Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 39

Hamilton-Jacobi Equations Part 3

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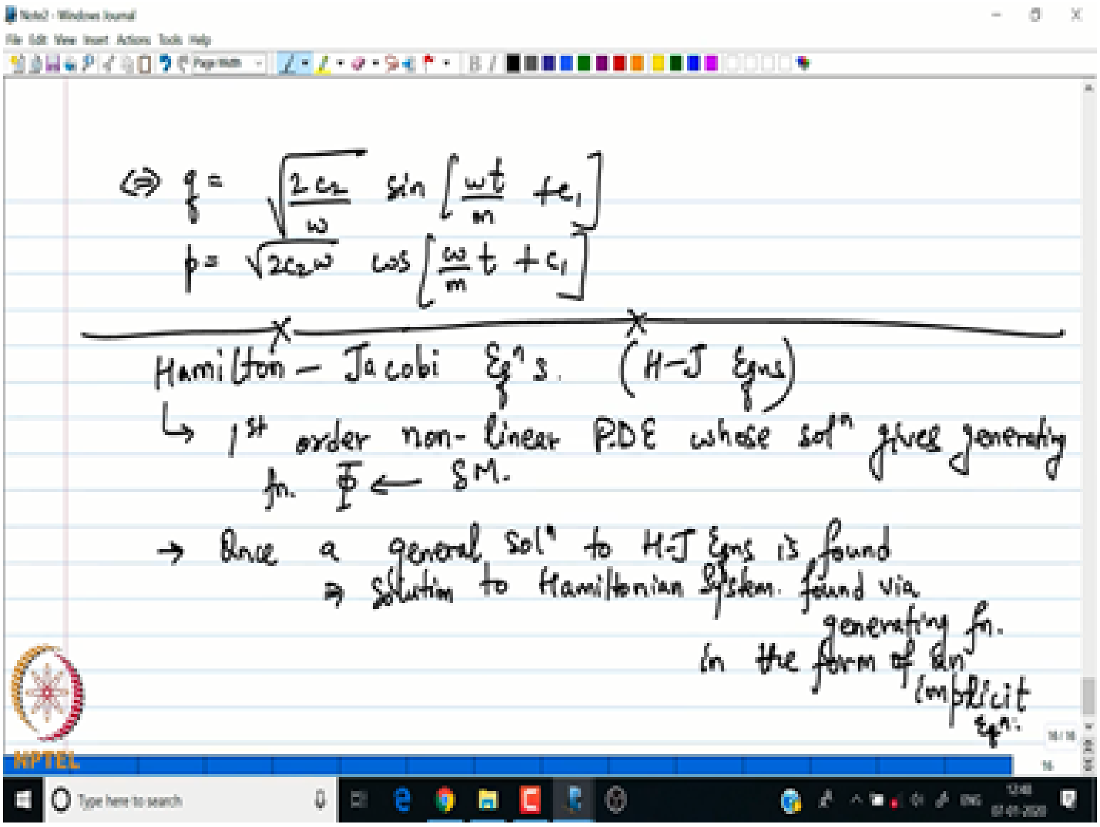

So, thus question is how to find this generating function that is the biggest hurdle. To do that we have to find out the generating function by the so called Hamilton-Jacobi equations, I call this as the H-J equations, so H-J equations are the first order nonlinear a partial differential equations whose solution gives the generating function, the generating function ϕ which gives the symplectic map, which converts from one Hamiltonian system to the other. So, thus question is what is this equation? once we have solved the Hamilton-Jacobi equation we can readily find the solution to the Hamiltonian via the set of implicit equations. So, what I said is the following.

A general solution to the H-J equations is found then the solution to the Hamiltonian system is found via the generating function in the form of an implicit equation. We are going to see what do I mean by this statement very soon through an example So, let me now describe this Hamilton-Jacobi equation.

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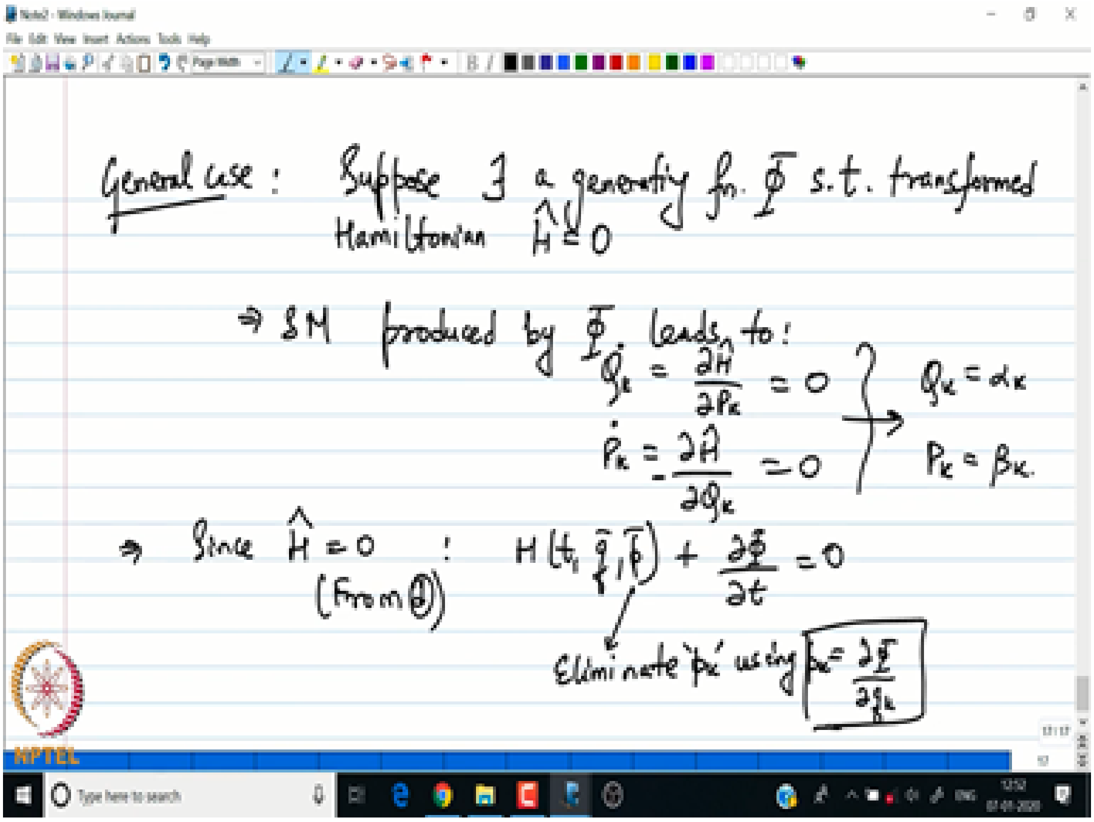

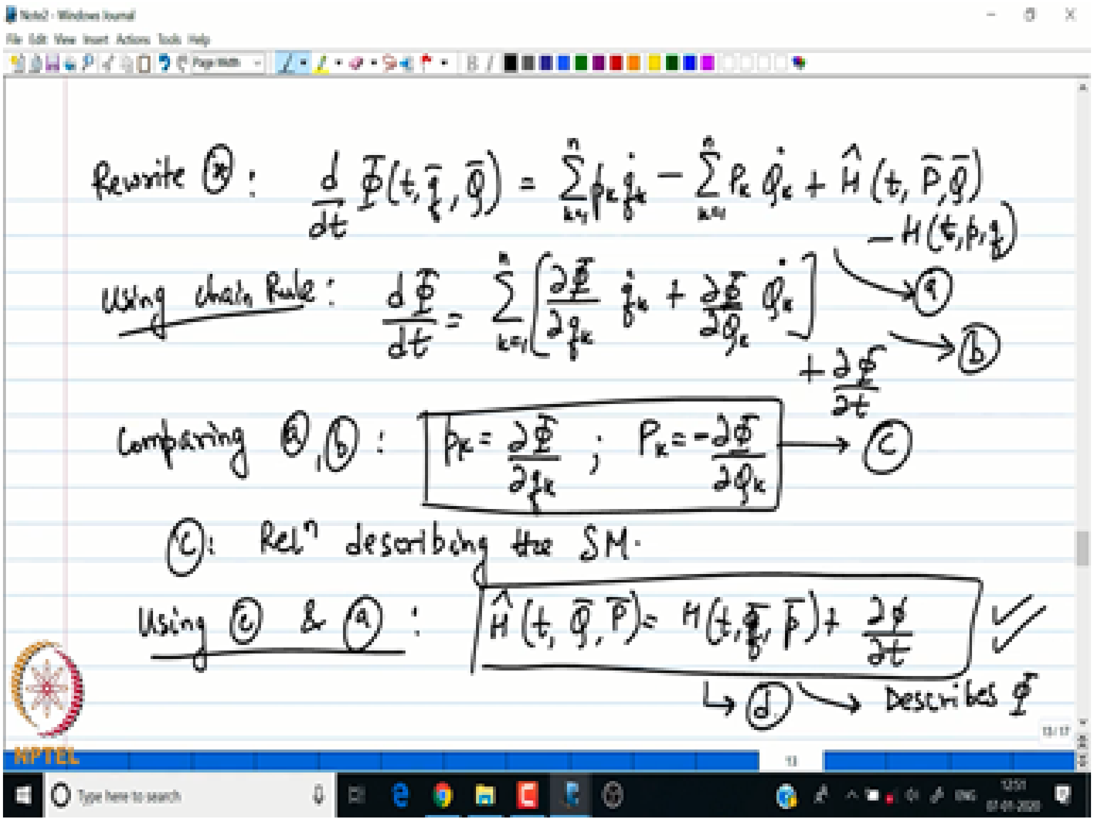

Note we are going to do that in the general case, suppose there exists a generating function ϕ such that the transformed Hamiltonian $\hat{H} = 0$ So, what am I saying is the generating function is such that the Hamiltonian in the new coordinate frame P, Q is identically equal to 0, as simple as that.

So, then it implies that the symplectic map produced by ϕ leads to the following set of equations

$$\dot{Q}_k = \frac{\partial \hat{H}}{\partial P_k} = 0 \quad \Rightarrow \quad Q_k = \alpha_k$$

$$\dot{P}_k = \frac{\partial \hat{H}}{\partial Q_k} = 0 \quad \Rightarrow \quad P_k = \beta_k$$

So, now I am saying in this equation I am going to plug $\hat{H} = 0$ and when we do that, we get this relation, let me call this relation as d.

$$\text{From d} \qquad H(t, \bar{q}, \bar{p}) + \frac{\partial \Phi}{\partial t} = 0$$

because $\hat{H} = 0$ and notice that, final relation should not contain any conjugate variables because conjugate variables are a new set of variables which we have to describe later.

So, we eliminate 'p_k' using $p_k = \frac{\partial \Phi}{\partial q_k}$ that is by our Hamilton's well, that is the relation obtained from the generating function itself.

To find the generating function we have to solve this first order non-linear partial differential equation.

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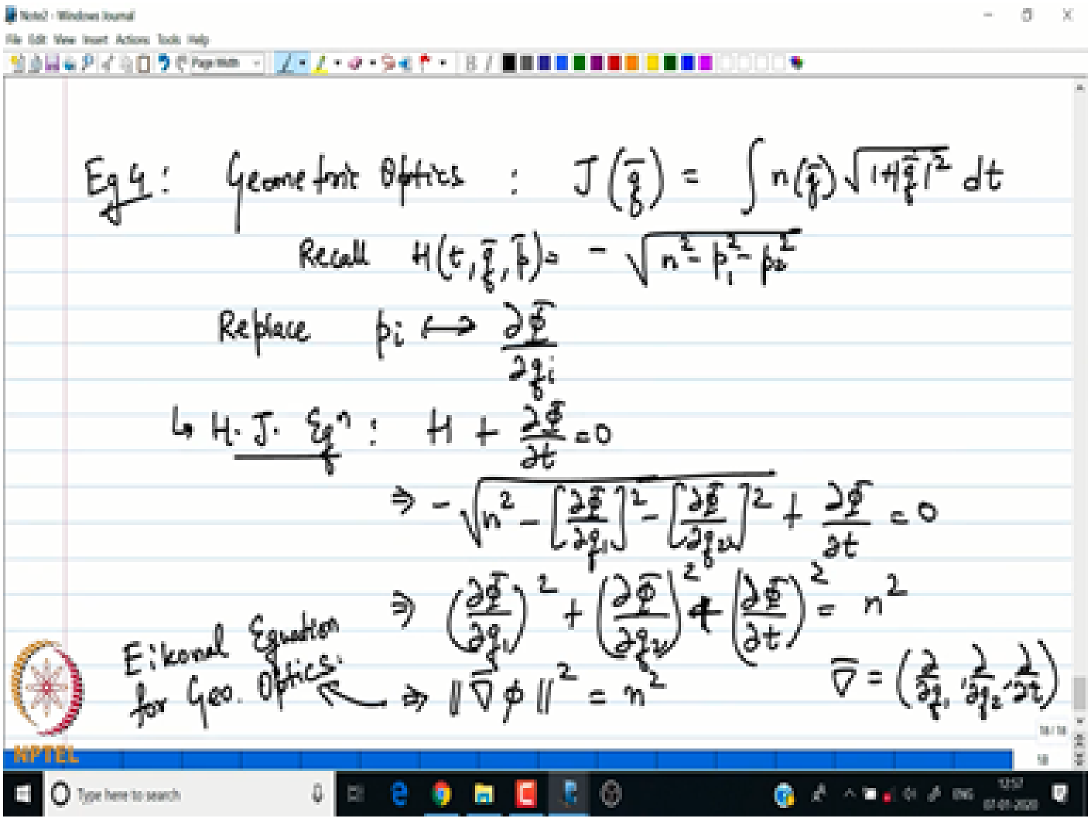

Let us look at an example. We revisit our example of geometric optics where our functional

$$J(\bar{q}) = \int n(\bar{q})\sqrt{1 + |\bar{q}|^2}dt$$

$$\text{Recall } H(t,\bar{q},\bar{p}) = -\sqrt{n^2 - p_1^2 - p_2^2}$$

And we are going to replace our p_i's by $\frac{\partial\Phi}{\partial q_i}$ and Hamilton-Jacobi equation in this case will be $H + \frac{\partial\Phi}{\partial t} = 0$

$$\Rightarrow \quad -\sqrt{n^2 - \left[\frac{\partial\Phi}{\partial q_1}\right]^2 - \left[\frac{\partial\Phi}{\partial q_2}\right]^2} + \frac{\partial\Phi}{\partial t} = 0$$

$$\Rightarrow \quad \left(\frac{\partial\Phi}{\partial q_1}\right)^2 + \left(\frac{\partial\Phi}{\partial q_2}\right)^2 + \left(\frac{\partial\Phi}{\partial t}\right)^2 = n^2$$

Now, it seems I can write down this equation in a more compact form. This is my Hamilton's Jacobi equation.

$$\Rightarrow \quad \left\|\bar{\nabla}\phi\right\|^2 = n^2, \quad \text{where} \quad \bar{\nabla} = \left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}\right)$$

Notice this equation is independent of t, ϕ is independent variable. Then this equation is nothing but the generalized Poisson equation.

But given the time derivative this is the well known Eikonal equation for geometric optics. So, people working in geometric optics will readily know this equation. So, I am not going to go beyond by and go ahead and solve this because it is quite complicated but we will look at a few more relations that should be true for ϕ and we will look at some simpler case in order to see the power of Hamilton-Jacobi equation. So, few more concepts I want to introduce.

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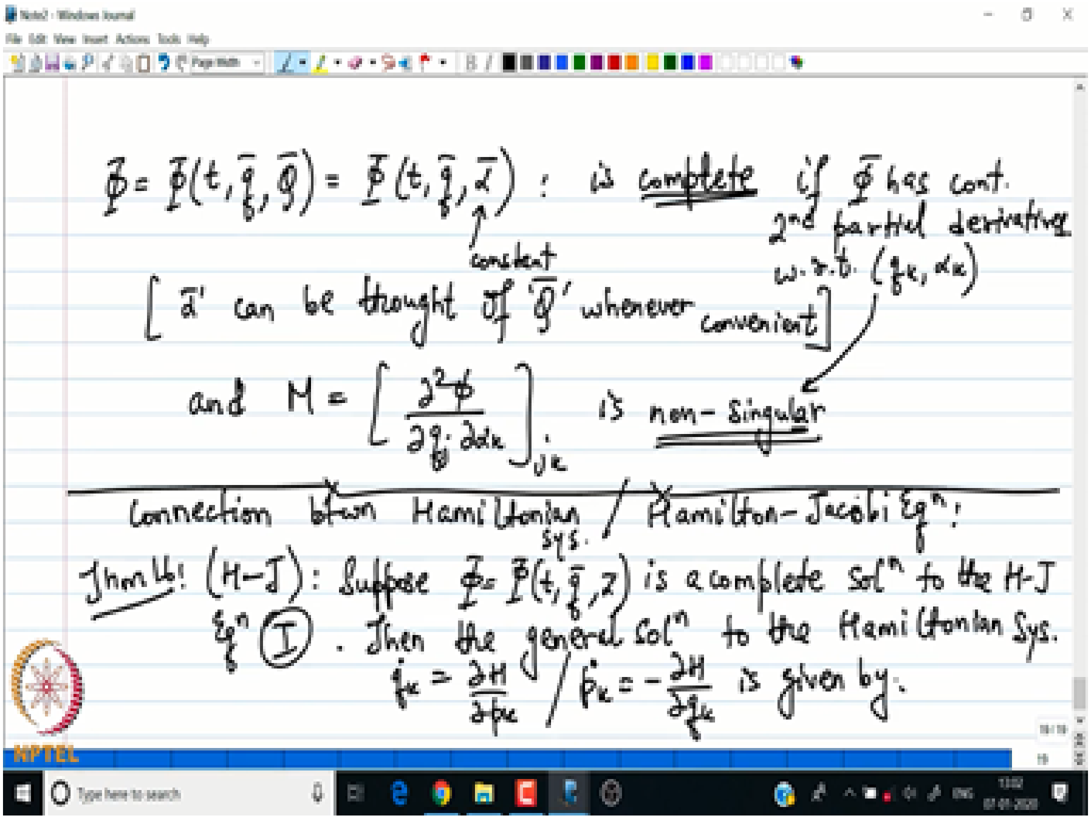
$\Phi = \Phi(t,\bar q,\bar Q) = \Phi(t,\bar q,\bar\alpha)$: is complete if Φ has cont. 2nd partial derivatives w.r.t. (q_k, α_k)
constant
['$\bar\alpha$' can be thought of '$\bar Q$' whenever convenient]
and $M = \left[\frac{\partial^2 \phi}{\partial q_j \partial \alpha_k}\right]_{jk}$ is non-singular
Connection b/wn Hamiltonian Sys. / Hamilton-Jacobi Eqn:
Thm 16: (H-J): Suppose $\Phi = \Phi(t,\bar q,\bar\alpha)$ is a complete Solⁿ to the H-J Eqⁿ (I). Then the general Solⁿ to the Hamiltonian Sys.
$\dot q_k = \frac{\partial H}{\partial p_k}$ / $\dot p_k = -\frac{\partial H}{\partial q_k}$ is given by:
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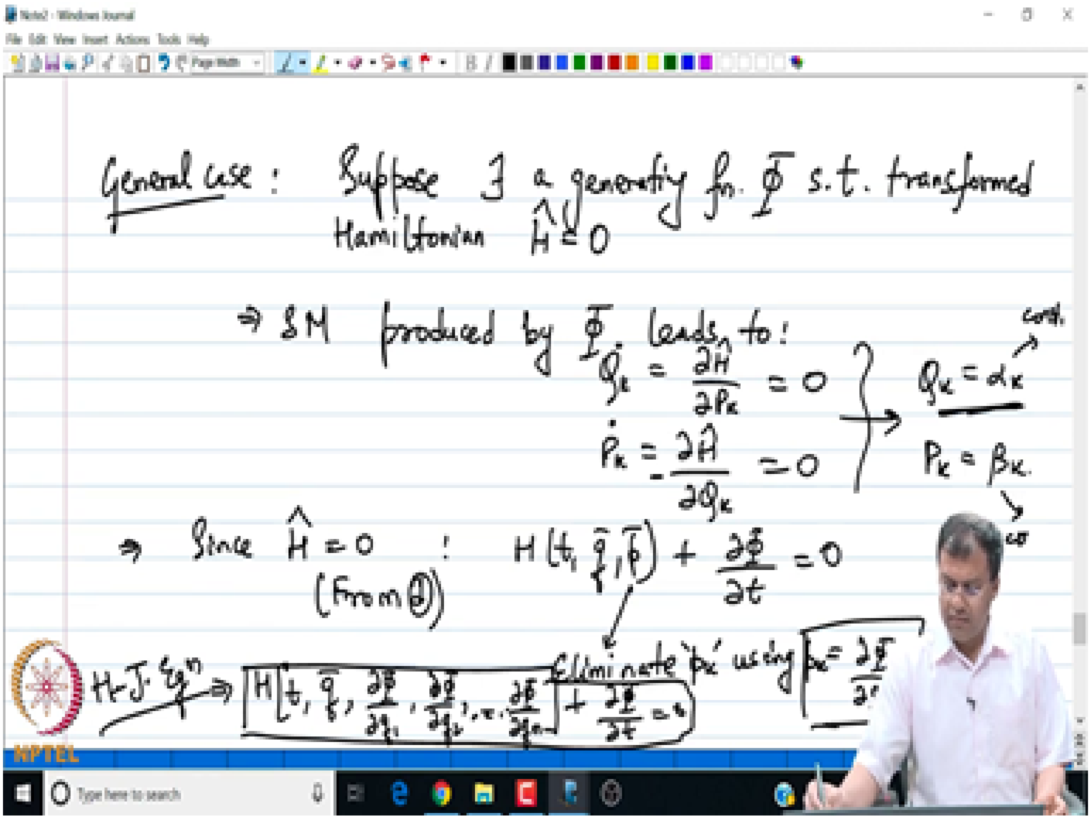
General case: Suppose ∃ a generating fn. Φ s.t. transformed Hamiltonian $\hat H = 0$
⇒ SM produced by Φ leads to:
$\dot Q_k = \frac{\partial \hat H}{\partial P_k} = 0$
$\dot P_k = \frac{\partial \hat H}{\partial Q_k} = 0$
$Q_k = \alpha_k$ const.
$P_k = \beta_k$
⇒ Since $\hat H = 0$ (From (2)) : $H(t,\bar q,\bar p) + \frac{\partial\Phi}{\partial t} = 0$
Eliminate 'p_k' using $p_k = \frac{\partial \Phi}{\partial q_k}$
H-J Eqⁿ ⇒ $H\left[t, \bar q, \frac{\partial\Phi}{\partial q_1}, \frac{\partial\Phi}{\partial q_2}, \dots, \frac{\partial\Phi}{\partial q_n}\right] + \frac{\partial\Phi}{\partial t} = 0$
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$$\Phi = \Phi(t, \bar{q}, \bar{Q}) = \Phi(t, \bar{q}, \bar{\alpha}) \quad \text{where } \bar{\alpha} \text{ is a constant}$$

Where did this substitution come from? Again, If we go back a few slides notice that the symplectic map for the Hamilton-Jacobi equation led to the following that Q is equal t α.

So, that is the reason that we substituted $\bar{Q}$to be $\bar{\alpha}$ where α and β are constants, So, the definition of complete has to do with the definition of the necessary existence of derivatives, we see that this generating function is complete. If Φ has continuous second partial derivatives with respect to its variables (q_k, α_k).

Notice that on some occasions I am conveniently saying that α_k is constant. And on some other occasions I am conveniently saying that α_k is variable. There is no confusion because α_k is nothing but our coordinates Q. So, when I take the derivative with respect to a variable they can be conveniently switched with capital Q.

So, what I am saying is $\bar{\alpha}$ can be thought of $\bar{Q}$ whenever convenient. So, if second partial derivative exists and what I have is the following Hessian matrix

$$M = \left[\frac{\partial^2 \phi}{\partial q_j \partial \alpha_k}\right]_{jk} \qquad \text{is non singular}$$

So, let us say we have a situation where we can invert the relation for Φ is complete, in that situation I can write down the relation between the Hamiltonian system and the Hamilton-Jacobi equation. So, what I am trying to do is writing down the connection between the Hamiltonian and the Hamiltonian system and the Hamilton-Jacobi equation, where does Hamilton-Jacobi equation fit in.

Theorem 16: Suppose given a function $\Phi = \Phi(t, \bar{q}, \bar{\alpha})$ is a complete solution to the Hamilton-Jacobi equation **I**. Then the general solution to the Hamiltonian system $\dot{q}_k = \frac{\partial H}{\partial p_k} / \dot{p}_k = -\frac{\partial H}{\partial q_k}$ is given by

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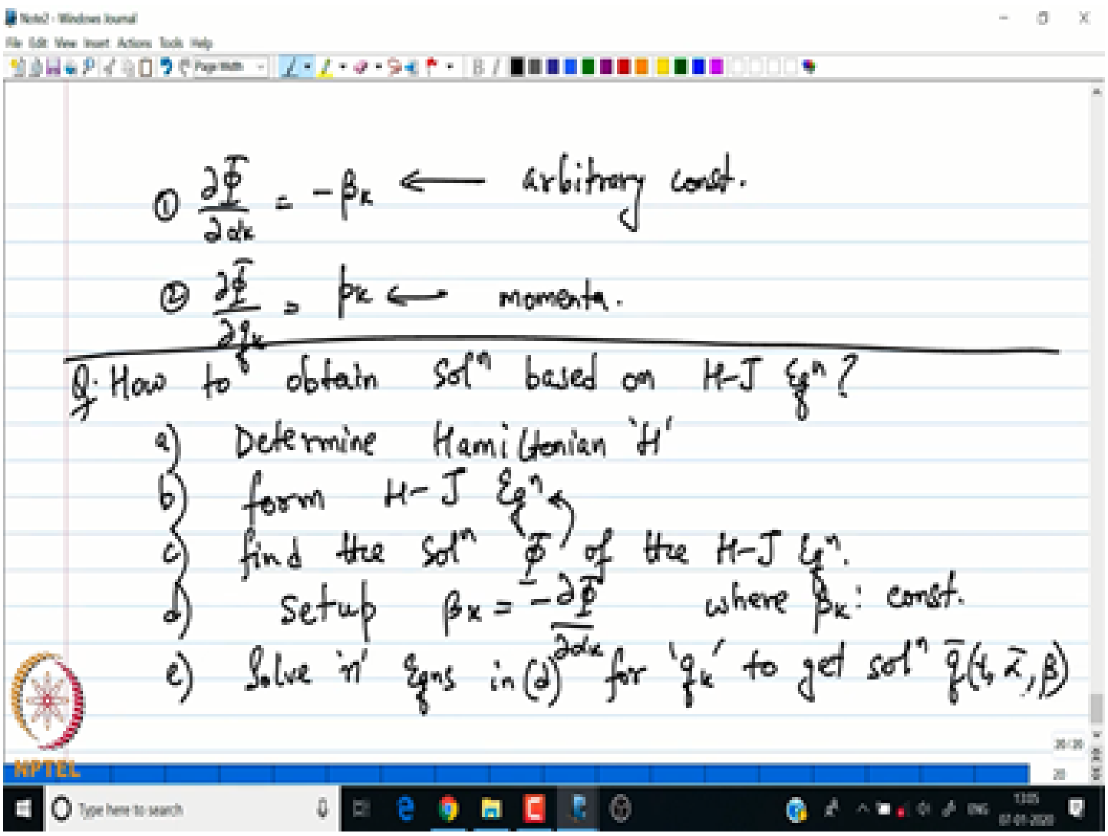

$$1) \qquad \frac{\partial \Phi}{\partial \alpha_k} = -\beta_k \qquad \text{arbitrary constant}$$

$$2) \qquad \frac{\partial \Phi}{\partial q_k} = p_k \qquad \text{momenta}$$

So, what this theorem says, I have stated this theorem without proof is that once we have solved the Hamilton-Jacobi equation and find the generating function then we can directly find the extremal solution q via the set of these two relations given by one and two from the generating function.

Thus question that we have to ask is how to obtain, so this is the gist of this theorem, how to obtain a solution based on the Hamilton-Jacobi equation. The answer is as follows, we determine Hamiltonian H function H. Then the next step is we form the Hamilton-Jacobi equation or the H-J equation with the help of the Hamiltonian.

The third step is we find the complete solution Φ of the Hamilton-Jacobi equation which is the generating of the H-J equation.

Then the fourth step is we set up the moment, we differentiate Φ with respect to the constant and set up these two equations, we set up especially the first equation $\beta_k = -\frac{\partial \Phi}{\partial \alpha_k}$ where β_k's are arbitrary constant.

And finally we solve for 'n' equations in d. So, these are my set of n equations to find my answer q_k which is my extremal solution for q_k to get the general solution $\bar{q}(t, \bar{\alpha}, \beta)$
Let us see this strategy with the help of an example.

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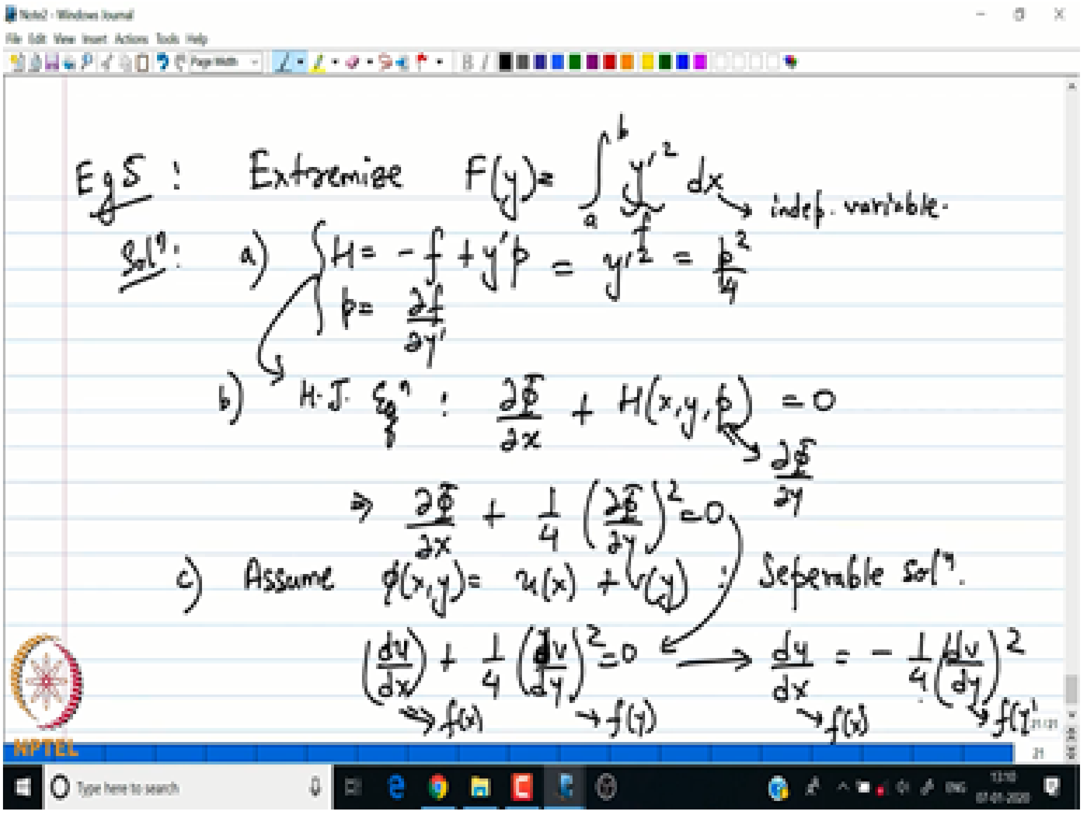

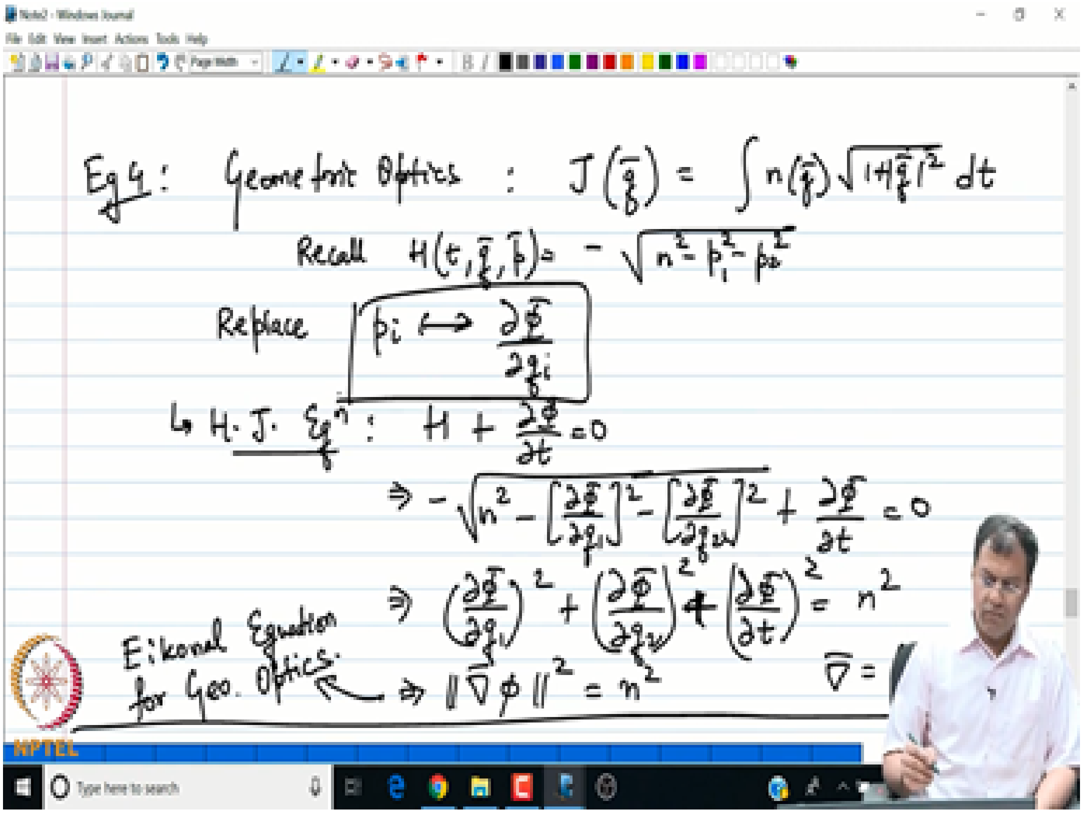

Example 5: Extremize $F(y) = \int_a^b {y'}^2 dx$ Now, this is a very very simple example and I do not want to

solve this example using Euler Lagrange but I want to use the Hamilton-Jacobi equation. And we will show that the solution obtained is identical to the one obtained by Euler Lagrange.

The first step
a) is to find the Hamiltonian. So, the Hamiltonian is given by in our previous lecture we are shown if this is my Hamiltonian is given by

$$H = -f + y^{'}p = y^{'2} = \frac{p^2}{4} \qquad \text{where} \qquad p = \frac{\partial f}{\partial y^{'}}$$

b) Hamilton-Jacobi equation is independent variables, this time is x. So, x is my independent variable,

$$\frac{\partial \Phi}{\partial x} + H(x, y, p) = 0 \qquad \text{where} \qquad p = \frac{\partial \Phi}{\partial y}$$

$$\Rightarrow \qquad \frac{\partial \Phi}{\partial x} + \frac{1}{4}\left(\frac{\partial \Phi}{\partial y}\right)^2 = 0$$

c) To find a solution to this Hamilton-Jacobi equation. We assume a solution which is variable separable.

$$\text{Assume} \qquad \Phi(x, y) = u(x) + v(y) \qquad \text{separable solution}$$

$$\left(\frac{du}{dx}\right) + \frac{1}{4}\left(\frac{dv}{dy}\right)^2 = 0 \qquad \Rightarrow \qquad \frac{du}{dx} = -\frac{1}{4}\left(\frac{dv}{dy}\right)^2$$

and that is only possible when both are constants, since they are equal to each other
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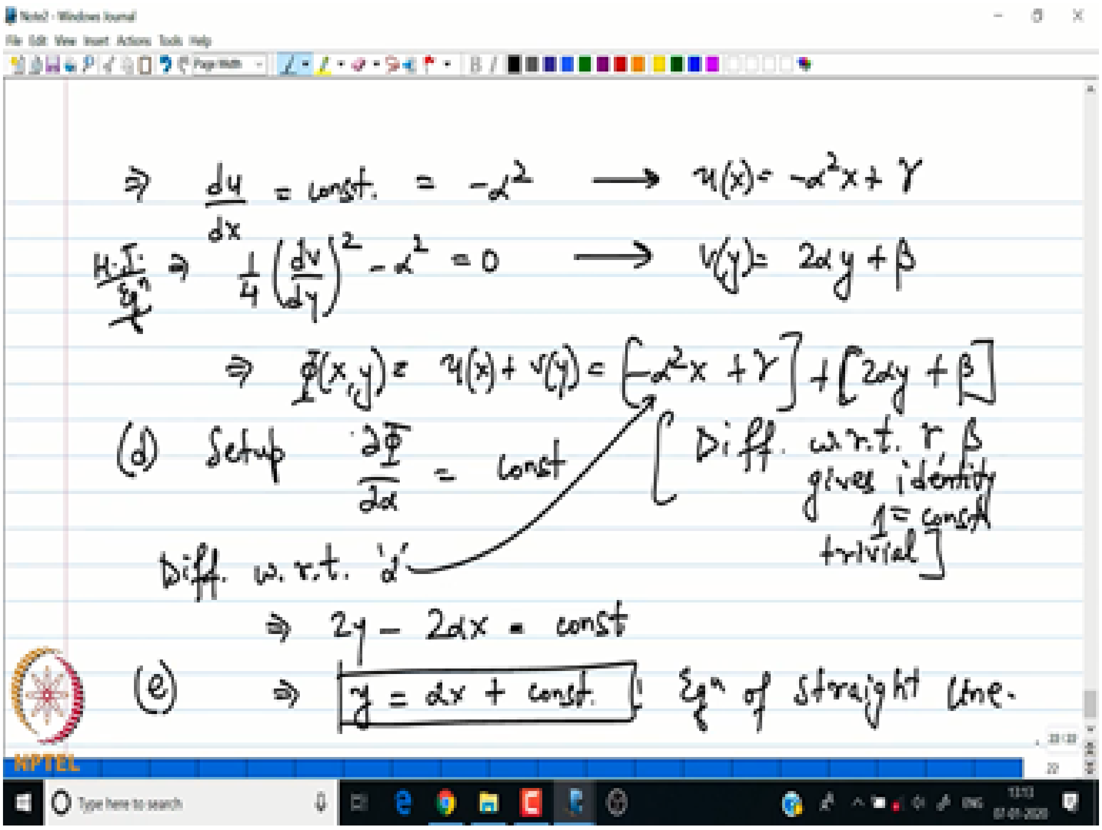

$$\Rightarrow \qquad \frac{du}{dx} = \text{ constant } = -\alpha^2 \qquad \Rightarrow \qquad u(x) = -\alpha^2 x + \gamma$$

$$\Rightarrow \quad \frac{1}{4}\left(\frac{dv}{dy}\right)^2 - \alpha^2 = 0 \quad \Rightarrow \quad v(x) = 2\alpha y + \beta$$

I do not care about the sign of α here because α is a constant it does not matter. So, I have retained the positive sign here.

$$\Rightarrow \quad \Phi(x, y) = u(x) + v(y) = \left[-\alpha^2 x + \gamma\right] + [2\alpha y + \beta]$$

Well, we have note that in Φ we have three constants α, β, γ. If you take the derivative of Φ with respect to γ or β we are going to get 1 is equal to a constant so there is no new relation. So, differentiation with respect γ and β gives an identity 1 is equal to a constant so there is no new information, this is quite trivial.

Now we are going to differentiate with respect to the constant α

$$\Rightarrow \quad 2y - 2\alpha x = \text{ constant}$$

$$\Rightarrow \quad y = 2\alpha x + \text{ constant}; \qquad \text{equation of a straight line}$$

that is the solution to the extremal that we have found directly from our Hamilton-Jacobi equation. Let us before we wrap up our lecture session, let us quickly look at the same equation or the same formulation via the Euler Lagrange condition. Let us find the extremal via Euler Lagrange.

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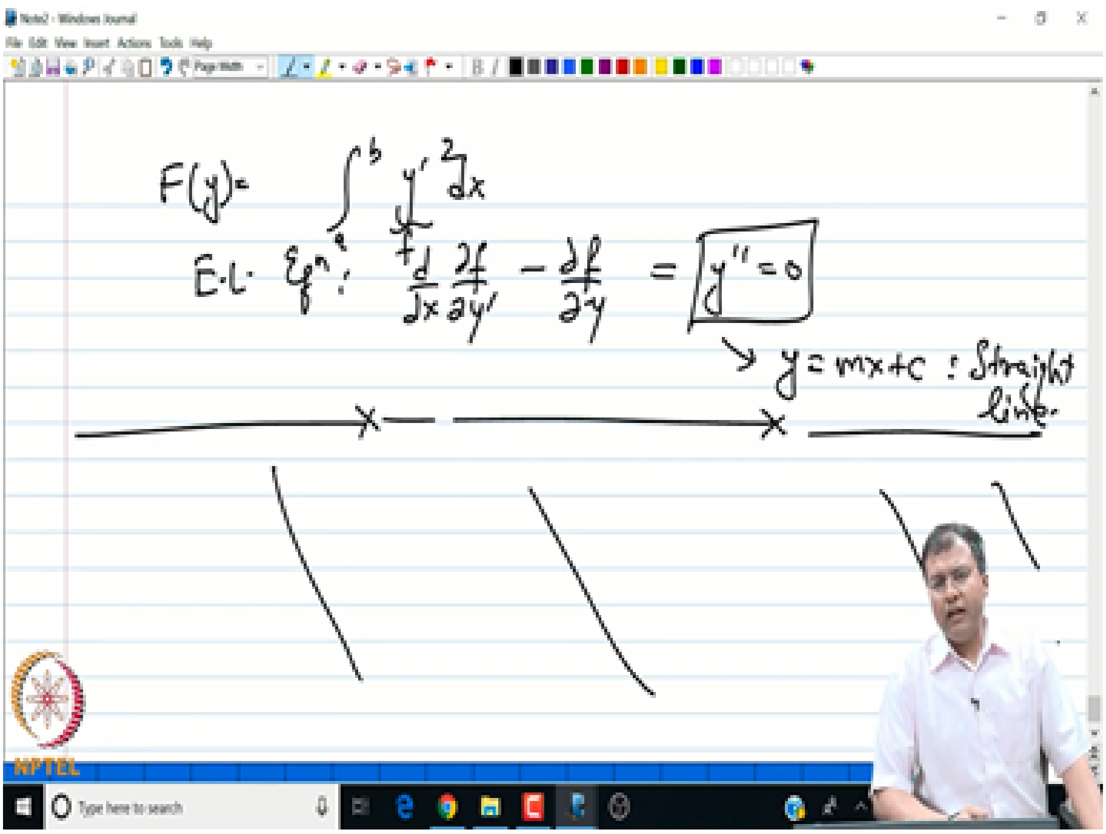

Notice that the functional $F(y) = \int_a^b {y'}^2 dx$ and note that the Euler Lagrange equation gives

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = y'' = 0$$

And the solution to this gives me $y = mx + c$, which is a straight line. So, the solution via the Hamilton-Jacobi matches with the solution via the Euler Lagrange.

Hence a Hamilton-Jacobi method is an alternative to the Euler Lagrange method, and it is quite a powerful alternative. Now, in the next lecture we are going to look at in more depth the power of Hamilton-Jacobi equation. And also look at the cases or the conditions under which we are able to separate variables in our generating function. So, thank you very much for listening.