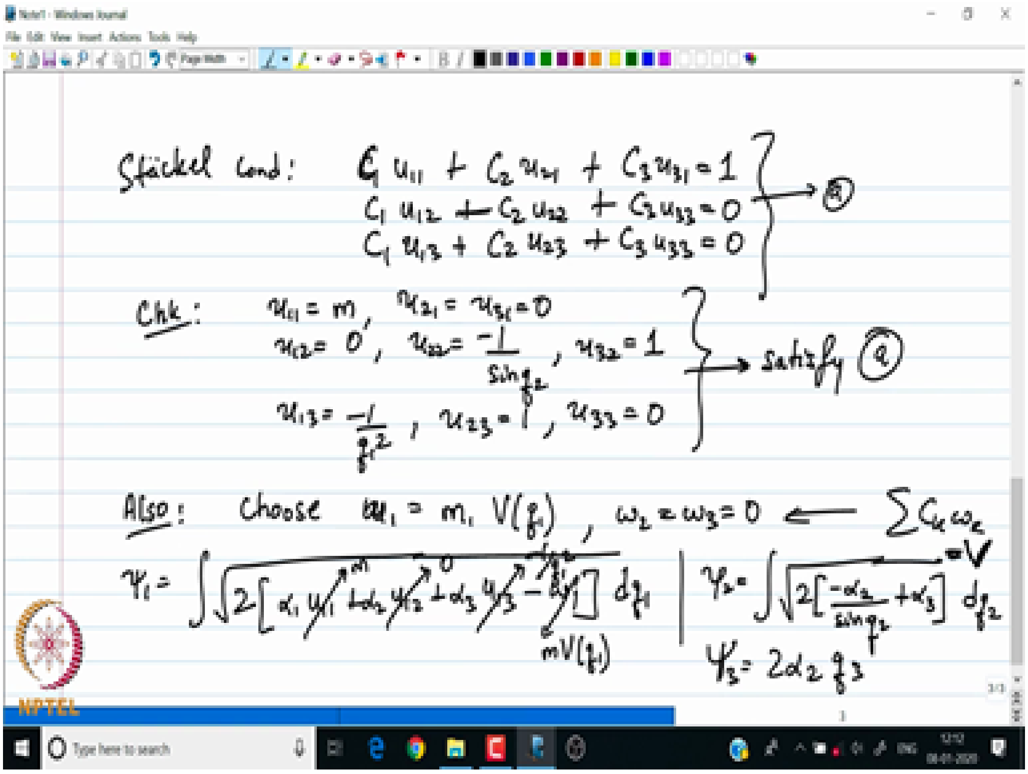
Stäckel Cond: C1 u11 + C2 u21 + C3 u31 = 1
C1 u12 + C2 u22 + C3 u32 = 0
C1 u13 + C2 u23 + C3 u33 = 0 → (a)
Chk: u11 = m, u21 = u31 = 0
u12 = 0, u22 = −1/sin q2, u32 = 1
u13 = −1/q1², u23 = 1, u33 = 0 → satisfy (a)
Also: Choose ω1 = m V(q1), ω2 = ω3 = 0 ← Σ Ck ωk = V
ψ1 = ∫√(2[α1 u11 + α2 u12 + α3 u13 − ω1]) dq1
ψ2 = ∫√(2[−α2/sin q2 + α3]) dq2
ψ3 = 2α2 q3
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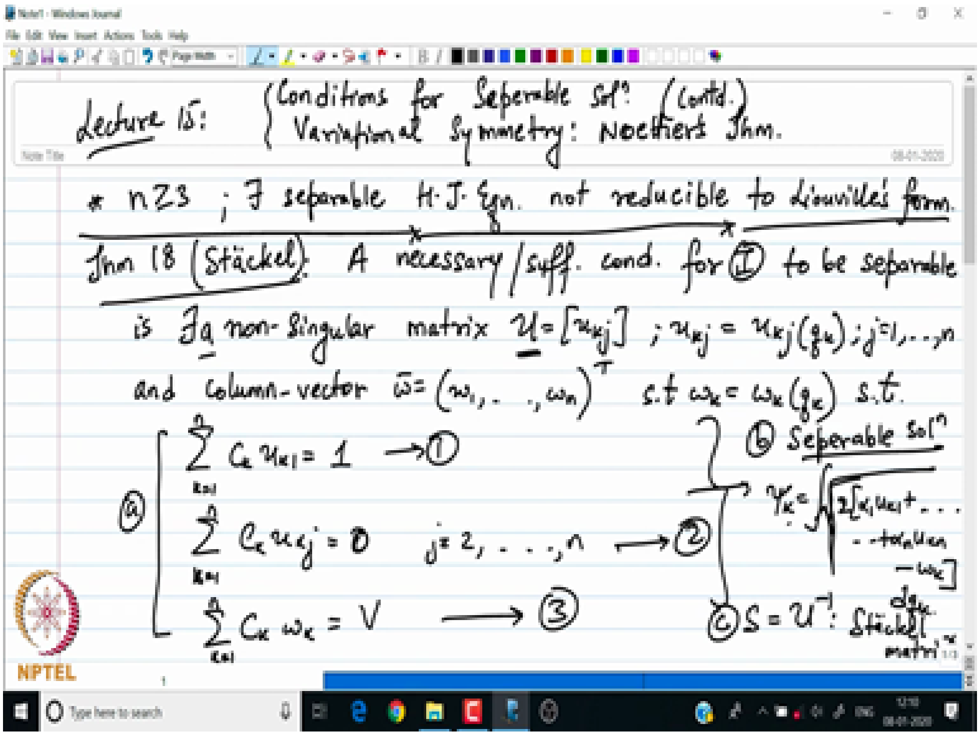
Lecture 15: Conditions for Seperable Sol^n (contd.)
Variational Symmetry: Noether's Thm.
* n ≥ 3 ; ∃ separable H-J Eqn. not reducible to Liouville's form.
Thm 18 (Stäckel): A necessary/suff. cond. for (I) to be separable
is ∃ a non-singular matrix U = [u_kj] ; u_kj = u_kj(q_k) ; j = 1, ..., n
and column-vector ω = (ω1, ..., ωn)^T s.t. ωk = ωk(qk) s.t.
(a) Σ_{k=1}^n Ck uk1 = 1 → (1)
Σ_{k=1}^n Ck ukj = 0 j = 2, ..., n → (2)
Σ_{k=1}^n Ck ωk = V → (3)
(b) Seperable Sol^n ψk = ∫√(2[α1 uk1 + ... + αn ukn − ωk]) dqk
(c) S = U⁻¹: Stäckel matrix
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So the Stackel equation condition says let us set up this matrix

$$C_1 u_{11} + C_2 u_{21} + C_3 u_{31} = 1$$

$$C_1 u_{12} + C_2 u_{22} + C_3 u_{32} = 0$$

$$C_1 u_{13} + C_2 u_{23} + C_3 u_{33} = 0$$

Well, so we have to find the solution to this problem. Note that the Stackel's result says that we need to find a solution. So there exists a non-singular matrix, it does not say that we need to find a unique solution. So we just need to find a solution or just one solution

There could be multiple solution. So check, if I call this system as my a, we can check that

$$u_{11} = m \quad , \quad u_{21} = u_{31} = 0$$
$$u_{12} = 0 \quad , \quad u_{22} = -\frac{1}{\sin q_2} \quad , \quad u_{32} = 1$$
$$u_{13} = -\frac{1}{q_1^2} \quad , \quad u_{23} = 1 \quad , \quad u_{33} = 0$$

From here you can check that these satisfy condition a. So what have we got here? So, these are my solutions. Also, if we choose $w_1 = mV(q_1), w_2 = w_3 = 0$

Then my second condition also holds. Then $\sum C_k w_k = V$

So once we have found u_i's and w_i's, we can directly write down our separable solution, my ψ_1 from the Stackel's result is

$$\psi_1 = \int \sqrt{2\left[\alpha_1 u_{11} + \alpha_2 u_{12} + \alpha_3 u_{13} - w_1\right]} dq_1$$

So $u_{11} = m$, $u_{12} = 0$, $u_{13} = -\frac{1}{q_1^2}$ and $w_1 = mV(q_1)$. So that will tell us ψ_1. Similarly, my ψ_2 is

$$\psi_2 = \int \sqrt{2\left[-\frac{\alpha_2}{\sin q_2} + \alpha_3\right]} dq_2 \quad , \quad \psi_3 = 2\alpha_2 q_3$$

So we end this example by showing what is our Stackel matrix.

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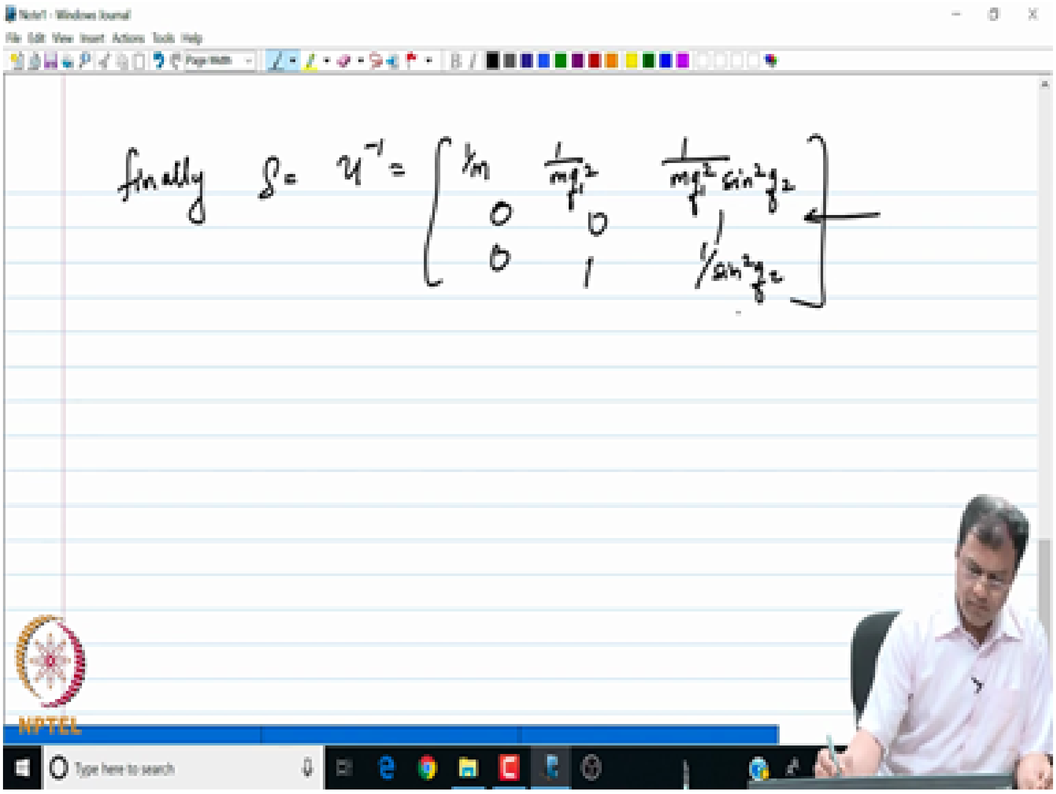

Finally, our Stackel matrix is

$$S = u^{-1} = \begin{bmatrix} \frac{1}{m} & \frac{1}{mq_1^2} & \frac{1}{mq_1^2 \sin^2 q_2} \\ 0 & 0 & 1 \\ 0 & 1 & \frac{1}{\sin^2 q_2} \end{bmatrix}$$

We see that this has special significance because it tells us, the determinant of this matrix is going to tell us whether the solution we have found is complete or not. Remember, we have found a solution not a unique solution.

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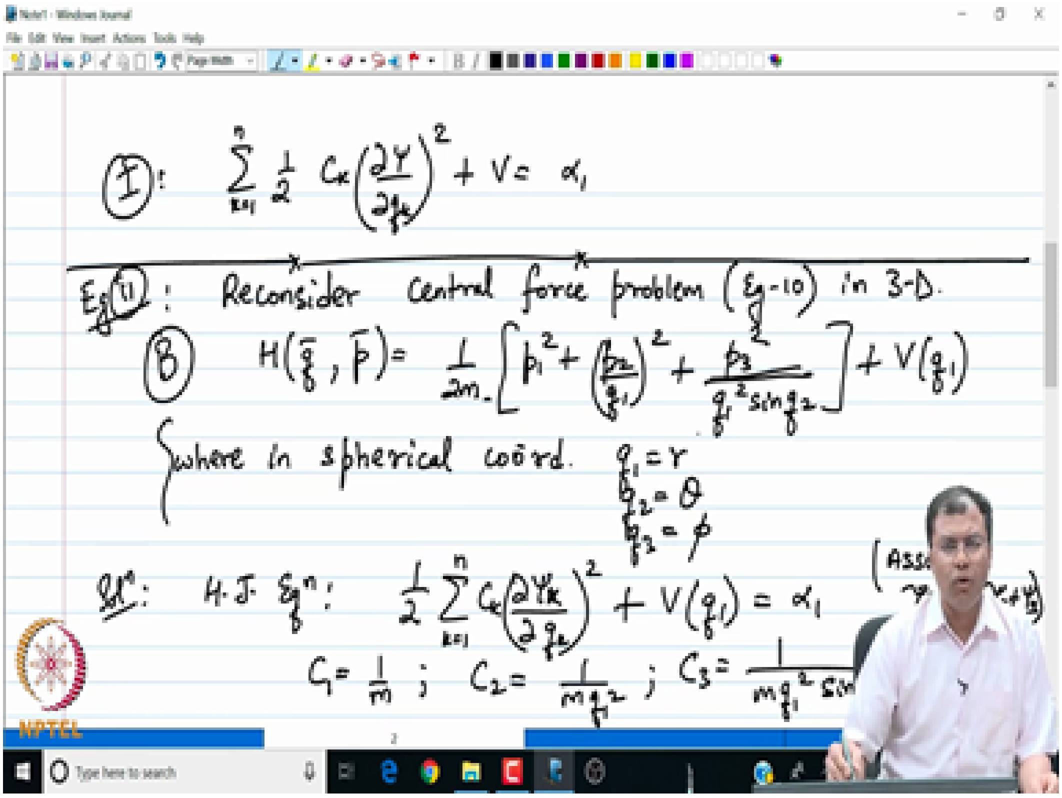

So let me just end the discussion on this example by stating the following: Notice the Hamiltonian that we began with. Let me call this expression as B. So saying that what we did when we were solving the Hamilton Jacobi equation corresponding to this Hamiltonian, we set the corresponding p_i's, C_i's equal to a constant. Setting p_1 equal to constant leads to the conservation of energy, setting p_2 is equal to a constant leads to the conservation of angular momentum and finally, setting this quantity equal to a constant will lead to the conservation of the angular momentum about the z-axis.

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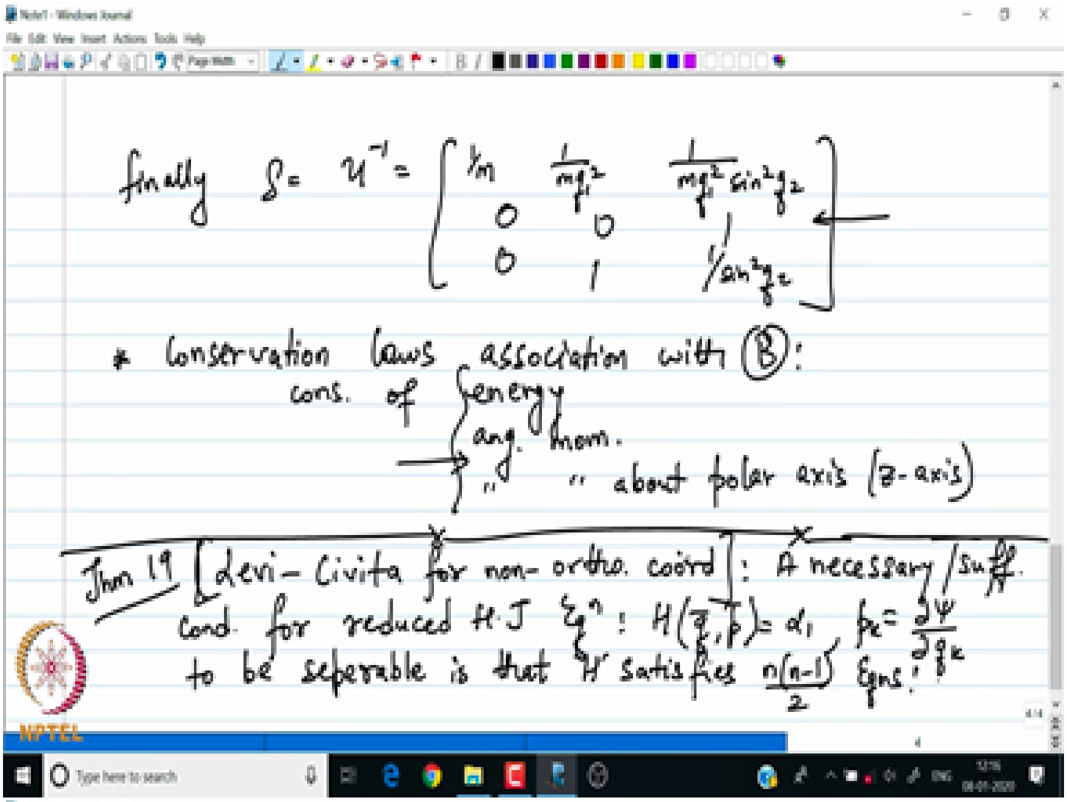

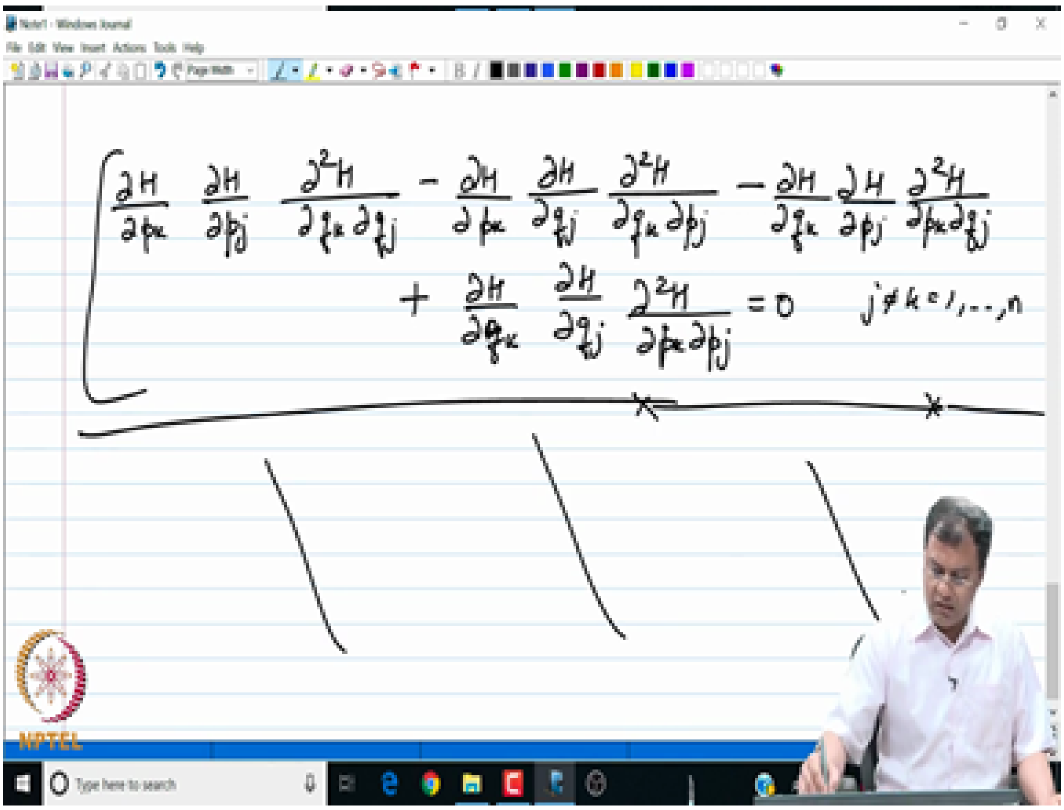

So when we were solving, the conservation laws, see I am regularly pointing out the importance of conservation laws because right now immediately after a few minutes I am going to talk all about conservation laws. So the conservation laws associated with, w our Hamiltonian system B are conservation of energy, conservation of angular momentum and conservation of angular momentum about the polar axis or the z axis.

So we saw some of the conservation laws being satisfied when we were solving the HJ equation using separation of variables. Finally, I am going to end the discussion of finding the solution via separation of variables with this final result by Levi-Civita. So what this result says on, essentially it tells us a necessary and sufficient condition for a system; whether the system is variable separable in the case of non-orthogonal coordinates. That is coordinates such that the Hamiltonian has the product term of the form $\dot{q}_i\dot{q}_j$

So, the Levi-Civita theorem for non-orthogonal coordinates says a necessary and sufficient condition for reduced Hamilton Jacobi equation is that

$$H\left(\bar{q},\bar{p}\right)=\alpha_1$$

And so this is my reduced Hamilton Jacobi equation, and where my p_k's are given to be

$$p_k=\frac{\partial\psi}{\partial q_k}$$

where ψ is the solution to this Hamilton Jacobi equation. So a necessary and sufficient condition for this setup to be separable, is that H satisfies the following set of $\frac{n(n-1)}{2}$ equations

Equations of the form

$$\frac{\partial H}{\partial p_k}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial q_k\partial q_j}-\frac{\partial H}{\partial p_k}\frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial q_k\partial p_j}-\frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_k\partial q_j}+\frac{\partial H}{\partial q_k}\frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial p_k\partial p_j}=0$$

where my $j\neq K=1,...,n$ So ,my necessary and sufficient condition nothing is said about the form of the variable separable solution, the result only says that such a solution exists. So we end our discussion in this setup.