Variational Calculus and its Applications in Control Theory and Nano mechanics
Professor Sarthok Sircar
Department of Mathematics
Indraprastha Institute of Information Technology, Delhi
Lecture 40
Hamilton-Jacobi Equations Part 4

In today's lecture I am going to continue our discussion on Hamilton-Jacobi equation which we started in the last lecture.

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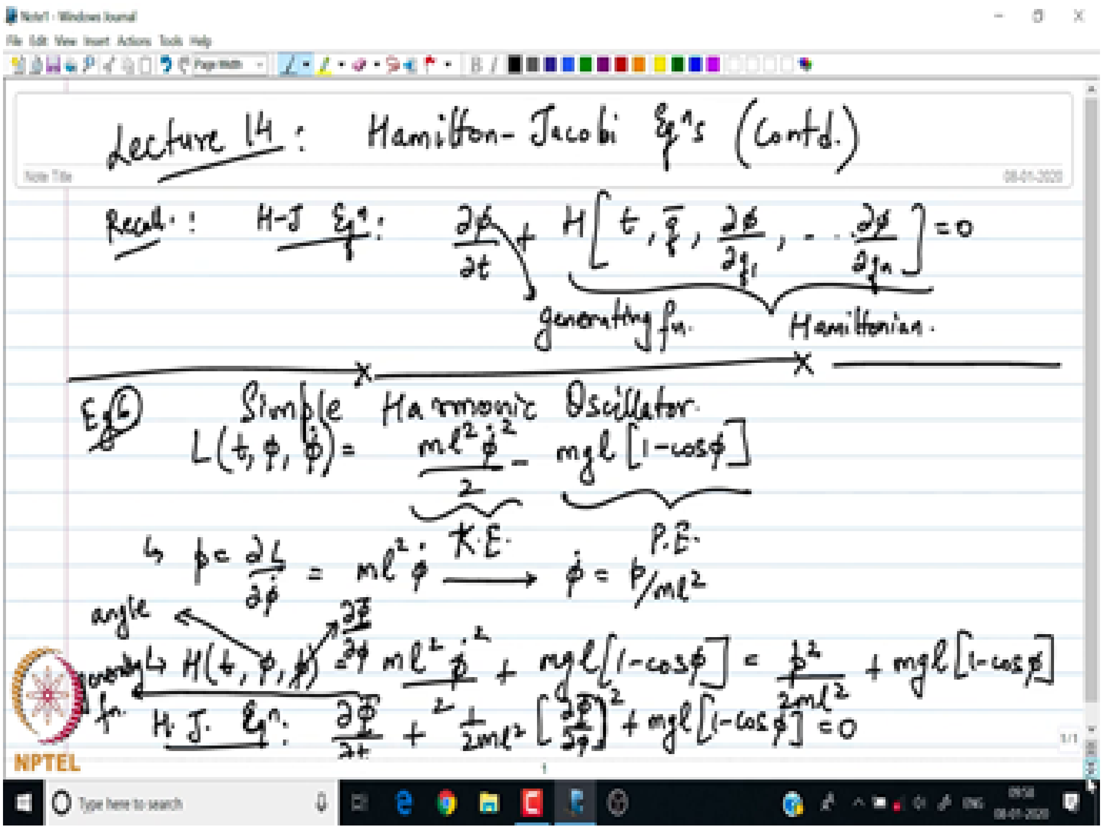

I am going to talk about specifically, how to solve the Hamilton-Jacobi equation for a special class of Hamiltonian problems, before I do that, I want to continue our discussion. let us recall what is our Hamilton-Jacobi equation. This is an equation through which we were able to solve for the generating function which leads us to the symplectic map between one Hamiltonian system to the other.

Recall the Hamilton-Jacobi equation was of the form

$$\frac{\partial \phi}{\partial t} + H\left[t, \bar{q}, \frac{\partial \phi}{\partial q_1}, ...\frac{\partial \phi}{\partial q_n}\right] = 0$$

where ϕ is generating function which gives us the symplectic map we are after and H is our Hamiltonian in the original frame of reference. So let us continue this discussion on the solution to this equation. I have an example to begin with, I am going to continue our numbering of the examples starting from our previous lecture.

Example 6: I am going to discuss Simple Harmonic Oscillators. So, in this case, the Lagrangian for the simple harmonic oscillator is as follows

$$L(t,\phi,\dot{\phi}) = \frac{ml^2\dot{\phi}^2}{2} - mgl[1-\cos\phi]$$

note now, the next stage of solving the Hamilton-Jacobi is to write the conjugate variables. To determine the conjugate variables we have

$$p = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{p}{ml^2}$$

and Hamiltonian is the sum of the kinetic plus the potential energy

$$H(t,\phi,p) = \frac{ml^2\dot{\phi}^2}{2} - mgl[1-\cos\phi] = \frac{p^2}{2ml^2} + mgl[1-\cos\phi]$$

Now notice that we have the variable p instead of $\dot{\phi}$

$$\text{From Hamilton-Jacobi equation} \frac{\partial \Phi}{\partial t} + \frac{1}{2ml^2}\left[\frac{\partial \Phi}{\partial \phi}\right]^2 + mgl[1-\cos\phi] = 0$$

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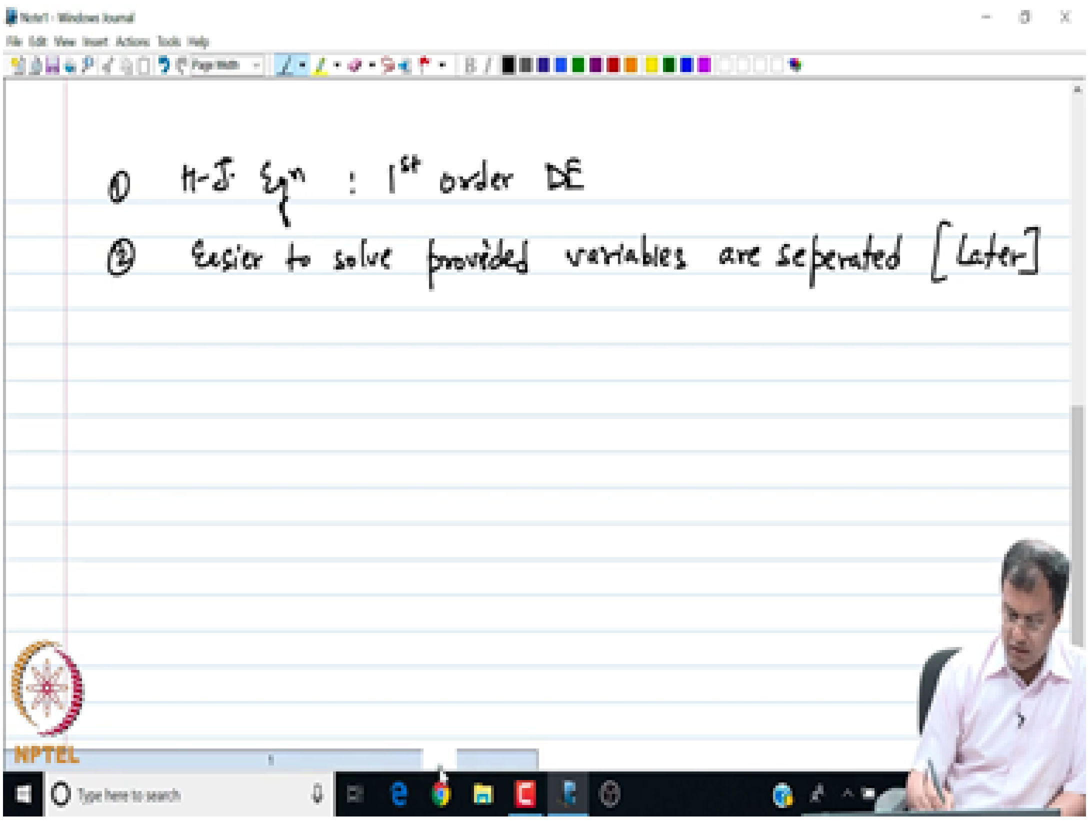
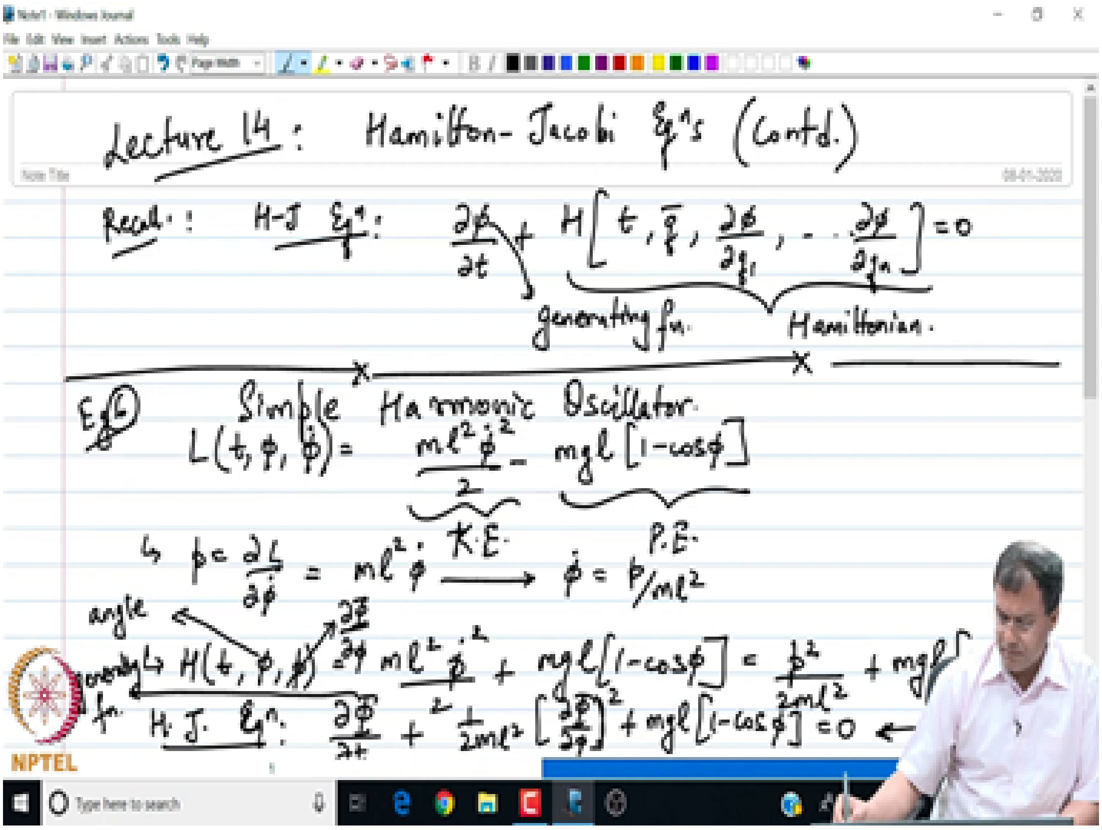


First of all, as seen in the previous lecture, the H-J equation is a first order differential equation. So it is a non-linear differential equation and later on we will see that this equation is easier to solve provided we are able to separate out the variables and we are going to talk a lot about this topic later, how to separate the variables and solve the Hamilton-Jacobi equation.

Right now, I am going to directly give my answer to the Hamilton-Jacobi equation in this example. (Refer Slide Time: 07:46)



So we have this equation that we want to solve and at this stage I am going to leave at this stage. I am going to revisit this problem in a few minutes, we will look at how to solve later for this problem. Instead, let me look at another example and we continue this discussion, arriving at a point where we have to solve for the Hamilton-Jacobi equation.

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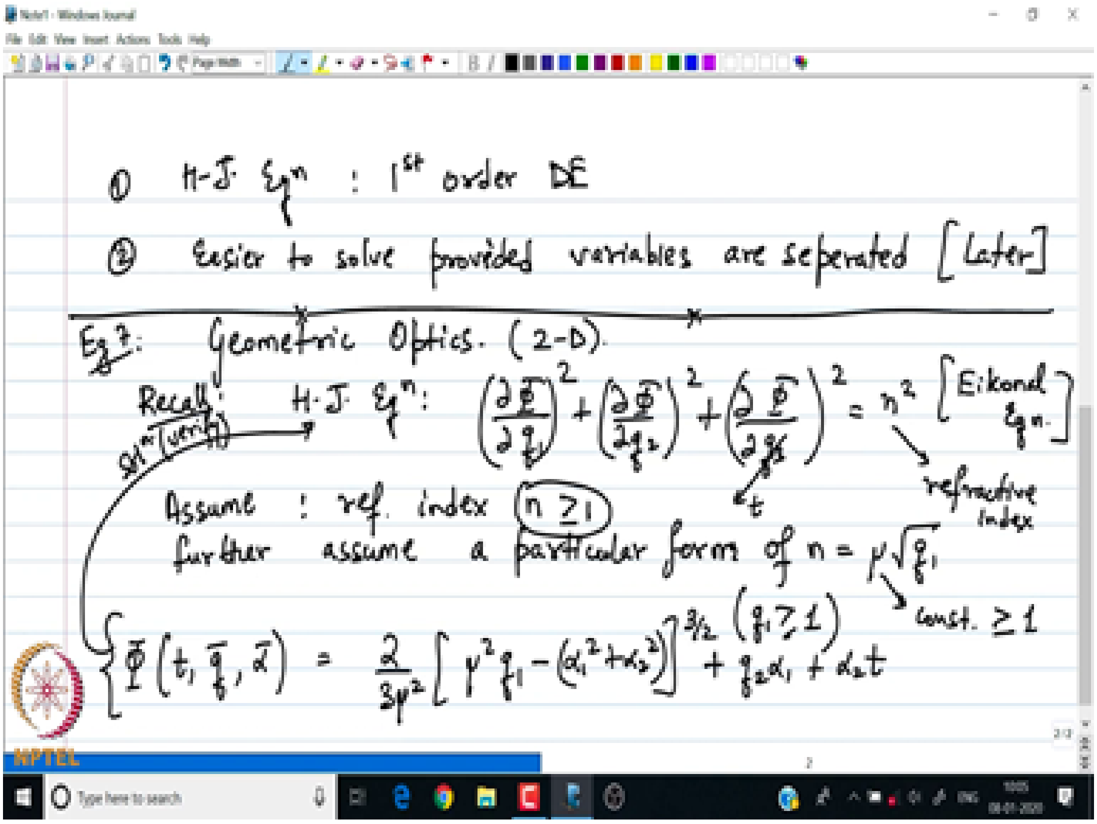

Let us continue our discussion with another example. The example that I am revisiting that I introduced in my previous lecture. So this is on the case of Geometric Optics, I am not going to write down the functional etcetera but recall that the Hamilton-Jacobi equation, in this case the so called Eikonal equation which we introduced.

$$\left(\frac{\partial \Phi}{\partial q_1}\right)^2 + \left(\frac{\partial \Phi}{\partial q_2}\right)^2 + \left(\frac{\partial \Phi}{\partial q_3}\right)^2 = n^2 \text{ where n is the refractive index}$$

This equation is the so called Eikonal equation of geometric optics, from physical considerations, we have that the right hand side, the refractive index is always greater than equal to 1. corresponds to the refractive index of vacuum or air, so that is the base index. so we assume that my refractive index n is greater than equal to 1. This is purely due to physical consideration and further we assume a particular form of the refractive index so that we are able to solve this equation, assume a particular form of $n = \mu\sqrt{q_1}$ where μ is a constant and taken to be greater than equal to 1.

Now further since n is greater than equal to 1, we assume the first coordinate q_1 is also greater than equal to 1, that is again coming from the physical constraint here. well, for this problem, I do have the solution. Again, I am not going to show how to solve the problem because I have devoted the solution to the Hamilton-Jacobi equation in the latter half of this lecture as well as the next lecture.

So I am going to right away write down the solution to example. The solution is

$$\Phi(t, \bar{q}, \bar{\alpha}) = \frac{2}{3\mu^2}\left[\mu^2 q_1 - (\alpha^2 + \alpha^2)\right]^{\frac{3}{2}} + q_2\alpha_1 + \alpha_2 t$$

students should check that this indeed satisfies my equation. So this is indeed a solution to this equation. Now what we can do is, we can check certain properties of this function, namely, first of all, we can check whether this solution is complete or not. Namely we have to construct the matrix or the Hessian matrix and find the determinant of this matrix, whether that determinant is non-zero or not.

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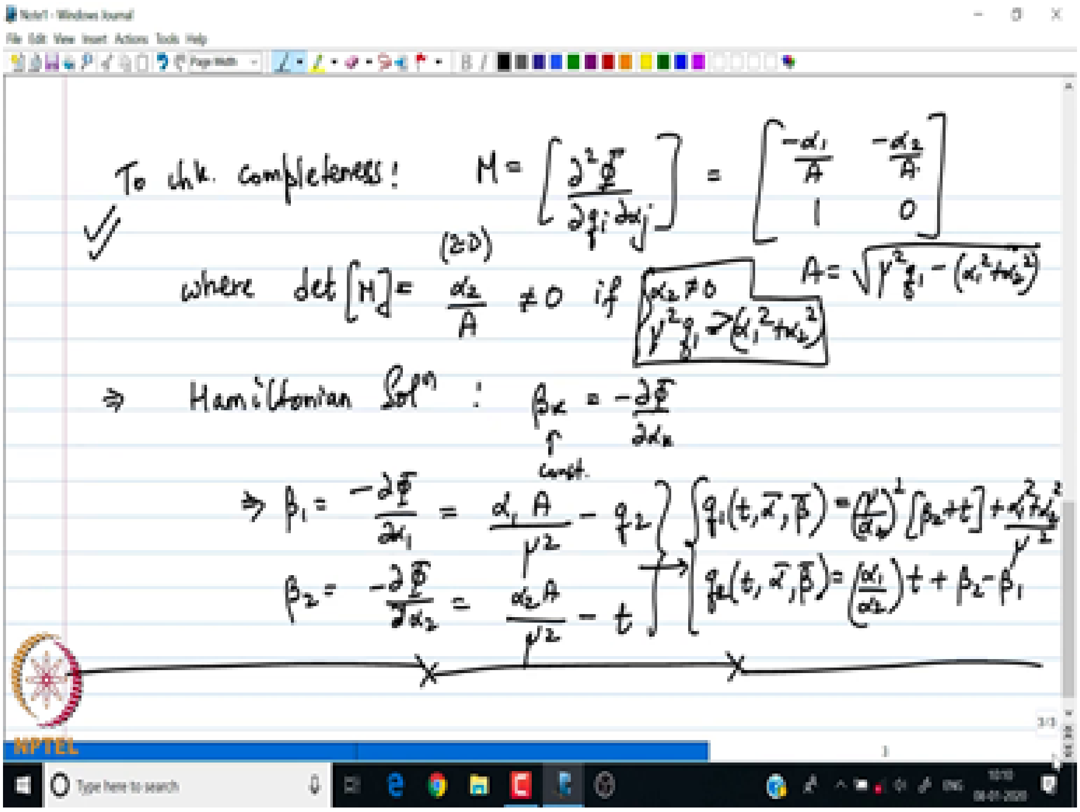

$$\text{To check completeness} \;\; M = \left[\frac{\partial^2 \Phi}{\partial q_i \partial \alpha_j}\right] = \begin{bmatrix} \frac{-\alpha_1}{A} & \frac{-\alpha_2}{A} \\ 1 & 0 \end{bmatrix} \quad textwhere A = \sqrt{\psi^2 q_1 - (\alpha_1^2 + \alpha_2^2)}$$

$$\text{where Det}[H] = \frac{\alpha_2}{A} \neq 0 \;\; \text{If} \;\; \alpha_2 \neq 0, \; \psi^2 q_1 > (\alpha_1^2 + \alpha_2^2)$$

Once we put these two sets of constraint, we can directly find out our extremal solutions q_1 and q_2 from this generating function ϕ it implies that my Hamiltonian solution is given by

$$\beta_k = -\frac{\partial \Phi}{\partial \alpha_k}$$

$$\Rightarrow \quad \beta_1 = -\frac{\partial \Phi}{\partial \alpha_1} = \frac{\alpha_1 A}{\psi^2} - q_2, \quad \beta_2 = -\frac{\partial \Phi}{\partial \alpha_2} = \frac{\alpha_2 A}{\psi^2} - t$$

From here I invert this relation to find q_1 and q_2

$$q_1(t, \bar{\alpha}, \bar{\beta}) = \left(\frac{\psi}{\alpha_2}\right)^2 [\beta_2 + t] + \frac{\alpha_1^2 + \alpha_2^2}{2}, \;\; q_2(t, \bar{\alpha}, \bar{\beta}) = \left(\frac{\alpha_1}{\alpha_2}\right) t + \beta_2 - \beta_1$$

At this point I have just shown some examples as to how to come to a point where we have the Hamilton-Jacobi equation and further for certain cases, how can we solve.

Well, we have not shown how can we solve, but once we have the solution to the Hamilton-Jacobi, how can we derive the extremal to the Hamiltonian system. So, the next set of discussions will involve how to solve the Hamilton-Jacobi equation, namely, we are going to see the solution of the Hamilton-Jacobi equation for a specific class of Hamiltonian system known as the conservative functions or conservative systems.

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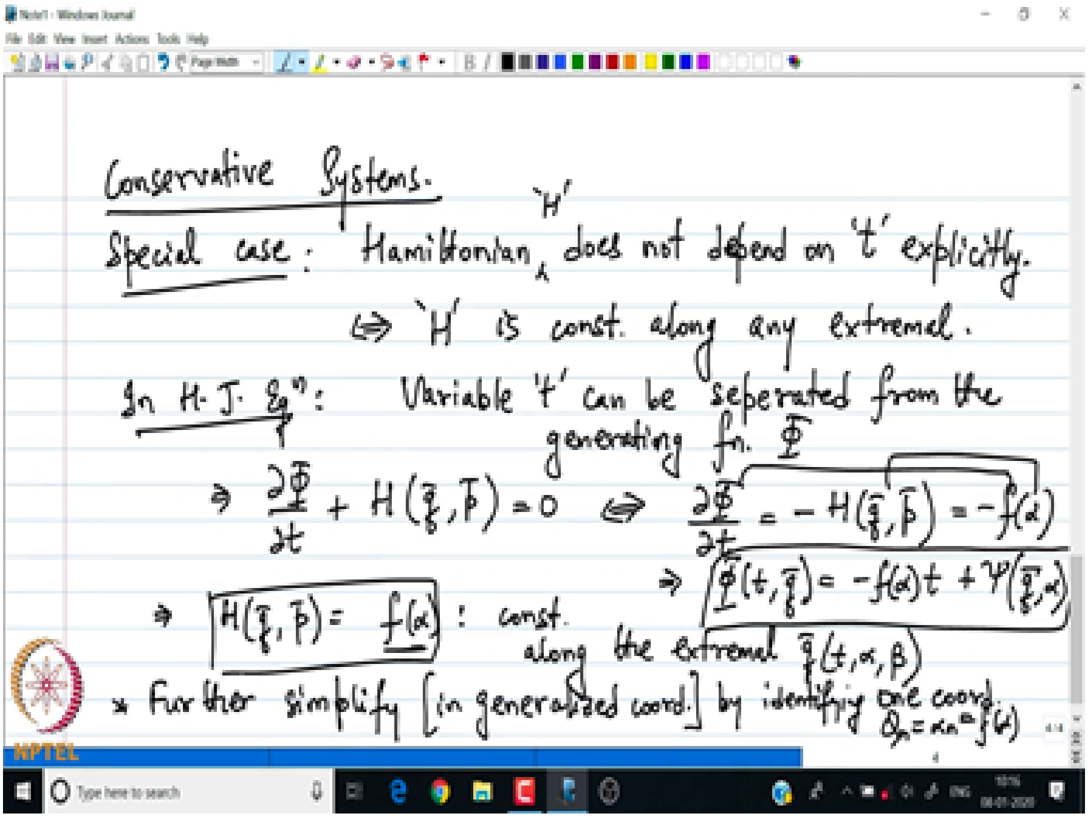

Let us see what do we mean by conservative systems, this is a special case or special class in which the Hamiltonian satisfies certain properties, when I have the Hamiltonian which does not depend on the independent variable, when the Hamiltonian does not depend on t explicitly, we see that we have arrived at the conservative system. Now, we have certainly seen few examples in this category namely the cases which involved Beltrami identity in which the integrand did not contain the independent variable x.

In this class, the moment H does not depend on the independent variable, it is easy to show that the Hamiltonian is described by H is constant along any extremal. So, we will now see the solution to this class of Hamiltonian system. Now let us write the Hamilton-Jacobi equation. So in H-J equation the variable t can now be separated from the generating function $\hat{\Phi}$

$$\Rightarrow \quad \frac{\partial \Phi}{\partial t} + H(\bar{q}, \bar{p}) = 0 \Leftrightarrow \frac{\partial \Phi}{\partial t} = -H(\bar{q}, \bar{p}) = 0 = -f(\alpha)$$

$$\Rightarrow \quad \Phi(t, \bar{q}) = -f(\alpha)t + \psi(\bar{q}, \alpha)$$

$$\Rightarrow \quad H(\bar{q}, \bar{p}) f(\alpha) \ \text{ constant along the extremal } \ \bar{q}(t, \alpha, \beta)$$

Further we can simplify by identifying one of the coordinates with this constant, this constant is special because this will help us to reduce our Hamilton-Jacobi equation.

Further simplify, we can further simplify, in generalized coordinates $Q_n = \alpha_n = f(\alpha)$ So, let us now look at what do we have here? We see that we have a special form of Hamilton-Jacobi equation namely the reduced form.

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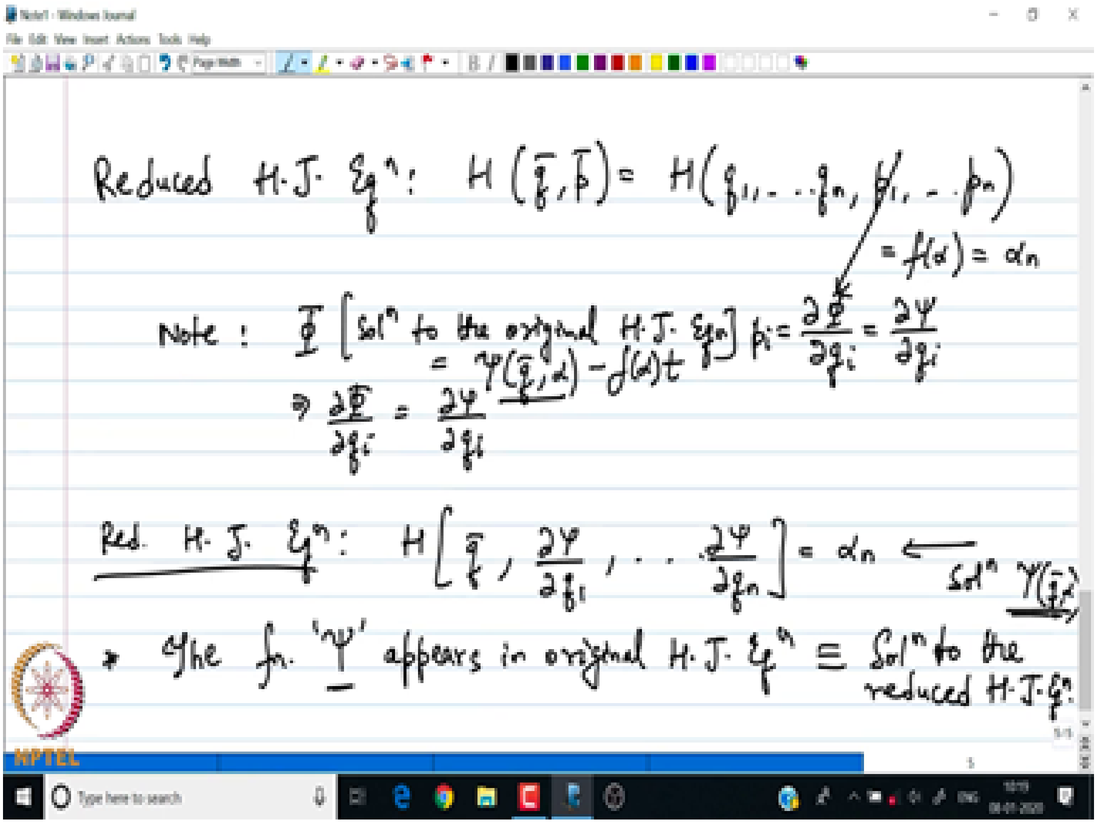

Reduced Hamilton-Jacobi equation, where my Hamiltonian does not depend on the independent variable

$$H(\bar{q},\bar{p}) = H(q_1,.......q_n,p_1,.....p_n) = f(\alpha) = \alpha_n$$

Now, further I have been saying that wherever we get momentum variables, we replace $p_i = \frac{\partial \Phi}{\partial q_i}$ because that is the relation for the generating function in terms of the conjugate variables. Note that Φ is the solution to the original Hamilton-Jacobi equation and we saw that $\Phi = \psi(\bar{q},\alpha) - f(\alpha)t$

$$\Rightarrow \quad \frac{\partial \Phi}{\partial q_i} = \frac{\partial \psi}{\partial q_i}$$

Reduced Hamilton-Jacobi equation now reads as follows

$$H\left[\bar{q},\frac{\partial \psi}{\partial q_1},........,\frac{\partial \psi}{\partial q_n}\right] = \alpha_n$$

solution to the reduced Hamilton-Jacobi equation is ψ. Note that the function ψ which is appearing in the original solution to the original Hamilton-Jacobi equation, this function ψ is the solution to the reduced Hamilton-Jacobi.

So what is it? What is the relation between the original Hamilton-Jacobi and reduced Hamilton-Jacobi? It turns out that the function psi which appears in the original mind us that ψ is not the solution to the original Hamilton-Jacobi. It only appears there but it is the solution to the reduced Hamilton-Jacobi, or the Hamilton-Jacobi corresponding to the conservative system. So then, some of the other issues to talk about is, how about the completeness of this solution?

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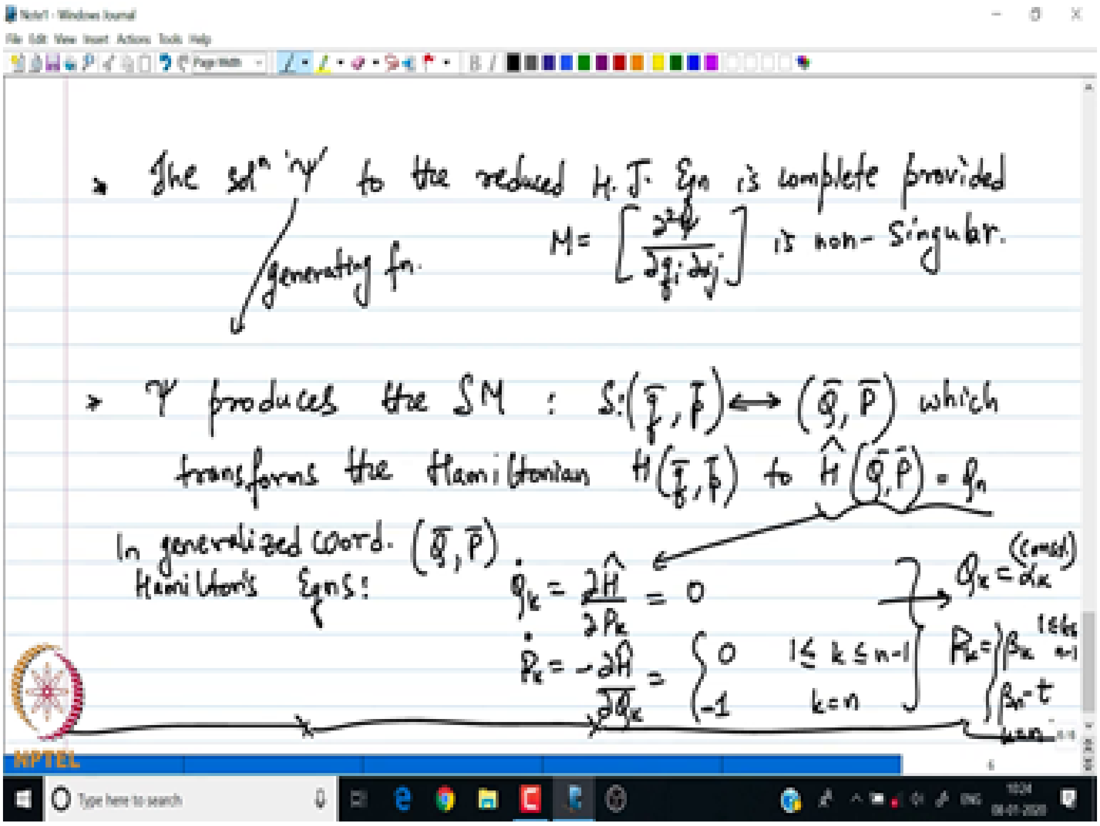

It turns out that the solution ψ reduced Hamilton-Jacobi equation is complete provided $M = \left[\frac{\partial^2 \psi}{\partial q_i \partial \alpha_j}\right]$ is non-singular, the same idea that we had shown for the general class of Hamilton-Jacobi equation. So it turns out that for this reduced case ψ is our generating function.

this is our generating function to the reduced Hamilton-Jacobi and so ψ produces the required symplectic map which is $S : (\bar{q}, \bar{p}) \leftrightarrow (\bar{Q}, \bar{P})$ which transforms the Hamiltonian $H(\bar{q}, \bar{p})$ to $\hat{H}(\bar{Q}, \bar{P}) = Q_n$ which means from here we can see what is my Hamilton's equation. So in generalized coordinate, when I say generalized coordinate, I talk about coordinates which are $(\bar{Q}, \bar{P})$.

My Hamilton's equation will provide the following set of constraints

$$\dot{Q}_k = \frac{\partial \hat{H}}{\partial P_k} = 0$$

$$\dot{P}_k = -\frac{\partial \hat{H}}{\partial Q_k} = \begin{cases} 0 & 1 \leq k \leq n-1 \\ -1 & k = n \end{cases}$$

$$Q_k = \alpha_k \ \text{ are constants } \ P_k = \begin{cases} \beta_k & 1 \leq k \leq n-1 \\ \beta_k - t & k = n \end{cases}$$

we are ready to look at examples in this Hamiltonian system of conservative functions, let me just summarize the solution strategy or how to find the solution for this class of problems.

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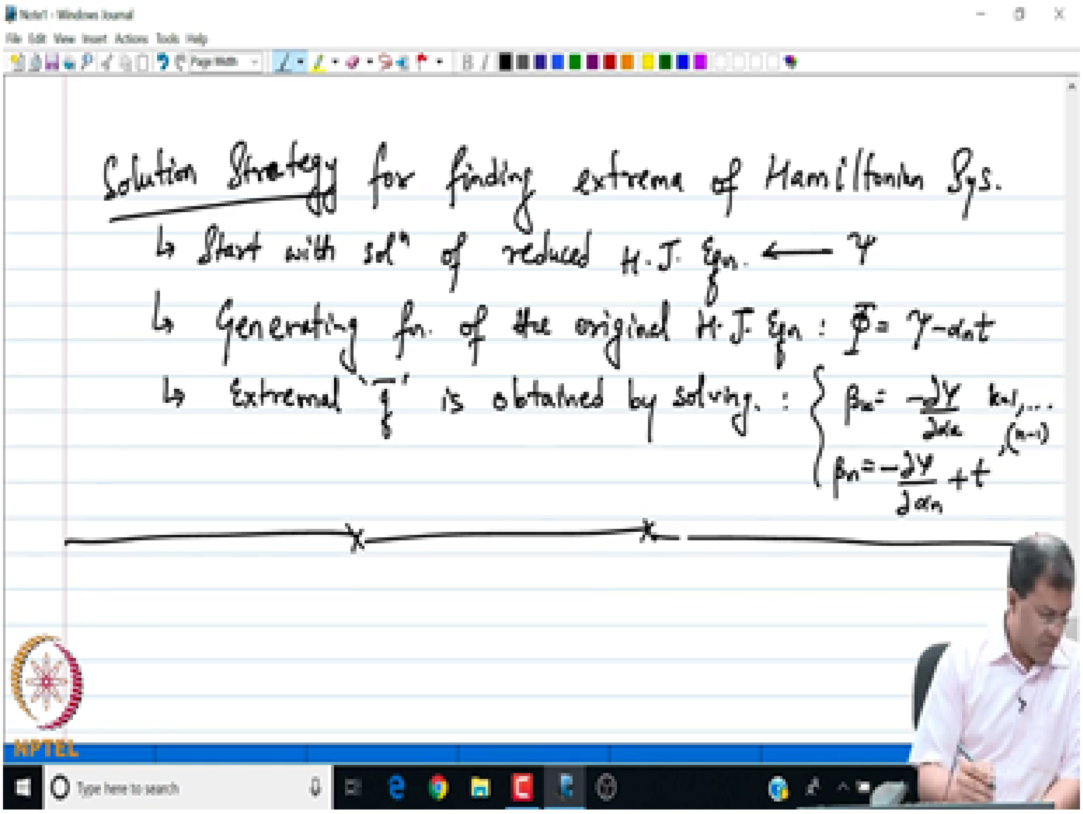

So solution strategy for finding extrema of the Hamiltonian system, we start with solution of reduced Hamilton-Jacobi equation and then the generating function, once we have the solution to the reduced Hamilton-Jacobi which is ψ, the generating function of the original Hamilton-Jacobi equation is $\phi = \psi - \alpha_n t$.

finally once we have the original generating function, we can derive the extremals from our relations as follows, the extremal, $\bar{q}$ is obtained by solving the following set of equations $\beta_k = -\frac{\partial \psi}{\partial \alpha_k}$, $k = 1, ...(n-1)$ and my solution $\beta_n = -\frac{\partial \psi}{\partial \alpha_n} + t$ the nth component, from here I can get my extremal. So let us look at some examples for this conservative system.