Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 38-Part 2

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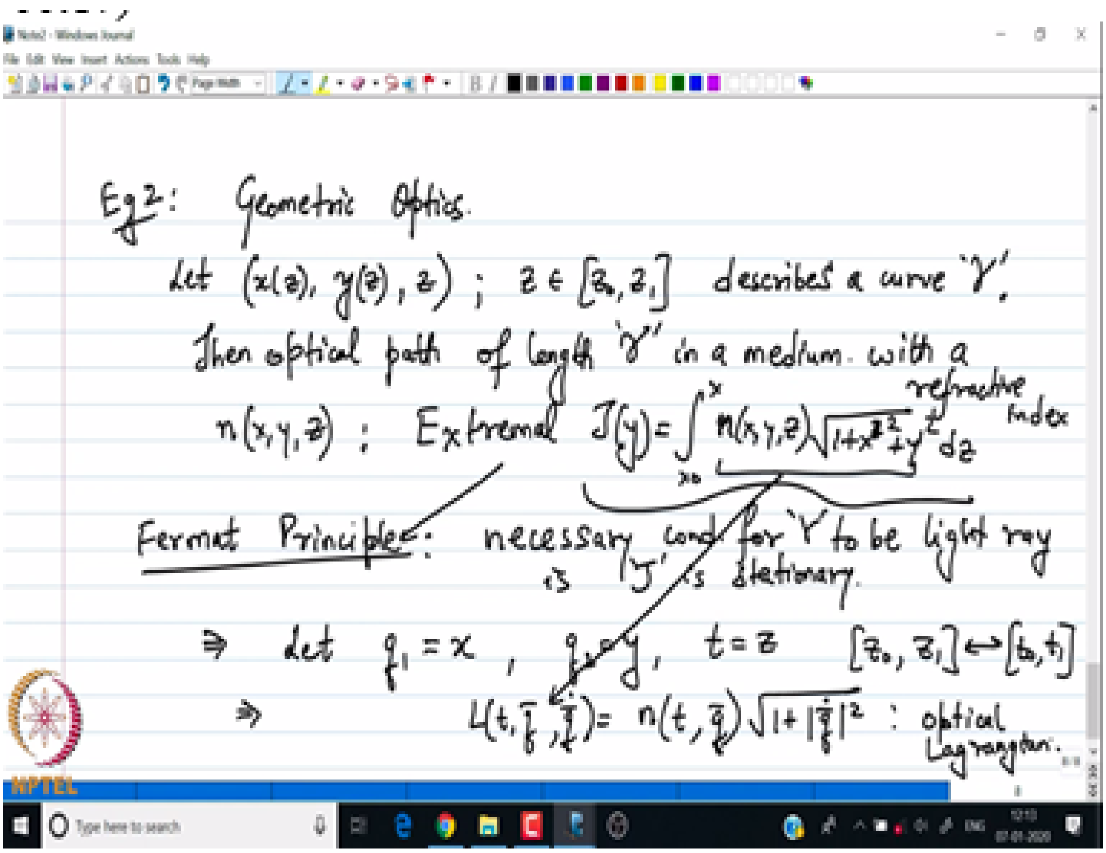

let us look at an example in Geometric Optic, let me say that I am given a coordinate system (x(z), y(z), z) where z now is the independent variable, and $z \in [z_o, z_1]$. It describes a curve γ

Then my optical path of length γ in a medium with refractive index n(x, y, z), so the optical path is given by the extremal $J(y) = \int_{x_o}^{x_1} n(x, y, z)\sqrt{1 + x'^2 + y'^2}dz$

To find extremal, make sure that the curve described by γ is the optical path length or the path length which is the shortest path length traveled by the particle of light, the answer to this is given by extremizing this path length or the arc length integral.

We can directly apply our Fermat's principle. Again, I am following our example done in lecture 3. We have used a simpler version of extremizing this, a simpler integral where n was taken to be 1, so by Fermat's principle, I see that the necessary condition for γ to be light ray is that J is stationary or J has an extremum.

let me in order to find the extremal of this functional, introduce new sets of variables $q_1 = x, q_2 = y, t = z$, now I have the length of the interval from $[z_o, z_1]$ is changed to $[t_o, t_1]$, further note that my Lagrangian $L(t, \bar{q}, \dot{\bar{q}}) = n(t, q)\sqrt{1 + |\dot{\bar{q}}|^2}$

So, this is the so-called optical Lagrangian, then the next step to solve this and find the extremal is to introduce our conjugate variables p and h.

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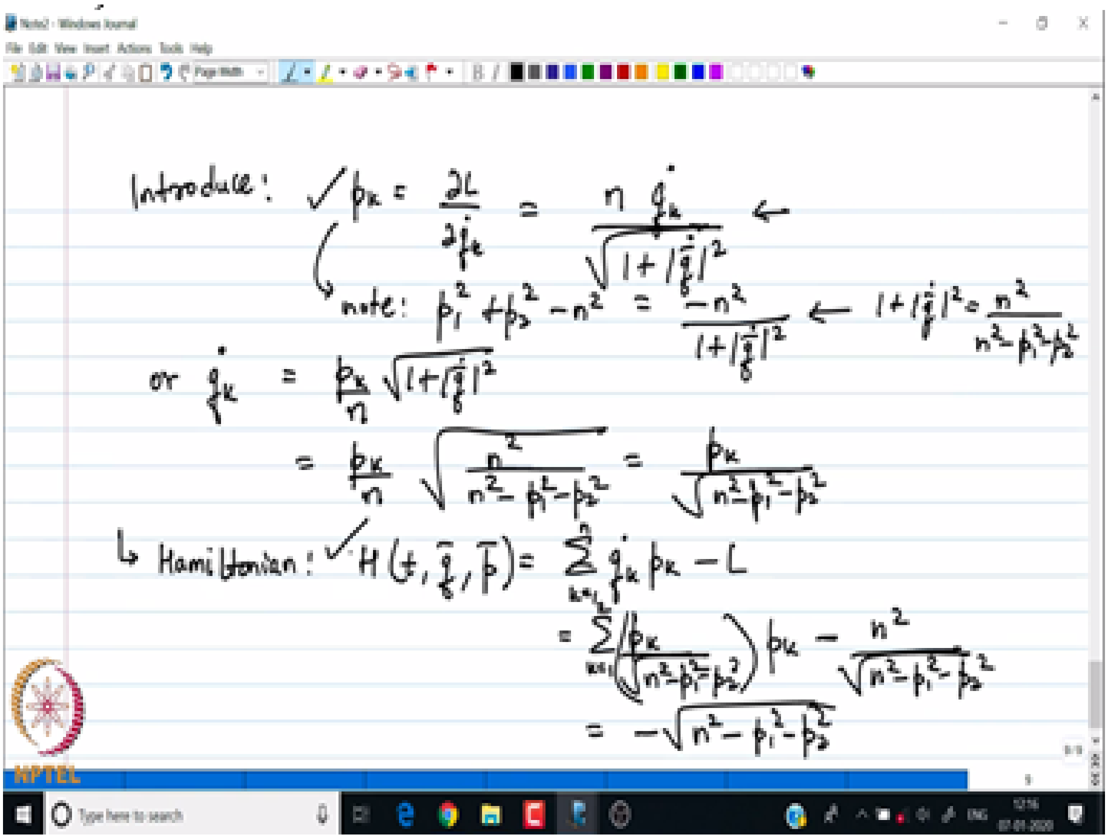

$$\text{Introduce} \quad p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{n\dot{q}_k}{\sqrt{1+|\dot{\bar{q}}|^2}}$$

Note that I have the relation

$$p_1^2 + p_2^2 - n^2 = \frac{-n^2}{1+|\dot{\bar{q}}|^2} \quad \Rightarrow \quad 1+|\dot{\bar{q}}|^2 = \frac{n^2}{n^2 - p_1^2 - p_2^2}$$

$$\text{Or} \quad \dot{q}_k = \frac{p_k}{n}\sqrt{1+|\dot{\bar{q}}|^2} = \frac{p_k}{n}\sqrt{\frac{n^2}{n^2-p_1^2-p_2^2}} = \frac{p_k}{\sqrt{n^2-p_1^2-p_2^2}}$$

$$\text{Hamiltonian} \quad H(t,\bar{q},\bar{p}) = \sum_{k=1}^{n} \dot{q}_k p_k - L = \sum_{k=1}^{n}\left(\frac{p_k}{\sqrt{n^2-p_1^2-p_2^2}}\right)p_k - \frac{n^2}{\sqrt{n^2-p_1^2-p_2^2}} = -\sqrt{n^2-p_1^2-p_2^2}$$

So, the moment I have defined my conjugate variables, I can immediately write the Hamilton's equation.

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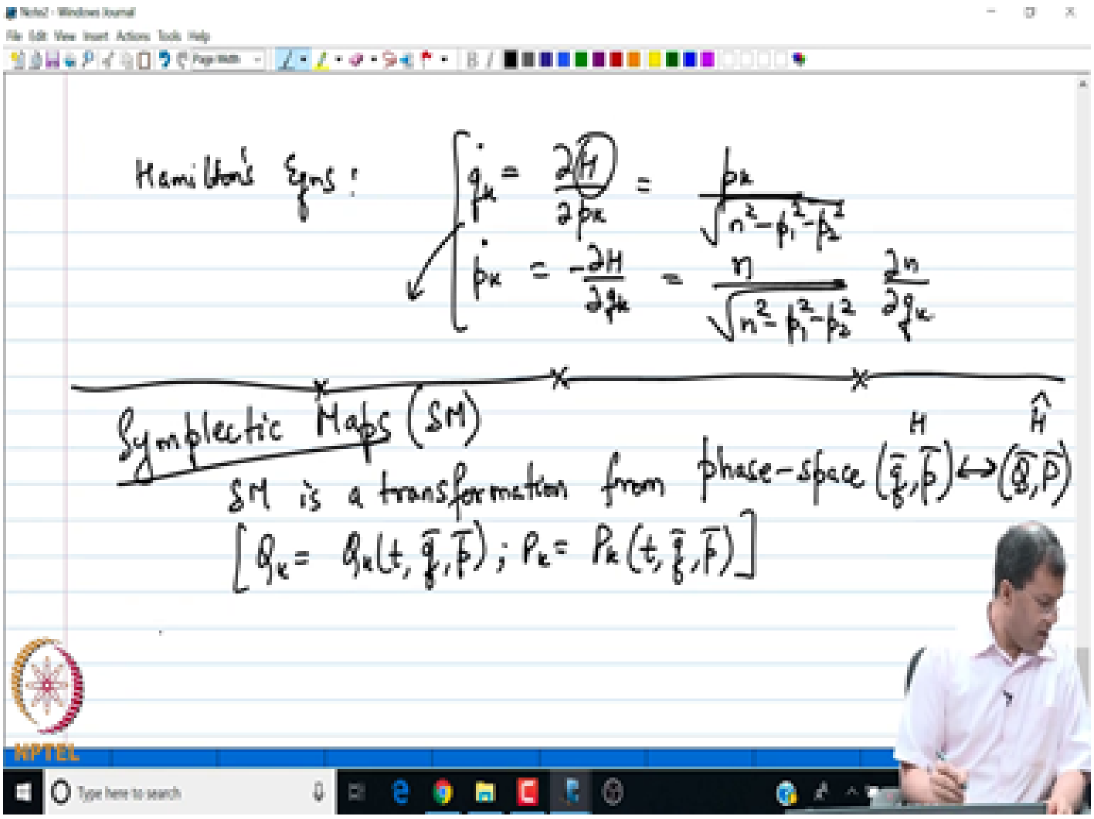

$$\text{Hamilton's equation} \quad \dot{q}_k = \frac{\partial H}{\partial p_k} = \frac{p_k}{\sqrt{n^2 - p_1^2 - p_2^2}}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \frac{n}{\sqrt{n^2 - p_1^2 - p_2^2}} \frac{\partial n}{\partial q_k}$$

From here all we need to do is to solve this system of equation and that can only be done once we have an exact relation for n. We do not know what is this function n. So, I leave this question at this point assuming that the Hamilton's equation can is solvable, these are my Hamilton's equation, at this stage we know that once we are in the Hamiltonian formulation we should be very easily be able to solve our Euler, we should be very easily be able to get our extremals via the Hamiltonian equation.

So, thus question is, is that true, given a Hamiltonian system, can we really solve, all the time can we really solve the Hamilton's equation? The answer is not really. It depends how complicated this function H is, so thus question is, suppose H is quite complex that we are not able to solve this system of Hamilton's equations, then can we reduce that system or reduce that Hamiltonian to a simpler case or a simpler Hamiltonian?

The answer is yes and to reduce from a more complex to a simpler Hamiltonian, we use a transformation or a map known as the symplectic maps. So, what are symplectic maps?

symplectic maps(SM) is a transformation from the phase space $(\bar{q}, \bar{p})$ to a new phase space $(\bar{Q}, \bar{P})$ and of course, there will be the Hamiltonian in this space is H and the Hamiltonian in this space is $\hat{H}$, but the idea is we are reducing a Hamiltonian which is far more complicated to a Hamiltonian which is simpler.

let SM be the transformation from one phase space to the other defined by $Q_k = Q_k(t, \bar{q}, \bar{p})$ and $P_k = P_k(t, \bar{q}, \bar{p})$
Let us go back a few slides. Where is my Hamiltonian mechanism? Hamilton's equation. Here.

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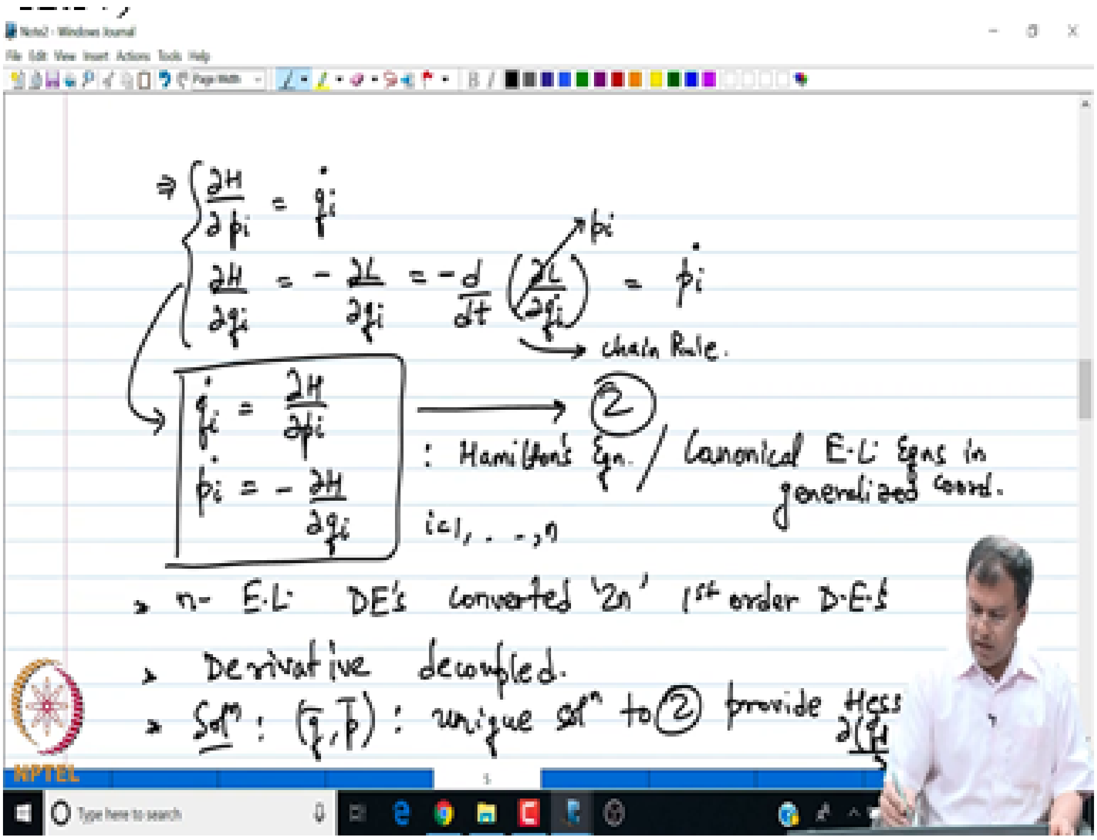

So, my Hamilton's, this set of equations or the Hamilton's equations are denoted by equation **2**. So we are going to use this.

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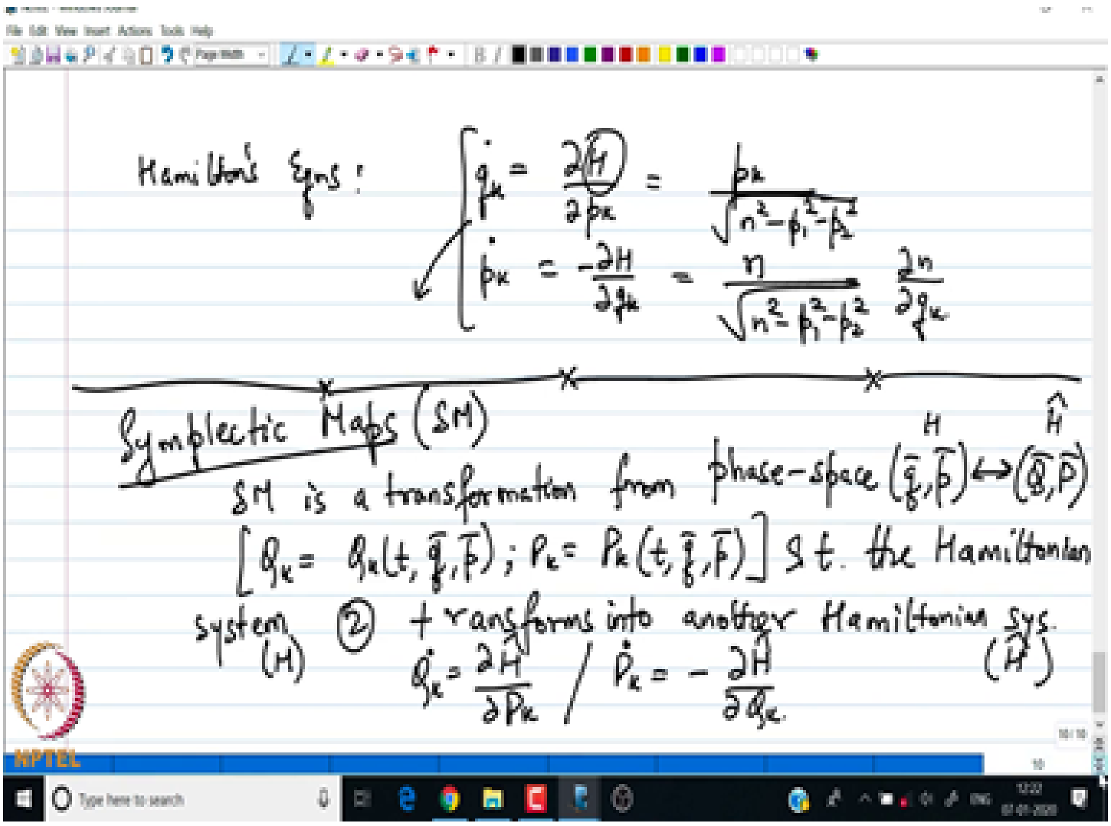

So this result says that, Hamilton's equation **2** or I would say the Hamiltonian system, such that the Hamiltonian system given by the Hamilton's equation **2**, transforms into another Hamiltonian system.

$$\text{Hamilton's equation}\ \ \dot{q}_k = \frac{\partial H}{\partial p_k} = \frac{p_k}{\sqrt{n^2 - p_1^2 - p_2^2}}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \frac{n}{\sqrt{n^2 - p_1^2 - p_2^2}}\frac{\partial n}{\partial q_k}$$

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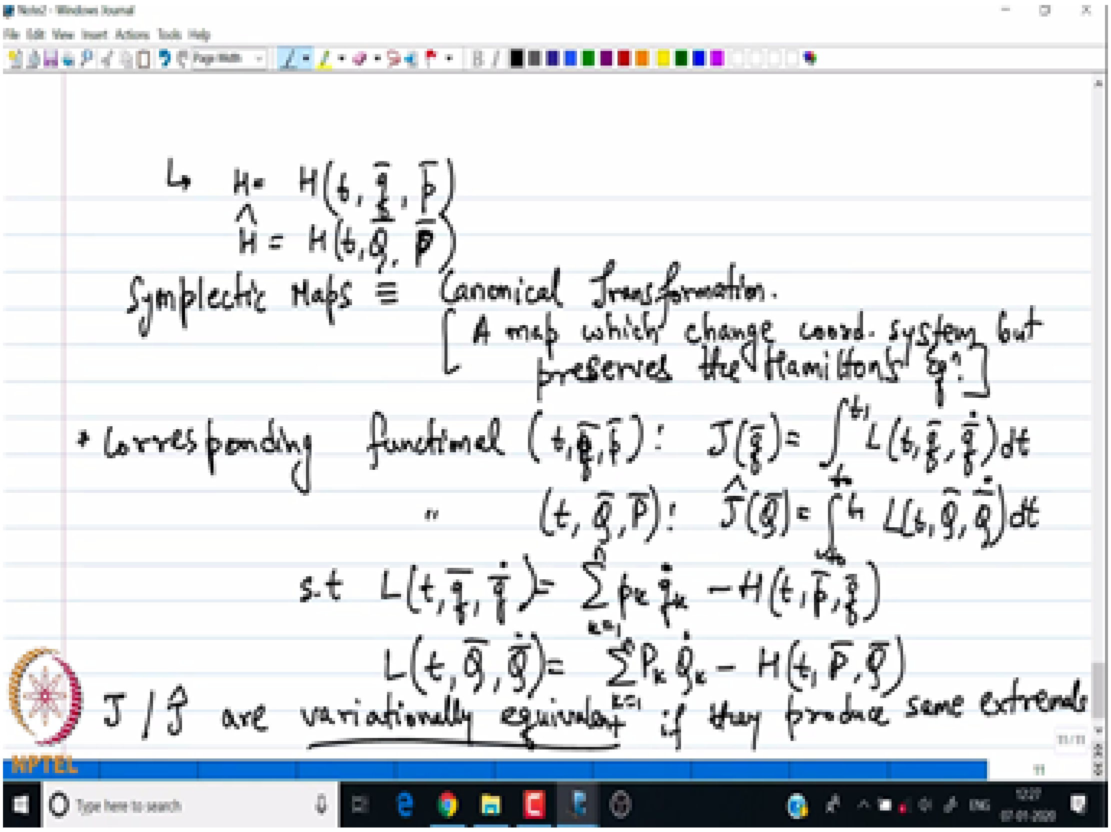

Here note that H, the original Hamiltonian system was a function of the variable p and q and my new Hamiltonian system is a function of Q and P. Note that symplectic maps are also known as canonical transformations, Symplectic maps are also canonical transformation. A map which changes the coordinate system but preserves the Hamilton equation.

What are canonical maps? A map which, this is in layman's term which changes the coordinate system but preserves the Hamilton's equation. Right? Hamilton's equation in the new system is still satisfied, which means that the corresponding extremals functionals $(t, \bar{q}, \bar{p})$ are given by

$$J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$$

and the corresponding functional in $(t, \bar{Q}, \bar{Q})$ is of the form

$$J(\bar{Q}) = \int_{t_o}^{t_1} L(t, \bar{Q}, \dot{\bar{Q}})dt \quad \text{Such that Lagrangian}$$

$$L(t, \bar{q}, \dot{\bar{q}}) = \sum_{k=1}^{n} p_k \dot{q}_k - H(t, \bar{p}, \bar{q})$$

and Lagrangian in the new coordinate system is $L(t, \bar{Q}, \dot{\bar{Q}}) = \sum_{k=1}^{n} P_k \dot{Q}_k - H(t, \bar{P}, \bar{Q})$

We introduce a new term known as a variationally equivalent, let me call this function as $\hat{J}$ to differentiate it from J. So, J and $\hat{J}$ are called as variationally equivalent.

They are variationally equivalent if they produce the same extremals. So, the symplectic map is the map which changes the functional J to functional $\hat{J}$ via changing the coordinate system.

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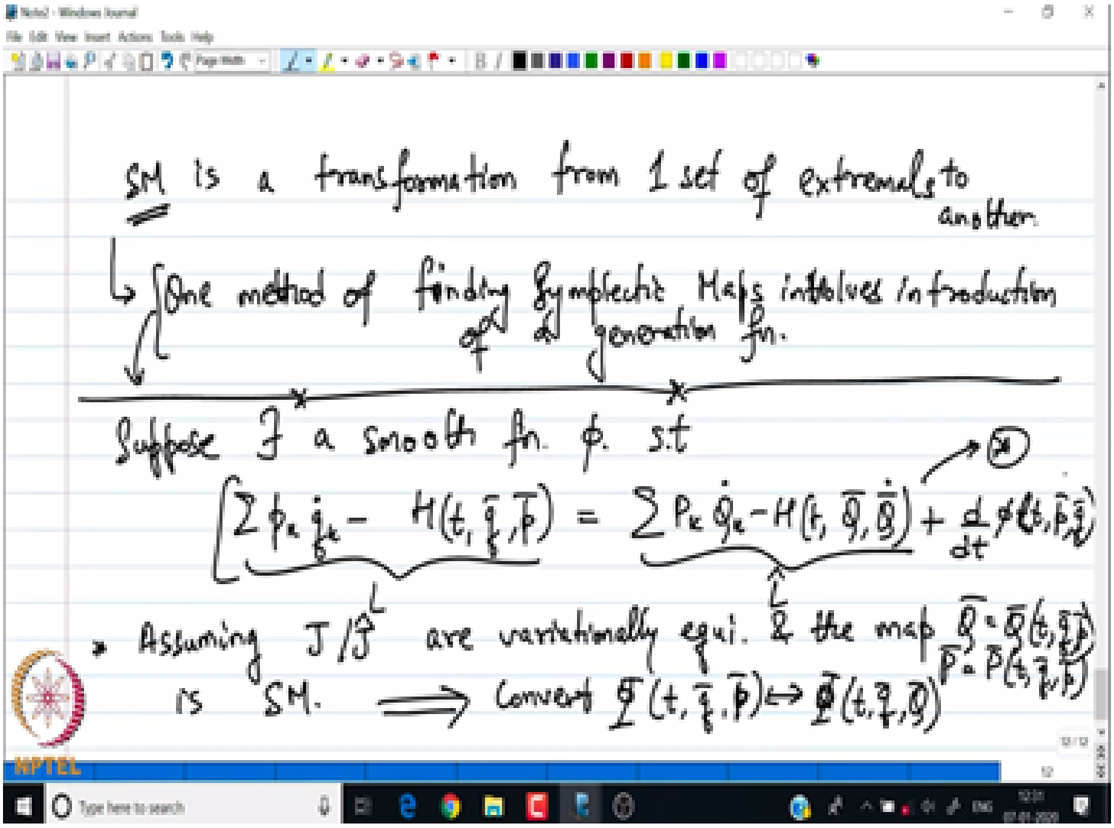

So the symplectic maps are quite important in our investigation of finding the Hamiltonian system. Why? Because symplectic maps is a transformation from one set of extremals to another. So, we definitely, in our description of our Hamiltonian system, finding symplectic maps is critical. It is a transformation from one set of extremals to another.

Now thus question is, having seen the importance, this question is how we are going to find the symplectic maps. One way of finding the symplectic maps involves the introduction of the so-called generating function and we will see what are those. So, one method, one method of finding symplectic maps involves introduction of a generating function.

It is the generating function, through the generating function later on, we will describe the famous Hamilton-Jacobi equation. So, I want to elaborate on this statement further, suppose there exists a smooth function ϕ such that

$$\sum p_k \dot{q}_k - H(t, \bar{q}, \bar{p}) = \sum P_k \dot{Q}_k - H(t, \bar{Q}, \dot{Q}) + \frac{d}{dt}\phi(t, \bar{p}, \bar{q}) \qquad *$$

Assuming that, one set of extremals could be converted to the other set via the symplectic map. So, suppose J and $\hat{J}$ are the functionals corresponding to these Lagrangians are variationally equivalent and the map given $\bar{Q} = \bar{Q}(t, \bar{q}, \bar{p})$ and $\bar{P} = \bar{P}(t, \bar{q}, \bar{p})$ of the old variables or variable in the older coordinate frame is symplectic.

Suppose this map is symplectic map, which means the moment this map is symplectic, it implies that we can convert my function ϕ . So,$\Phi(t, \bar{q}, \bar{p}) \leftrightarrow \Phi(t, \bar{q}, \bar{Q})$, we have just replaced one conjugate variable p with Q.

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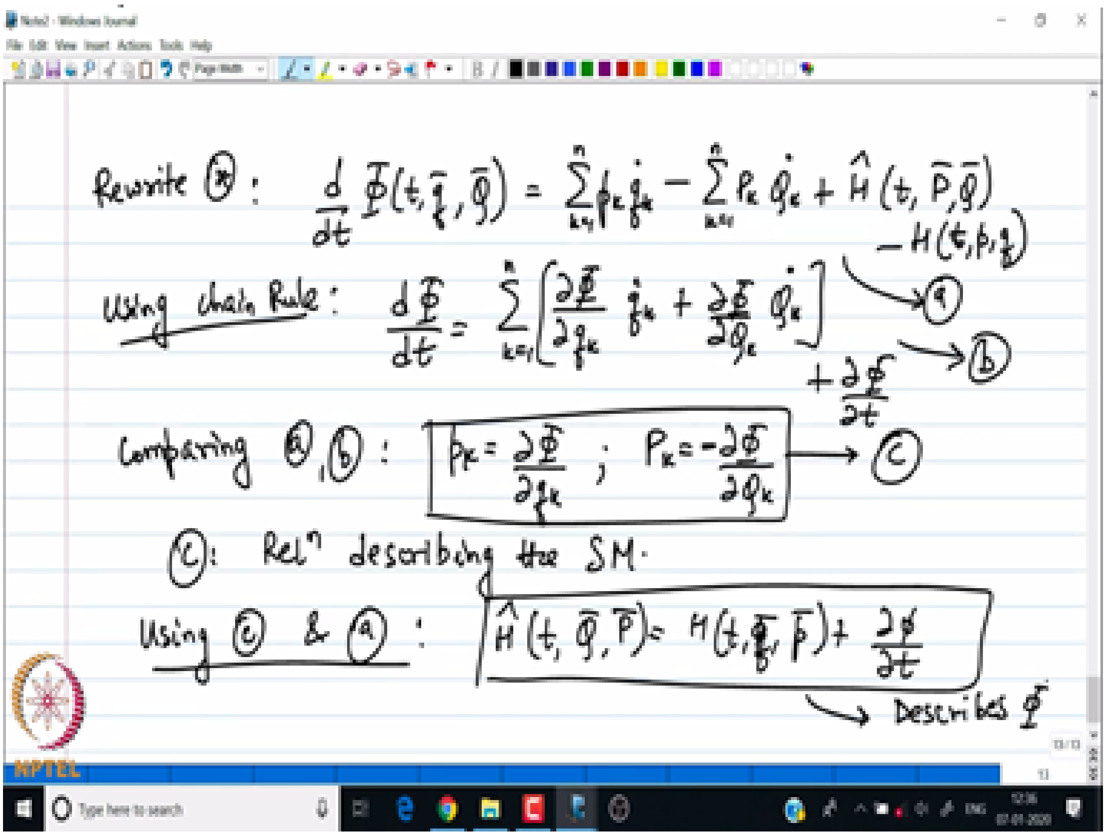

We rewrite our equation $*$ in the generalized coordinate or in the new coordinate frame.

$$\frac{d}{dt}\Phi(t,\bar{q},\bar{Q}) = \sum_{k=1}^{n} p_k\dot{q}_k - \sum_{k=1}^{n} P_k\dot{Q}_k + \hat{H}(t,\bar{P},\bar{Q}) - H(t,p,q) \qquad \mathbf{a}$$

$$\frac{d\Phi}{dt} = \sum_{k=1}^{n}\left[\frac{\partial\Phi}{\partial q_k}\dot{q}_k + \frac{\partial\Phi}{\partial Q_k}\dot{Q}_k\right] + \frac{\partial\Phi}{\partial t} \qquad \mathbf{b}$$

comparing equation **a** and **b** $p_k = \dfrac{\partial\Phi}{\partial q_k}; P_k = \dfrac{\partial\Phi}{\partial Q_K}$ **c**

Using **c** and **a** $\hat{H}(t,\bar{Q},\bar{P}) = H(t,\bar{q},\bar{p}) + \dfrac{\partial\phi}{\partial t}$

So equation **c** gives the relation, describing the generating function through which we find out the generating function and that leads to our symplectic map. We will see through an example

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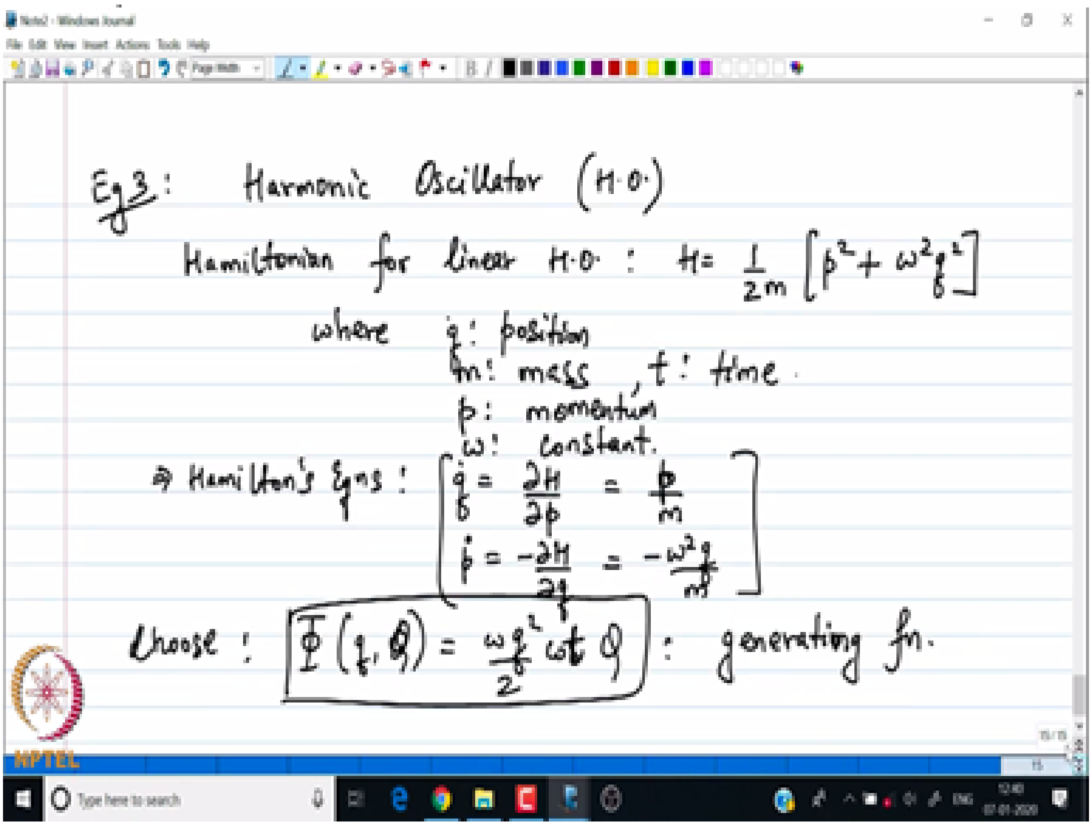

Example 3 is of harmonic oscillator (H.O), the Hamiltonian for the linear harmonic oscillator is H given to be the sum of the kinetic energy plus the potential energy and I write it in terms of the conjugate variable p and q.

$$H = \frac{1}{2m}\left[p^2 + \omega^2 q^2\right]$$

where we see that q is the position, p is the momentum, 'm' is the mass and t is the time of integration which we will see soon and ω denotes a physical quantity known as the angular velocity. Here we will denote ω to be a constant without going much into detail assume it is a constant.

which means my Hamilton's equation,

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = \frac{-\omega^2 q}{m}$$

To describe this Hamilton's equation in another Hamiltonian, we need the symplectic map and so on so forth and then we need the so-called generating function. Later on, we will see that finding generating function is very methodical. Right now, using some hit and trial, we assume a form of generating function and the form is chosen so that the Hamiltonian in the new coordinate frames is very simple.

We choose our generating function

$$\Phi(q, Q) = \frac{\omega q^2}{2}\cot Q$$

right now, the choice seems very arbitrary. In the next topic of discussion, we will see that the choice is not random but follows a very specific set of equations Or the Hamilton- Jacobi equation.

So, let us assume the generating function of this form, from here I can describe my variables. Note that the moment I have the generating function, I can describe using this condition **c**, I can describe my momenta variables p_k and P_k.

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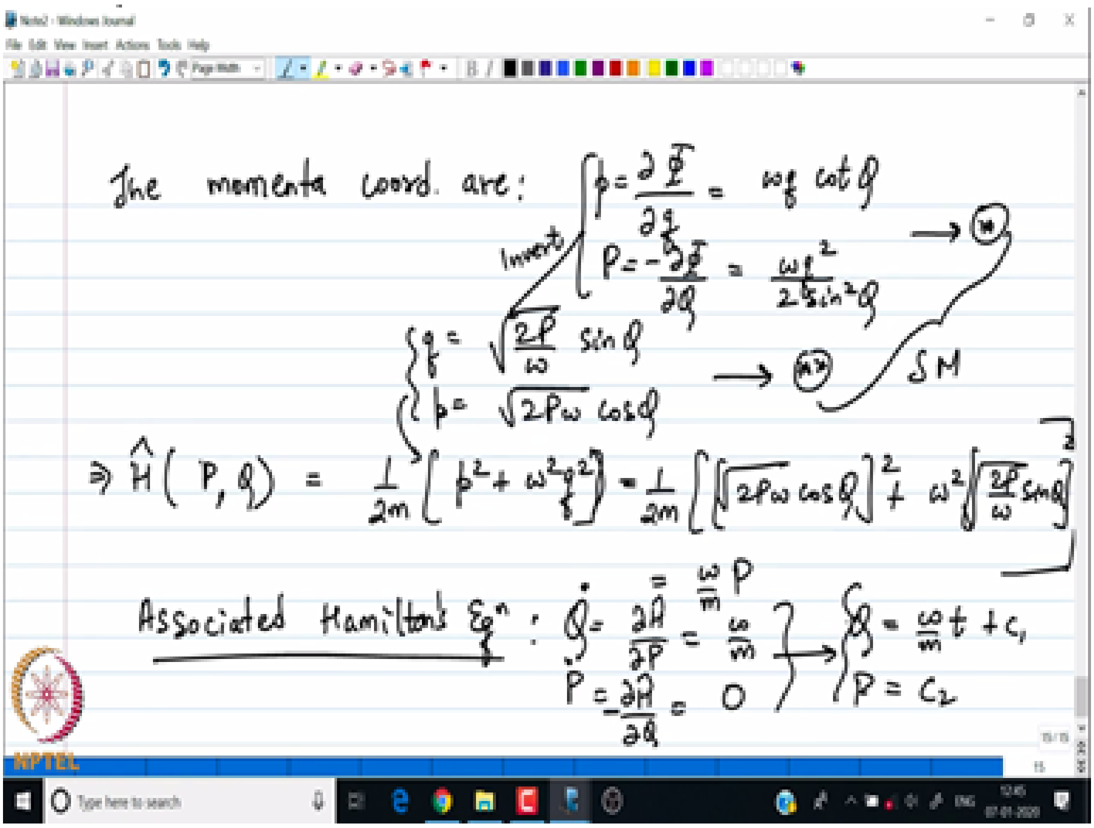

So the generalized momenta coordinates are

$$p = \frac{\partial \Phi}{\partial q} = \omega q \cot Q, \qquad P = -\frac{\partial \Phi}{\partial Q} = \frac{\omega q^2}{2\sin^2 Q} \qquad *$$

We can also invert to find the relation between q and Q in the original coordinate frame the coordinates q and p can be written in terms of Q and P.

$$q = \sqrt{\frac{2P}{\omega}} \sin Q, \quad p = \sqrt{2P\omega} \cos Q \qquad **$$

This can be readily checked from this relation and these are my symplectic maps which means now I am ready to describe my Hamiltonian in the new coordinate frame.

$$\Rightarrow \hat{H}(P, Q) = \frac{1}{2m}\left[p^2 + \omega^2 q^2\right] = \frac{1}{2m}\left[\left[\sqrt{2P\omega}\cos Q\right]^2 + \omega^2\left[\sqrt{\frac{2P}{\omega}}\sin Q\right]^2\right] = \frac{\omega}{m}P$$

Notice now the Hamiltonian in this new frame P and Q is relatively simple compared to the Hamiltonian in the original frame p and q, so which means my Hamilton's equation in this frame, let me denote it by the associated Hamilton's equation in this new frame P and Q

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{\omega}{m} \to Q = \frac{\omega}{m}t + C_1$$

$$\dot{P} = -\frac{\partial H}{\partial Q} = 0 \to P = C_2$$

Note that now, I have found the solution to the Euler-Lagrange in a frame Q and P. So, all I do is, we plug this variables into our expression ** to find q and p.

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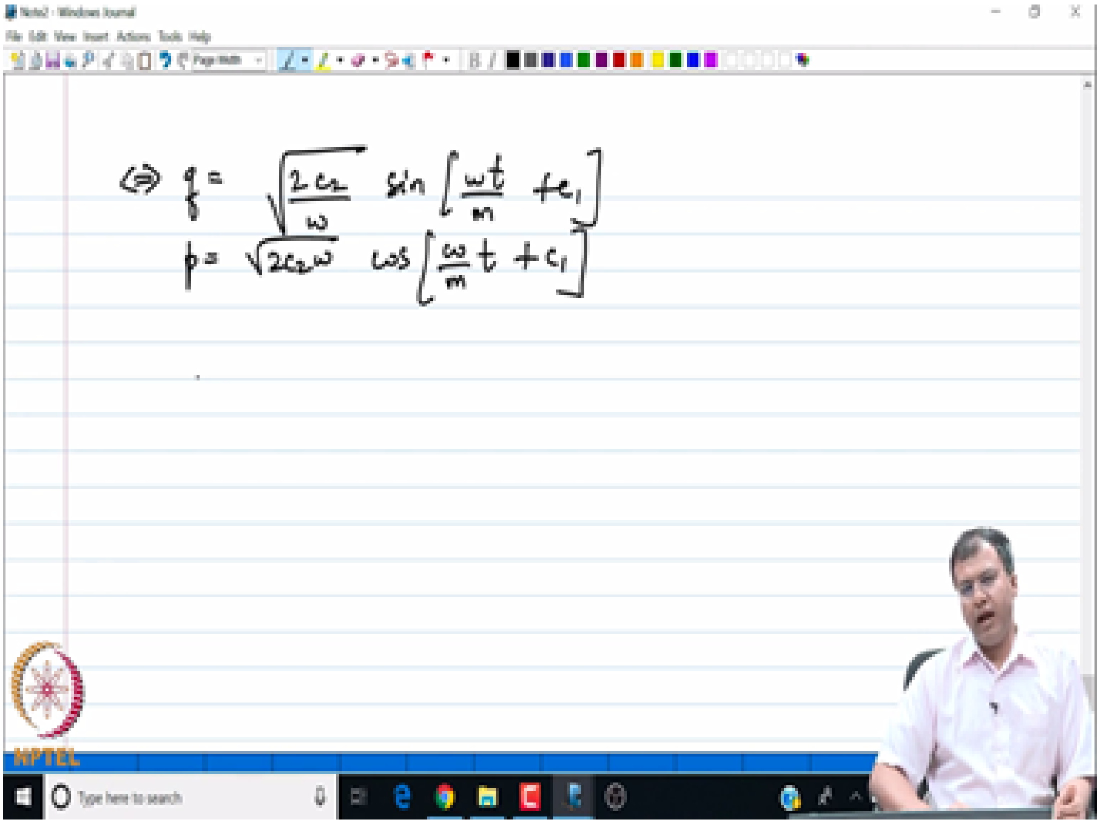

$$q = \sqrt{\frac{2C_2}{\omega}} \sin\left[\frac{\omega t}{m} + C_1\right]$$

$$p = \sqrt{2C_2\omega} \cos\left[\frac{\omega t}{m} + C_1\right]$$

So, these are my variables in the original frame and these are my solution to the Euler-Lagrange equation. so this exercise in this example has shown how can we use the power of Hamilton's equation, symplectic maps and generating function to quickly arrive at the extremals which would have been very difficult to solve had we used the Euler-Lagrange approach.