Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture – 42

Hamilton-Jacobi Equations Part 6

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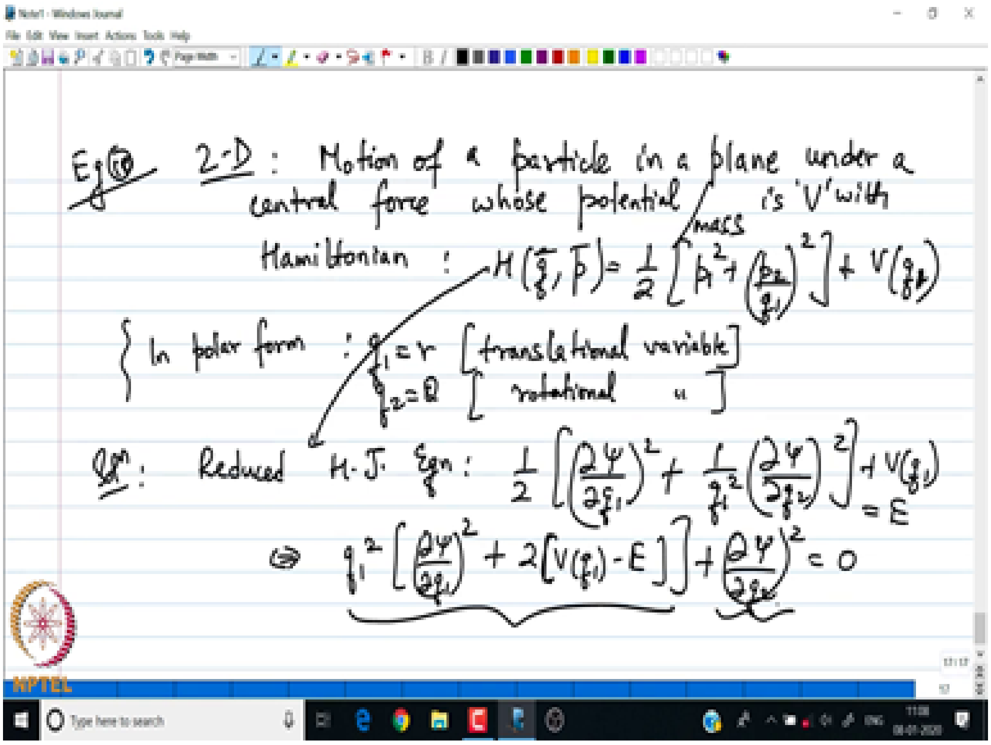

So, let us look at one more example. The other example that I have, example number 10. So this is a problem in 2-D and the problem says that we have to look at the motion of a particle in a plane. So, this is a problem in 2-D, plane under a central force, whose potential per unit mass is V with the Hamiltonian given

$$H\left(\bar{q}, \bar{p}\right) = \frac{1}{2}\left[p_1^2 + \left(\frac{p_2}{q_1}\right)^2\right] + V\left(q_1\right)$$

Suppose the 2-D problem would have been in the polar setup. Suppose we were describing the problem in polar coordinates, then people can recognize my coordinate q_1 as my radial coordinate. So, so just a trivia, in polar form my coordinate q_1 is r, which is the translational variable and my coordinate q_2 which is theta is the rotational variable. So, let us move ahead. From here , I can write down my solution to the reduced Hamilton-Jacobi equation.My equation looks like following:

$$\frac{1}{2}\left[\left(\frac{\partial \psi}{\partial q_1}\right)^2 + \frac{1}{(q_1)^2}\left(\frac{\partial \psi}{\partial q_2}\right)^2\right] + V\left(q_1\right) = E$$

From here, I can multiply throughout by q_1. So, I get a alternate form

$$q_1^2\left[\left(\frac{\partial \psi}{\partial q_1}\right)^2 + 2\left[V\left(q_1\right) - E\right]\right] + \left(\frac{\partial \psi}{\partial q_2}\right)^2 = 0$$

The reason for writing it in this form is notice that this is purely a function of q_1, while this quantity is purely a function of q_2 and that will help us to select our quantities g_1 and g_2.

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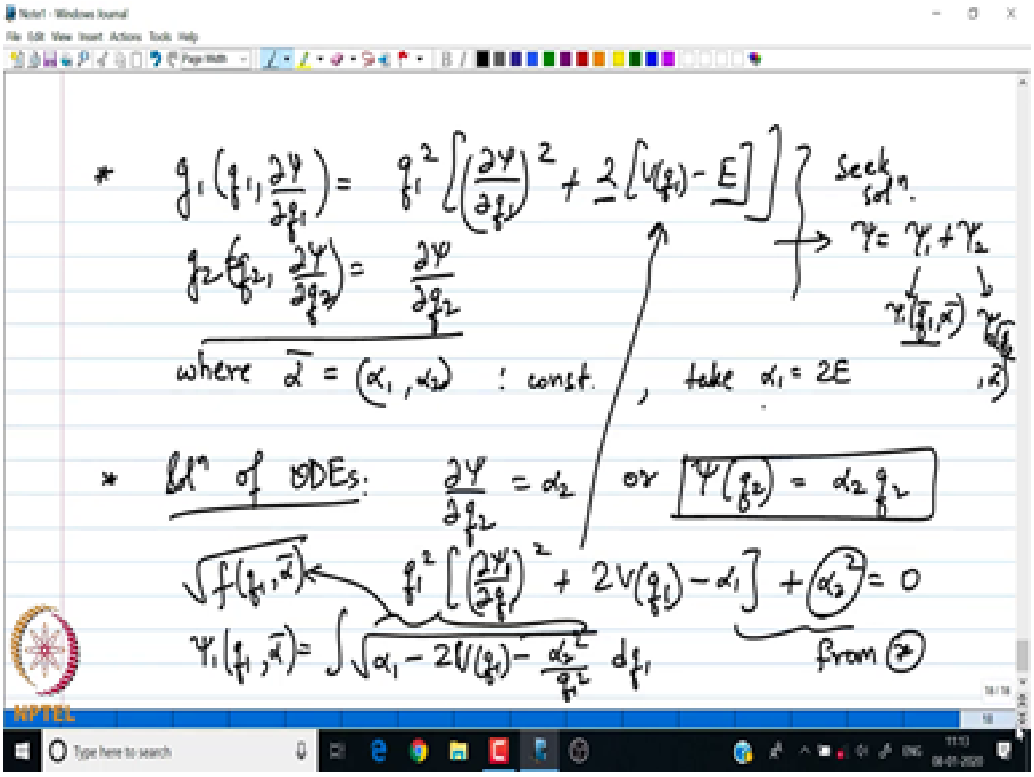

So in the solution strategy, let me select

$$g_1\left(q_1, \frac{\partial\psi}{\partial q_1}\right) = q_1^2\left[\left(\frac{\partial\psi}{\partial q_1}\right)^2 + 2\left[V\left(g_1\right) - E\right]\right] \quad \text{and}$$

$$g_2\left(q_2, \frac{\partial\psi}{\partial q_2}\right) = \frac{\partial\psi}{\partial q_2}$$

So what we do is we seek solutions of the form $\psi = \psi_1 + \psi_2$ So ψ_1 is a function of (q_1, α) So α comes out to be constant or the functions of Q which is the generalized coordinate.

And ψ_2 is a function of (q_2, α) where $\bar{\alpha} = (\alpha_1, \alpha_2)$ these are my constant vectors. So what we do is, we are going to take, one constant α_1 to be 2E. So, now, the next stage involves the solution of ODEs that we have, g is equal to C_i's.

Let us setup the ODEs. The second ODE is quite easy to setup.

$$\frac{\partial\psi}{\partial q_2} = \alpha_2 \ \ or \ \ \psi\left(q_2\right) = \alpha_2 q_2$$

And why we have avoided the constant of integration because we want to express our final answer in terms of only two constants, α_1 and α_2, 2-D problem, two constants. Also for the first case, we have

$$q_1^2\left[\left(\frac{\partial\psi}{\partial q_1}\right)^2 + 2\left[V\left(g_1\right) - \alpha_1\right]\right] + \alpha_2^2 = 0$$

Now I have to solve for ψ_1. So I can write down the expression directly.

$$\psi_1\left(q_1, \bar{\alpha}\right) = \int \sqrt{\alpha_1 - 2V\left(q_1\right) - \frac{\alpha_2^2}{q_1^2}} dq_1$$

Let me call the quantity inside square root be f . And, I leave the solution in this form where f is a quantity inside the integral, in general $\sqrt{f\left(q_1, \bar{\alpha}\right)}$ can be integrated.

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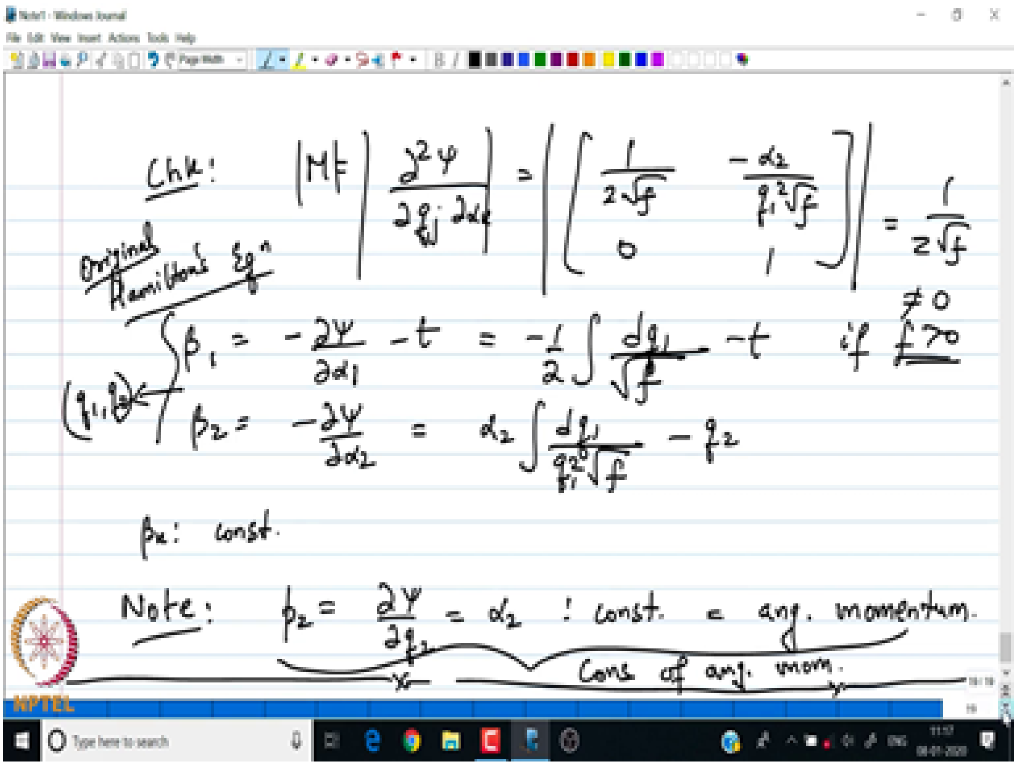

All I do is I check whether the solution is complete or not. So, check the determinant of the matrix

$$M = \begin{bmatrix} \frac{1}{2\sqrt{f}} & -\frac{\alpha_2}{q_1^2\sqrt{f}} \\ 0 & 1 \end{bmatrix}$$

I can see that the determinant of this matrix is equal to $\frac{1}{2\sqrt{f}}$ This is not going to be 0 provided my function f is well defined and positive

Then in that case, I can directly write down my Hamilton's equation to give me the extremals are as follows:

$$\beta_1 = -\frac{\partial \psi_1}{\partial \alpha_1} - t = -\frac{1}{2}\int \frac{dq_1}{\sqrt{f}} - t$$

$$\beta_2 = -\frac{\partial \psi}{\partial \alpha_2} = \alpha_2 \int \frac{dq_1}{q_1^2\sqrt{f}} - q_2$$

Where my $\beta - k$'s are constants Note that, the end of the problem. The problem final step involves inverting these two relations to find my q_1 and q_2 which is my extremal to the Hamiltonian system. So, note one thing, note that p_2, the second component of the momentum, $p_2 = \frac{\partial \psi}{\partial q_2}$, but $\frac{\partial \psi}{\partial q_2}$ is a constant, by the setup of our problem.

This is also equal to α_2 which is a constant.Now, notice that, to begin with, polar coordinates, we had assumed that the second component involved represented the angle or the rotational component? So p_2 is the angular momentum in the physical language and we are showing here that, in this setup, the angular momentum is constant. So this is nothing but our angular momentum or the rate of change.The change in the angular component.

And we are saying that the angular momentum is constant, meaning that, we are saying that the law of conservation of angular momentum is satisfied in this problem. So,far we have shown that if we are able to separate variables for conservative system, you should be able to solve and find the generating function which will give us the extremal to the original functional. The question is, can we really separate out the variables?

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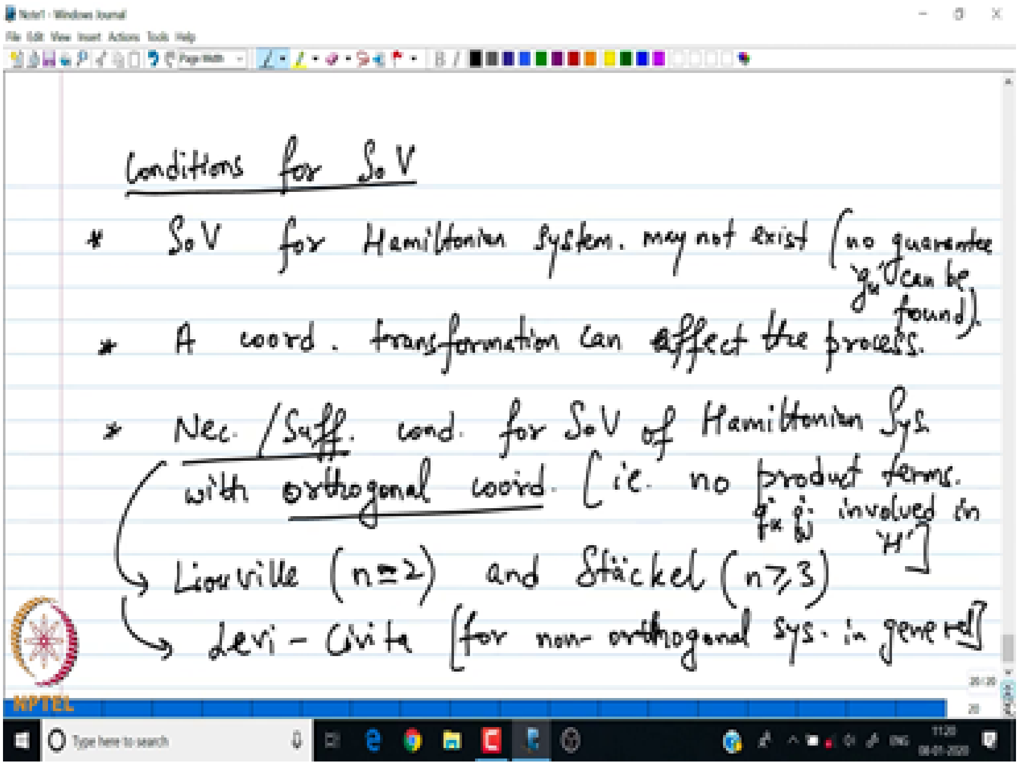

Now I am going to state some conditions under which the reduced Hamilton-Jacobi can be variable separated. So I am going to write some criterias which is true and I am going to state some results in the form of theorem without proof. So, separation of variables for Hamiltonian system may not exist

There is no guarantee that g_k's can be found. And it turns out that, we will see that a coordinate transformation can affect the process, which means, that in one coordinate, the separation of variable can be done, while in another, for the same problem, it cannot be done. We will see some examples.

Also, it turns out that the necessary and the sufficient condition for separation of variables of Hamiltonian system with there are three results that I am going to state for different coordinate systems. So, with orthogonal coordinates, so what do I mean by orthogonal coordinate system? That is system in which no product terms $\dot{q_j}.\dot{q_k}$ involved in our Hamiltonian H.

So, those are my orthogonal coordinate system. It turns out that there is a necessary and sufficient condition given by Liouville, I am going to state these results, Liouville for n= 2 and by Stackel, the German mathematician for $n \geq 3$. And finally, by Levi-Civita for non-orthogonal systems in general. So, we have the condition for all the cases. So, let us now look at these.

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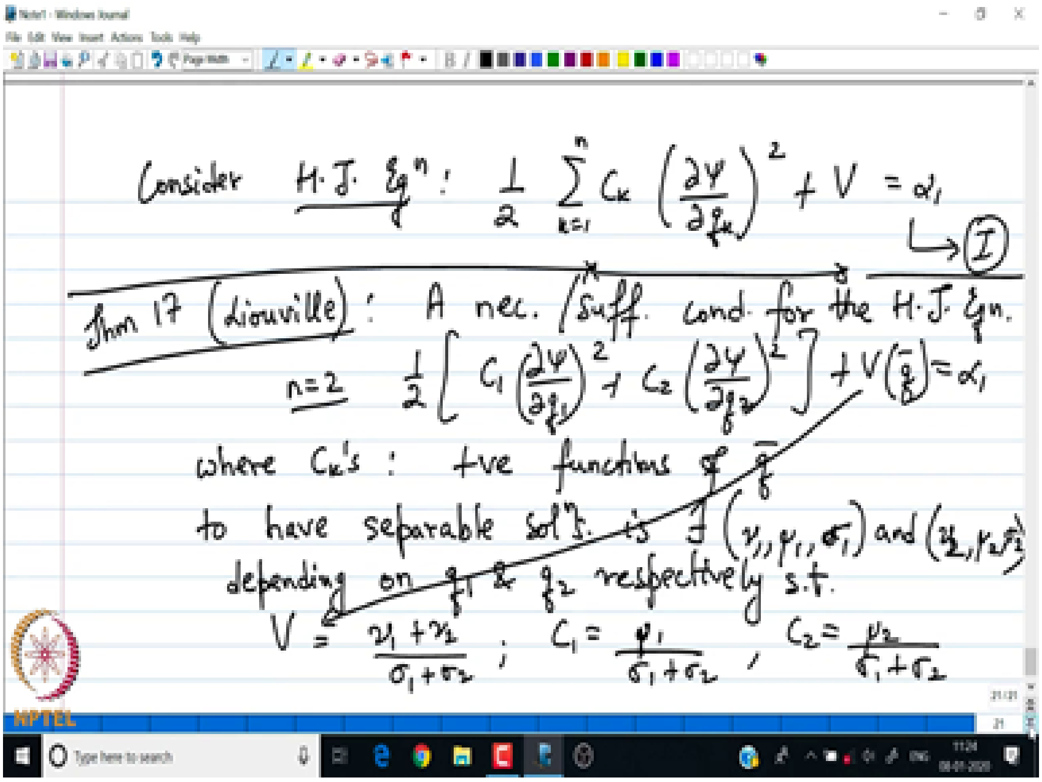

I am going to end my discussion by stating few of these results. We consider our Hamilton-Jacobi equation of the form that we have seen in the, our separation of variable. So, let us say we have the following form

$$\frac{1}{2}\sum_{k=1}^{n} C_k\left(\frac{\partial\psi}{\partial q_k}\right)^2 + V = \alpha_1 \qquad \textbf{(1)}$$

So the first result by Liouville,this is my theorem 17.

This is for 2-dimensional problem, it says that a necessary and a sufficient condition for the Hamilton-Jacobi equation

$$\frac{1}{2}\left[C_1\left(\frac{\partial\psi}{\partial q_1}\right)^2 + C_2\left(\frac{\partial\psi}{\partial q_2}\right)^2\right] + V\left(\bar{q}\right) = \alpha_1$$

So this is a problem in 2-D. So, n = 2. Where my constants C_k's are positive functions of my variable $\bar{q}$ to have separable solutions, to have separable solutions is the existence of (ν_1, μ_1, σ_1) *and* (ν_2, μ_2, σ_2)

So, the existence of these two pairs of functions depending on q_1 and q_2 respectively, so the first pair depends on q_1 and the second depends on qq_2 such that my function V can be written as

$$V = \frac{\nu_1+\nu_2}{\sigma_1+\sigma_2} \quad ; \quad C_1 = \frac{\mu_1}{\sigma_1+\sigma_2} \quad ; \quad C_2 = \frac{\mu_2}{\sigma_1+\sigma_2}$$

The moment we are able to write these functions of the generalized coordinates Q and this function V in this form, then we are guaranteed to have a separable solution that is what Liouville stated and proved. And then, let us look at an example. The example that I have in mind is the example that we just considered, of the central force problem, example number 10, few slides back.

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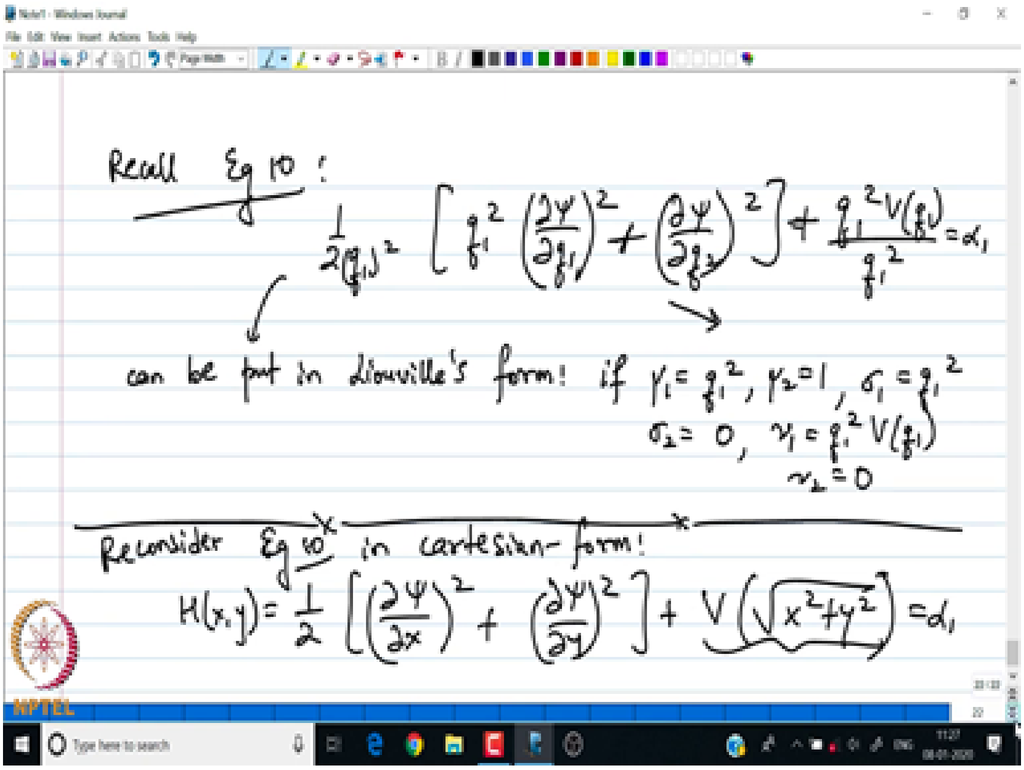

So I am going to reconsider my problem, this one. So this is the problem which is example 10 here. So, recall example 10 that was done few minutes back. See that my Hamiltonian leads to the following Hamilton-Jacobi equation which is

$$\frac{1}{q_1^2}\left[q_1^2\left(\frac{\partial\psi}{\partial q_1}\right)^2+\left(\frac{\partial\psi}{\partial q_2}\right)^2\right]+\frac{q_1^2V(q_1)}{q_1^2}=\alpha_1$$

And that can be put in the Liouville's form if I take

$$\mu_1 = q_1^2 \quad , \quad \mu_2 = 1 \quad , \quad \sigma_1 = q_1^2$$

$$\sigma_2 = 0 \quad , \quad \nu_1 = q_1^2 V\left(q_1\right) \quad , \quad \nu_2 = 0$$

. The moment we take all these quantities as follows, we will see that this equation reduces to the Liouville's form as specified by the result in theorem 17. So,that is the end of this discussion, but let us also reconsider example 10 in Cartesian form

So, in Cartesian form, notice what is happening in the Cartesian form. My Hamiltonian, $H\left(x,y\right)$ now can be written in the following form

$$H\left(x,y\right)=\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]+V\left(\sqrt{x^2+y^2}\right)=\alpha_1$$

Notice that the central force V_1 here depends on the $\sqrt{x^2+y^2}$.

So, in this case, we cannot separate the variable x and y in this function, as such, unless and until this function itself is a constant. Or in other words, this function cannot be written in the Liouville's form, or we cannot use the separation of variable, when the same Hamiltonian is written in the Cartesian form. So the moral of the story here is, that the transformation process from one coordinate frame to the other may affect the solution methodology. So, finally, let me write down the statement

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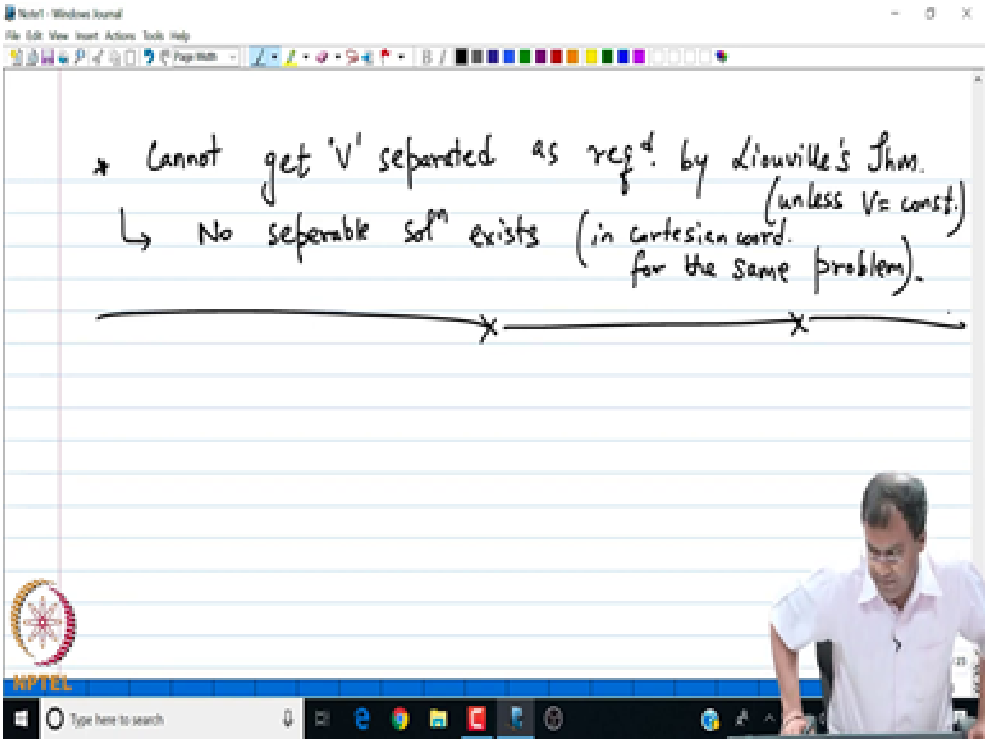

So, what I said is the following in this example we cannot get V separated as required by Liouville's theorem, unless V is a constant. So, which means, no separable solution exists, for the same problem in Cartesian coordinates, for the same problem. However, in polar, we saw that a separable solution exists.

So, I end my discussion here due to the lack of time, but we are going to continue and finish our discussion on the results of how can we separate solutions for higher dimensions and also for non-orthogonal coordinate. But more importantly, we are going to look at a very very important result by Nother which gives the relation between finding the conservation loss and the so called transformations, which reduces our functional, the function or the integrand in the functional. So, thank you very much for listening. Thanks a lot