Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 37- Part 1

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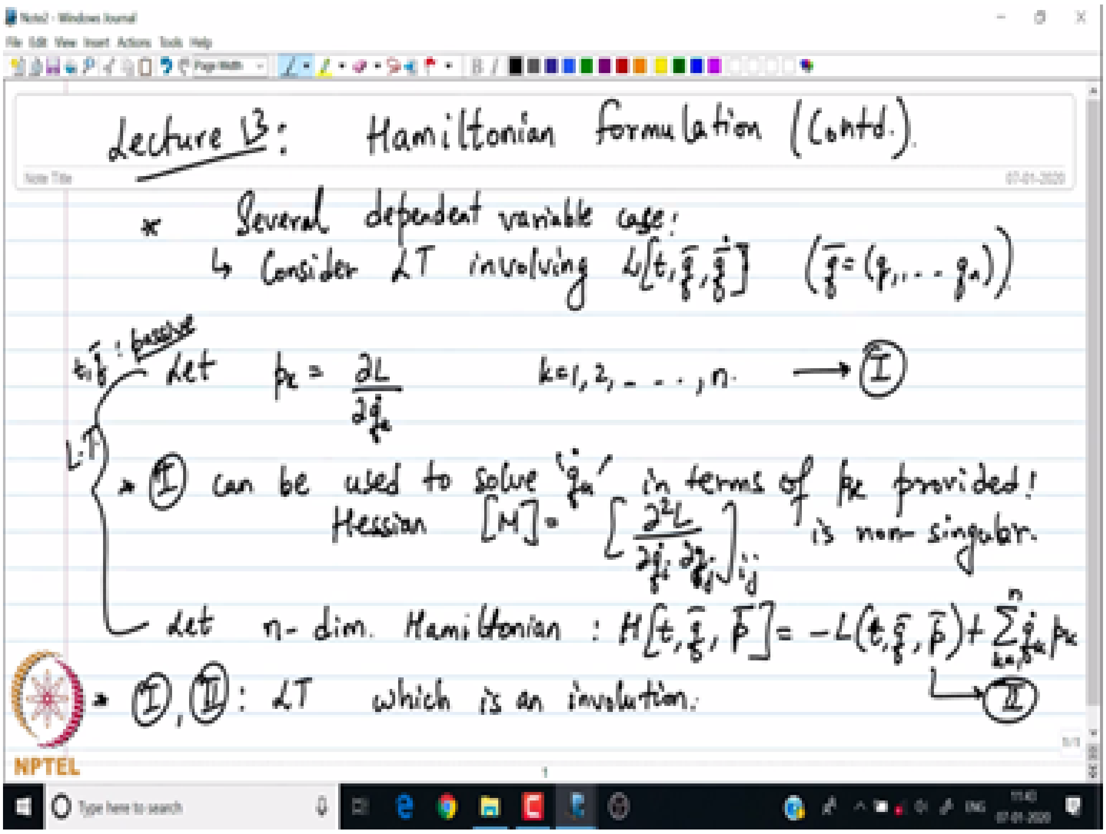

In today's lecture, I am going to continue our discussion on the Hamiltonian formulation of the condition for finding the extremals, we will continue our discussion on the Hamiltonian formulation, this is a continued series of discussion, I am going to start our discussion right away by introducing the Hamiltonian formulation for functions of several variables.

So that will include the special case of function with one dependent variable. So, several dependent variable case. Right? So, I am talking about, so let us consider a Legendre transformation involving the Lagrangian of the form $L(t, \bar{q}, \dot{\bar{q}})$ where vector $\bar{q} = (q_1, ..., q_n)$, we will see that we are going to see if our starting function is of this form, the first set of steps for defining the Legendre transformation is to introduce new sets of variables.

Let me introduce the new variable p, let the variable $p_k = \frac{\partial L}{\partial \dot{q}_k}$ where k = 1, 2,......,n. **I** using this **I**, we can invert this relation to find q_K in terms of p_K provided this inversion is possible. What I mean by that is certain higher derivatives are non-zero. So, equation **I** can be used to solve for q_K in terms of p_K provided the Hessian matrix has a non-determinant

$$\text{Hessian matrix} \quad [M] = \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right]_{ij} \quad \text{is non-singular or it has non-zero determinant}$$

this is a standard inversion argument for functions of several variable.

Next we have to define another variable which is our Hamiltonian variable, note that while introducing the variables, p and H, we have used the fact that t and q are passive, let me now introduce another variable. Let the n dimensional Hamiltonian is defined by

$$H[t,\bar{q},\bar{p}] = -L(t,\bar{q},\bar{p}) + \sum_{k=1}^{n} \dot{q}_k p_k \qquad \textbf{II}$$

I have assumed that the variable t and $\bar{q}$ are passive variables i.e. they are going to appear implicitly. So, this Legendre transformation defined by equation **I** and **II** is an involution.

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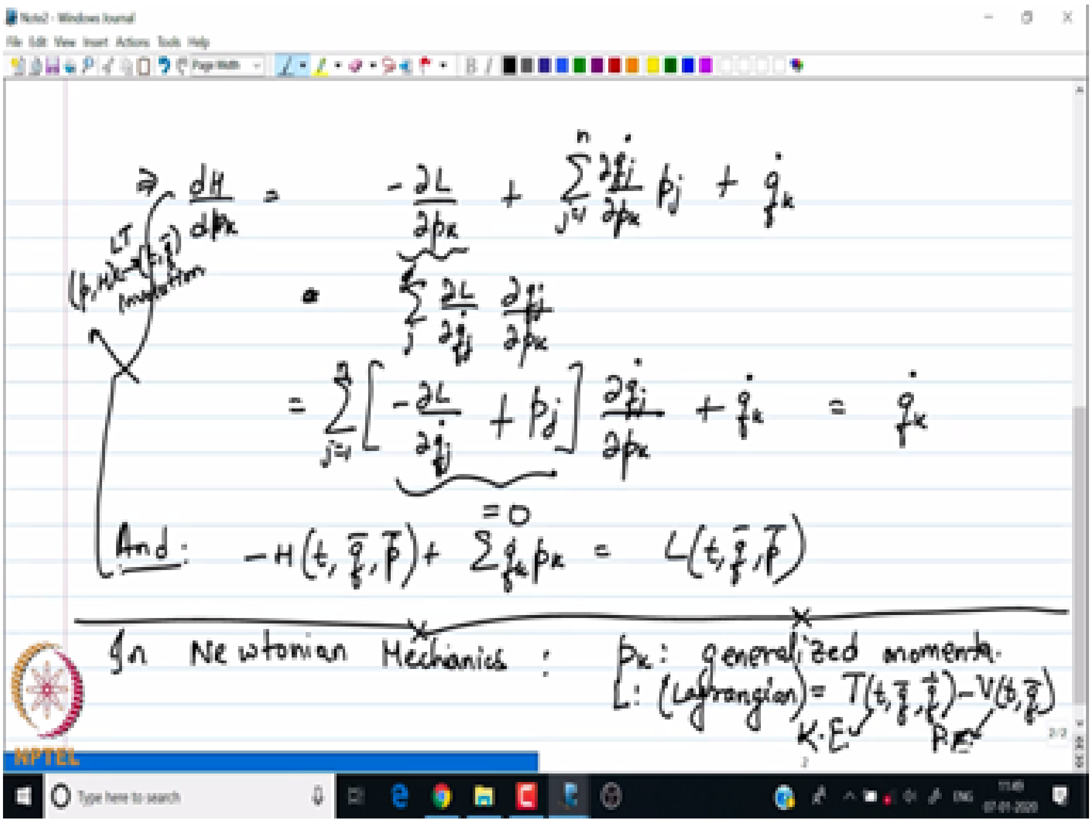

We take the derivative of the Hamiltonian with respect to p.

$$\Rightarrow \frac{dH}{dp_k} = -\frac{\partial L}{\partial p_k} + \sum_{j=1}^{n} \frac{\partial \dot{q}_j}{\partial p_k} p_j + \dot{q}_k$$

$$= \sum_{j=1}^{n} \left[-\frac{\partial L}{\partial \dot{q}_j} + p_j \right] \frac{\partial \dot{q}_j}{\partial p_k} + \dot{q}_k = \dot{q}_k$$

$$\text{And } \; -H(t,\bar{q},\bar{p}) + \sum_{k=1}^{n} \dot{q}_k p_k = L(t,\bar{q},\bar{p})$$

Let us look at this example, more applied framework. In Newtonian mechanics, my L, which is the Lagrangian, is the sum of the kinetic plus potential energy. So, in Newtonian mechanics my p_k's are denoted as the generalized momenta variable and L which is the Lagrangian is defined to be $T(t,\bar{q},\bar{p}) - V(t,\bar{q})$, where the first quantity in the continuum mechanics is the kinetic energy and the second quantity is the potential energy.

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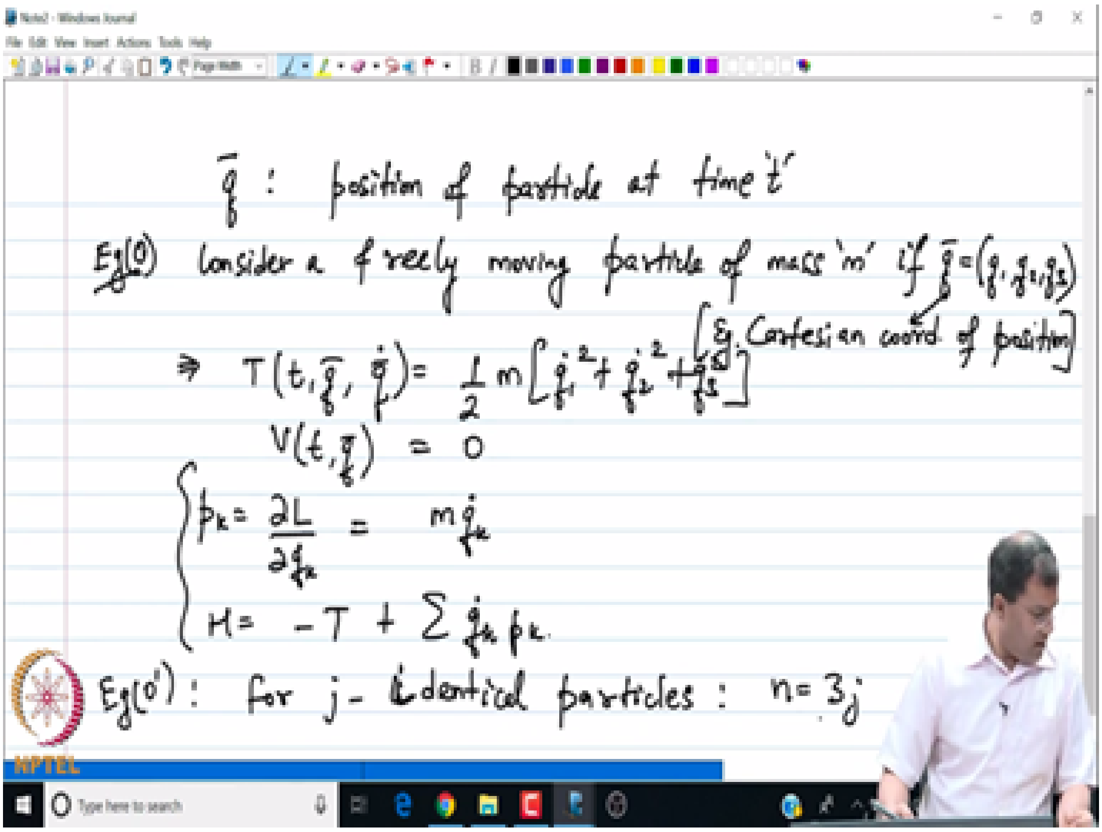

Vector $\bar{q}$ is the position of the particle at time t, for example, consider a freely moving particle of mass 'm' If $\bar{q} = (q_1, q_2, q_3)$ and these are my Cartesian coordinates of the position of the particle.

$$\Rightarrow T(t, \bar{q}, \dot{\bar{q}}) = \frac{1}{2}\left[\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right], \quad V(t, \bar{q}) = 0$$

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = m\dot{q}_k \quad H = -T + \sum \dot{q}_k p_k$$

So, for j identical particles, for each particles. So, I have that n = 3j For each particle we have 3 coordinate positions and hence we have the total degrees of freedom being 3j, Where j is standing for each particle, so we are now in a position to describe another form of Euler-Lagrange equation. Right?

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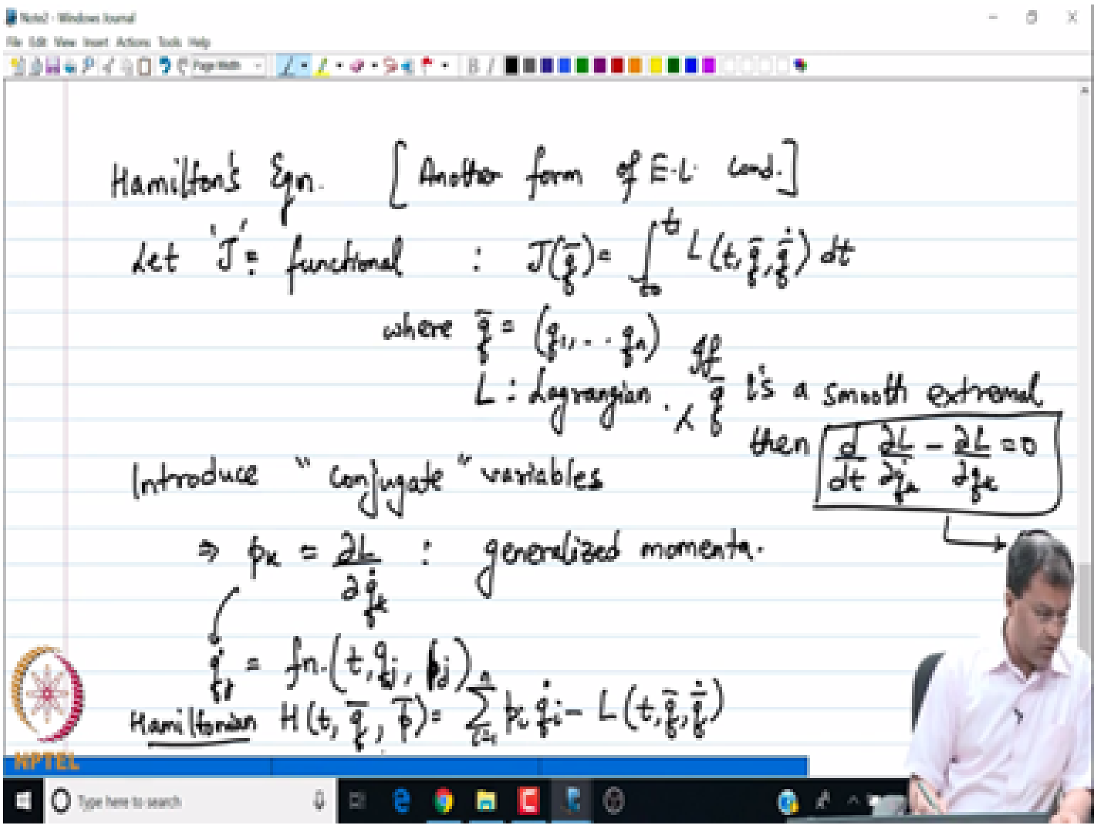

So, let me now start with the so-called Hamilton's, Hamilton's, This is another form of Euler-Lagrange conditions. We will see what is the advantage of Hamilton's equation soon. So now let us describe the Hamilton's equation, let J be a functional such that $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ where $\bar{q} = (q_1, ...q_n)$ and function L is the Lagrangian and if q is a smooth extremal then it must satisfy the n Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$$

Now I'm going to introduce the so-called conjugate variables which are p and H

$$\Rightarrow p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ generalized momenta}$$

And we can always invert this relation to find $\dot{q}_j = f(t, q_j, p_j)$

$$\text{Hamiltonian } H(t, \bar{q}, \bar{p}) = \sum_{i=1}^{n} p_i \dot{q}_i - L(t, \bar{q}, \dot{\bar{q}})$$

Now, let us see some necessary derivatives out of this Hamiltonian. So, we see that if we differentiate Hamiltonian with respect to p_i, p_i is an independent variable. It only appears in the first sum i from 1 to n. So, the derivative of H with respect to p_i will give me $\dot{q}_i$ and the derivative of H with respect to q_i will give me the derivative of L with respect to q_i with a minus sign because q_i appears only in the second term.

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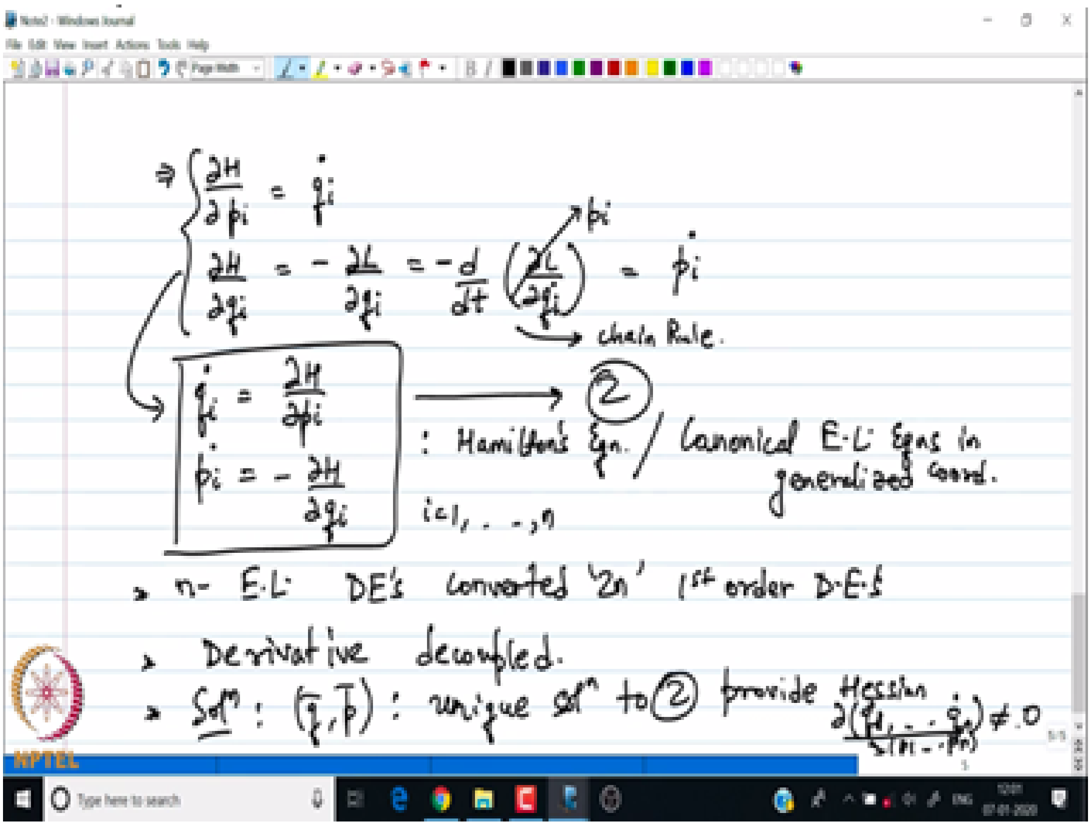

$$\Rightarrow \quad \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \dot{p}_i$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \;\; \dot{p}_i = -\frac{\partial H}{\partial q_i} i = 1, ...n \qquad \mathbf{2}$$

These two are the relations that we were after. So, the idea is to derive the Hamilton's equation, we start with our Euler-Lagrange condition. From there I transform those equations into the conjugate variables and from there I derive the necessary equations, which are these two set of equations. Notice now, we have two sets of decoupled equation for each i rather than having one coupled equation in terms of Euler-Lagrange condition.

By the way, this is Hamilton's equation or also known as the Canonical Euler-Lagrange equations in generalized coordinates and we see that these are n Euler-Lagrange differential equations Which are now for i equal to 1 to n and so the n Euler-Lagrange differential equations have been converted into 2n 1st order differential equations.

Note that the derivatives of q and p have been decoupled. So, this set of equations are quite easy to solve and comparatively easier to solve then the Euler-Lagrange equation, then finally, I end this discussion by mentioning that the solution that we obtained $(\bar{q}, \bar{p})$ is a unique solution to equation **2** provided the Hessian matrix has a nonzero determinant

$$\frac{\partial(\dot{q}_1, ..., \dot{q}_n)}{\partial(p_1, ...p_n)} \neq 0$$

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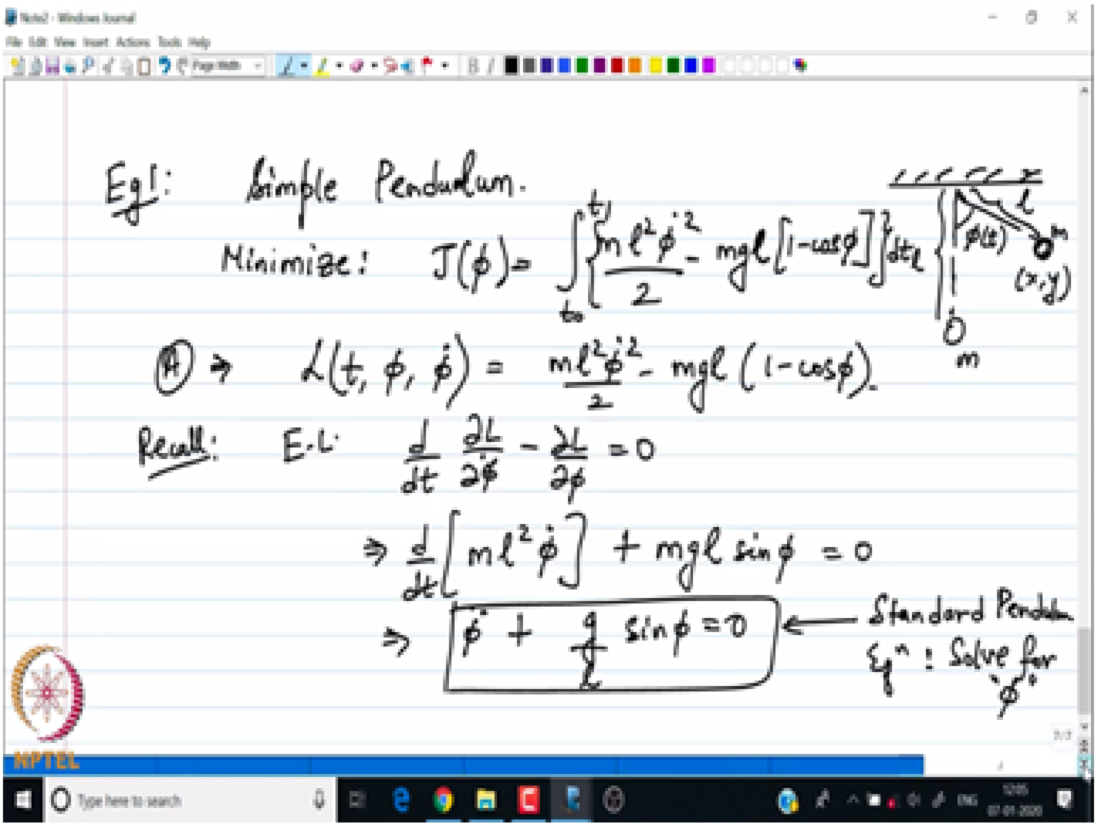

Let us quickly look at how to use Hamilton's equation in finding the extremal of a functional.

Example that I have in mind is that of the simple pendulum. So again, we have a pendulum standard blob of mass m which is hanging by a rope of length l and the angle that it subtends $\phi(t)$ and let us say that the position coordinate is (x, y) at any point of time So, I am not going to write down the statement but rather state the functional directly.

I have to minimize the action integral which was described few lectures back.

$$J(\phi) = \int_{t_o}^{t_1} \left\{ \frac{ml^2\dot{\phi}^2}{2} - mgl(1 - \cos\phi) \right\} dt$$

$$L(t, \phi, \dot{\phi}) = \frac{ml^2\dot{\phi}^2}{2} - mgl(1 - \cos\phi)$$

let us look at the solution via the Euler-Lagrange
Recall that has already been done earlier few lectures back. So, recall the Euler-Lagrange machinery provided this following relation.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\Rightarrow \quad \frac{d}{dt}[ml^2\dot{\phi}] + mgl\sin\phi = 0$$

$$\Rightarrow \quad \dot{\phi} + \frac{g}{l}\sin\phi = 0$$

Standard pendulum equation that we derived from Euler-Lagrange earlier, we solve for the angle ϕ We have done that earlier. So, let us see what happens when we use the Hamilton's equation.

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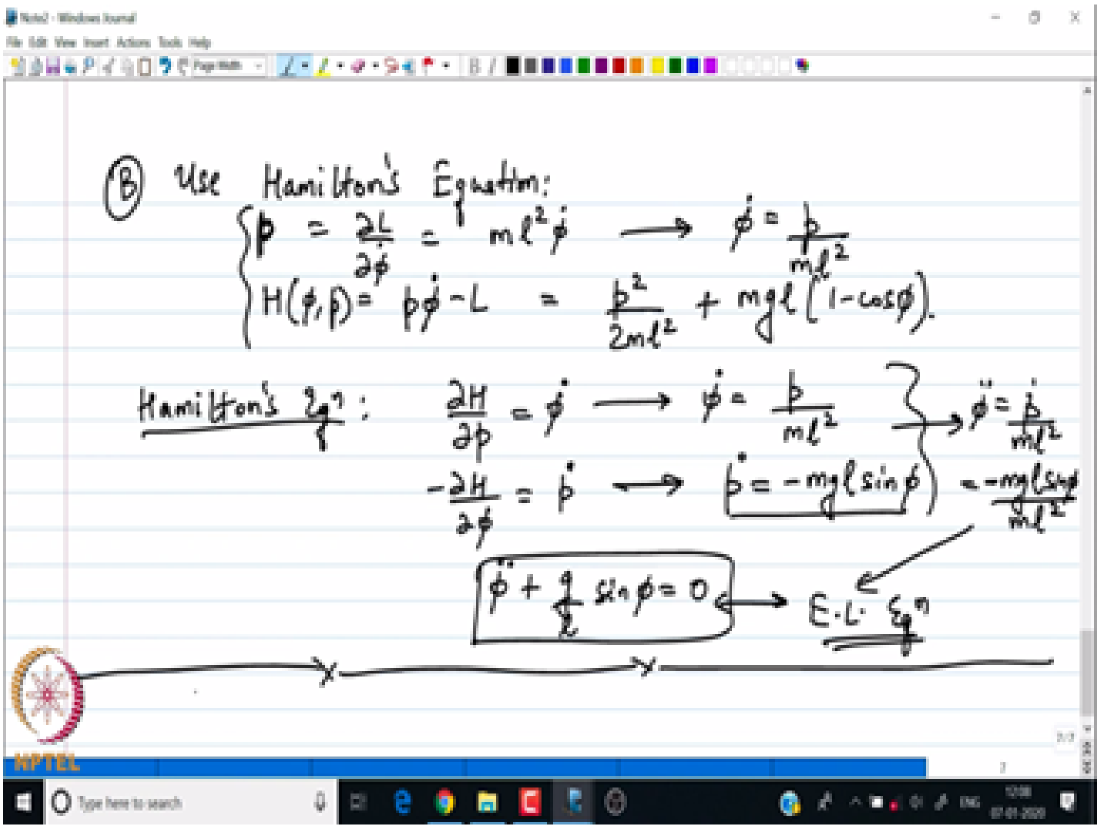

Now I am going to use the Hamilton's equation, for that, I have to set up the conjugate variables.

$$p = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} \Rightarrow \qquad \dot{\phi} = \frac{p}{ml^2}$$

$$H(\phi, p) = p\dot{\phi} - L = \frac{p^2}{2ml^2} + mgl(1 - \cos\phi)$$

$$\text{Hamilton's equation} \qquad \frac{\partial H}{\partial p} = \dot{\phi} \qquad \Rightarrow \qquad \dot{\phi} = \frac{p}{ml^2}$$

$$-\frac{\partial H}{\partial \phi} = \dot{p} \qquad \Rightarrow \qquad \dot{p} = -mgl\sin\phi$$

Then we solve this equation simultaneously, we differentiate these equation and plug the answer in the first one to come to a point that

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0$$

And that is the same equation as the Euler-Lagrange equation. So, the Hamilton's equation in this case gives us the extremal which is originally given by the Euler-Lagrange machinery, let us look at another example in which the Hamilton's principle is especially useful. The Euler Lagrange machinery is going to give me an equation which is very very complicated.