Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 36

Broken extremals / Hamiltonian Formulation Part 6

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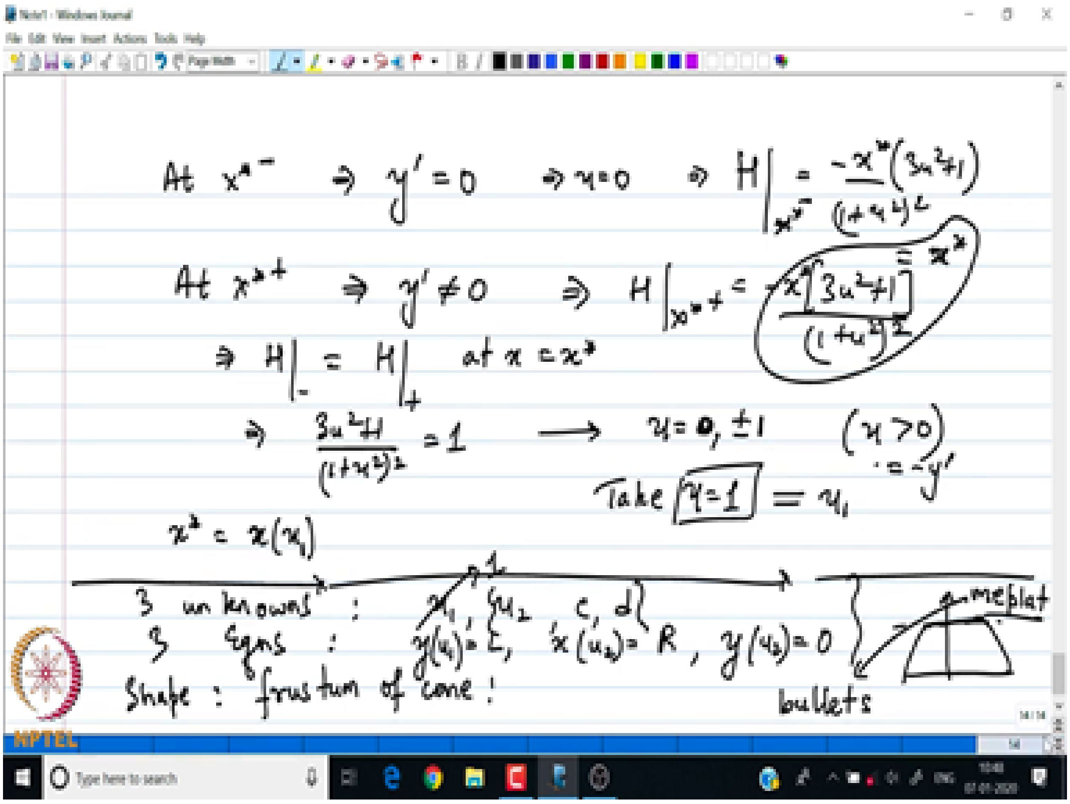

Let us now end the discussion on broken extremals.

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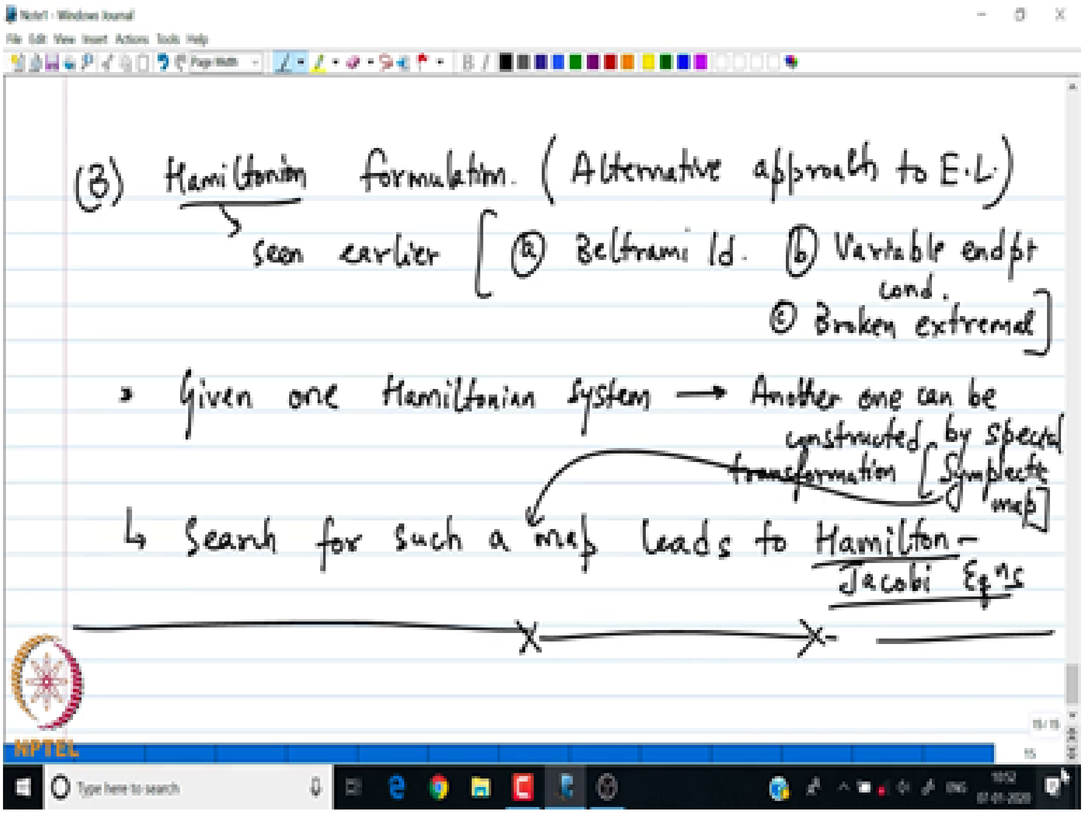

And let us now look at another formulation, a very vital formulation of reformulation of Euler Lagrange. Namely, the Hamiltonian Formulation. So, what have we got? We have seen Hamiltonian earlier, where have we seen? We have seen in several places. For example, in Beltrami identity, it is not completely new to us.

We have also seen when we were describing variable end point conditions and of course when we were using in broken extremals, so H appears in lot of places. It seems that this this quantity is useful. But we have not explored the full power of the Hamiltonian Formulation, it is an alternative approach to my Euler Lagrange equation.

We will see that the variable H and why is it that we have to consider alternative approach because sometimes using the Hamiltonian approach will reduce the problems significantly simplify the problem and further the Hamiltonian approach is coordinate free. We will see all these issues. So, those are my advantages of Hamiltonian Formulation.

There is another issue we will see later on that given one Hamiltonian system, we can see another one can be constructed by a special transformation known as the Symplectic map.
which means that suppose the Hamiltonian Formulation is quite complicated in one coordinate system we can go to another coordinate system via the Symplectic map to go towards a much more simpler Hamiltonian Formulation. So, that is also one of the advantages of using Hamiltonian, we can shift from one coordinate to the other.

And also we will see later on the search for such a Symplectic map, when I say map I am talking about these maps, Symplectic maps will lead to a first order PDE well known as the Hamilton-Jacobi equation or later on as I will say that this is the well known H-J equation.

So, I am going to talk about all these issues namely the Symplectic map, Hamilton-Jacobi and so on step by step. But before that let me start by introducing what is a point transformation.

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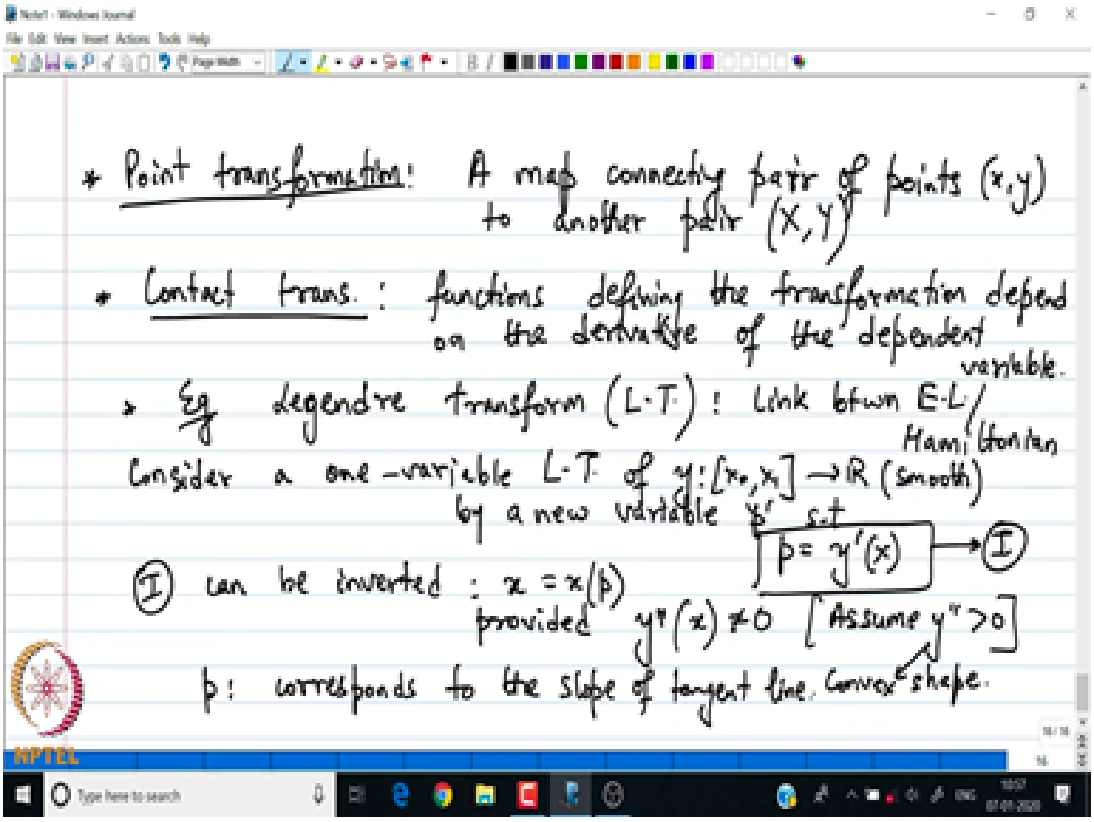

So, what is a point transformation? So, let us build the theory step by step. So, a point transformation is a map connecting connecting pair of points (x, y) to another pair (X, Y), so point transformation is a map from one point to the other.

Alternatively we could also have a contact transformation which is quite easy to understand. On the other hand we have the so called contact transformation which is a function defining the transformation which depends on the derivative of the dependent variable. So, I have said lot of things, let me describe in detail. So, my contact transformation are functions defining the transformation which depend on the derivative of the dependent variable.

We will see one such very important contact transformation known as the Legendary transformation and we will use that transformation. So, an example of contact transformation is the so called Legendary transformation, we will see all these definitions immediately.

Legendary transform is a contact transform, I denote this as L.T and we will see that the Legendary transformation provides a link between my Euler Lagrange formulation and the Hamiltonian Formulation, this is the link that we are after, once we have found the solution in one form that is the Hamiltonian form we have to change back into Euler Lagrange form, that is via the Legendary transformation.

Let us consider a function, a one-variable Legendary transform of $y : [x_o, x_1] \to \mathbb{R}$, this is a smooth function by a new variable p such that $p = y^{'}(x)$ **I**

Note that this is a contact transformation because we have defined a function which depends on the derivative of the dependent variable, here the dependent variable is y and we have defined the function as a derivative of the dependent variable. So, equation **I** can potentially be inverted, we can essentially find x as a function of p provided the second derivative of y exists and it is non-zero and that is via a standard inversion argument. So, we can assume without loss of generality that the derivative is positive Because in this case we are assuming that the shape of the profile is strictly convex upward or we are assuming a convex shape of the function.

So, what is p? p is the derivative of the dependent variable or geometrically it is the slope of the curve, p corresponds to the slope of the tangent line, let us also introduce another new function namely H.

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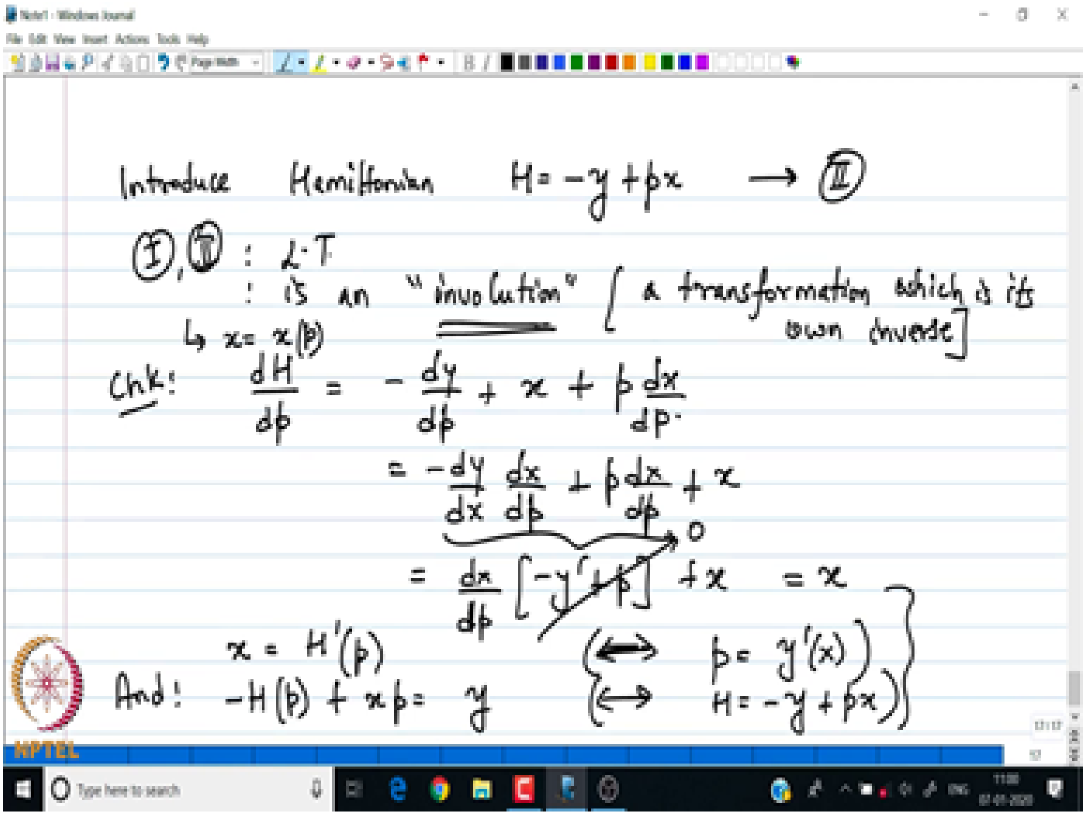

We introduce another new function or the Hamiltonian H = -y + p x **II**
Now, it turns out equation **I** and **II** are of course Legendary transforms by definition, these are contact transformations and further equation **I** and **II** is an involution, what is an involution?

An involution is a transformation which is its own inverse or a transformation we will see how it is, so we can check how it is an involution. so note that, x is a function of p, so it is p which is a independent variable now in this new variable system (p, H)

$$\frac{dH}{dp} = -\frac{dy}{dp} + x + p\frac{dx}{dp} = -\frac{dy}{dx}\frac{dx}{dp} + p\frac{dx}{dp} + x$$
$$= \frac{dx}{dp}[-y^{'} + p] + x = x$$

So, what I have found is that $x = H^{'}(p)$ this relation is very similar to $p = y^{'}(x)$, Also notice notice that $-H(p) + xp = y$ and this relation is very similar to saying that H = -y + p x , hence we can see that this set up is an involution.

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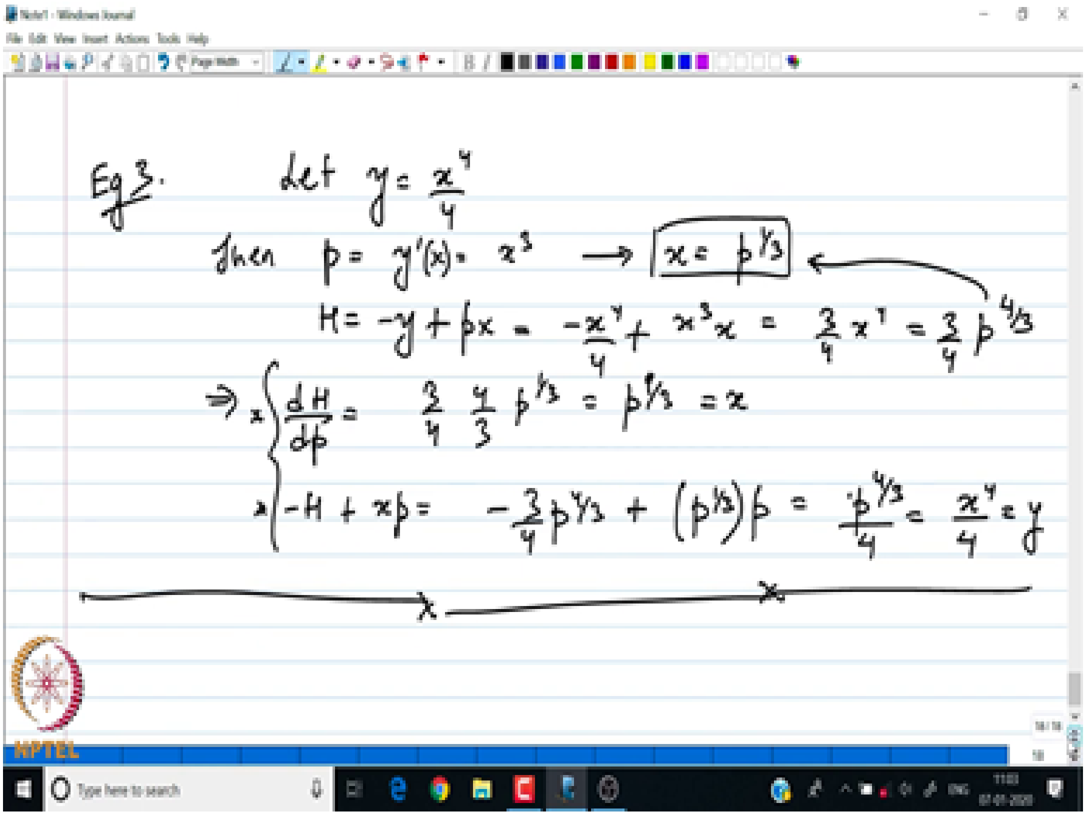

Let us look at a quick example, the example that I have to see how this legendary transformation work. let function y = $\frac{x^4}{4}$
Then we see that $p = y^{'}(x) = x^3 \Rightarrow x = p^{\frac{1}{3}}$

$$H = -y + px = \frac{-x^4}{4} + x^3 x = \frac{3}{4}x^4 = \frac{3}{4}p^{\frac{4}{3}}$$

$$\Rightarrow \frac{dH}{dp} = \frac{3}{4}\frac{4}{3}p^{\frac{1}{3}} = p^{\frac{1}{3}} = x$$

$$-H + xp = -\frac{3}{4}p^{\frac{4}{3}} + p^{\frac{1}{3}}p = \frac{p^{\frac{4}{3}}}{4} = \frac{x^4}{4} = y$$

So, I see that this definition is an involution. So, now let me wrap up this lecture by giving the Legendary transformation with respect to the integrand of the function.

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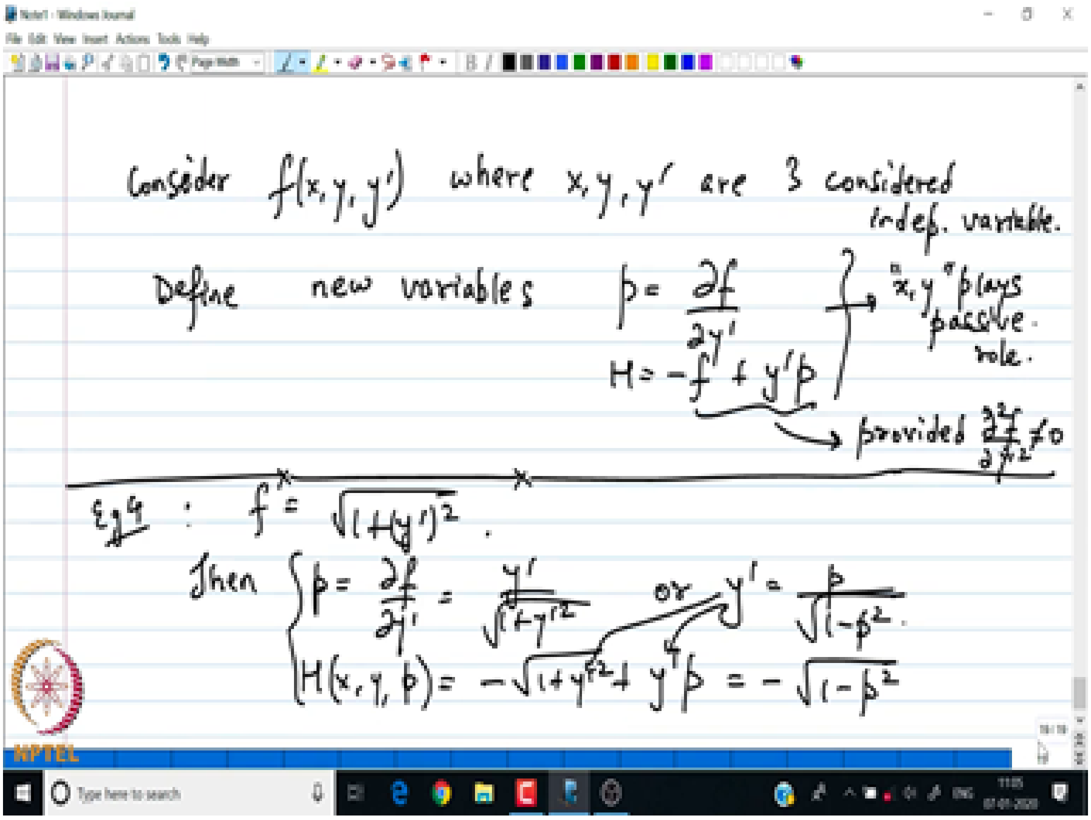

Let us now consider a more general function, consider $f(x, y, y^{'})$ which is nothing but the integrand in our functional in the Euler Lagrange formulation, where x, y and $y^{'}$ are 3 independent variables and we define new variables p = $\frac{\partial f}{\partial y^{'}}$ and $H = -f + y^{'}p$. Note that the role of y is replaced by the role of f and the role of x is replaced by $y^{'}$ and here in this relation x and y plays passive role, they are all implicitly hidden in this definition of the new variable.

And also this quantity is valid provided second derivative of f with respect to $(y^{'})^2$ is not 0, Let me quickly look at an example in this category, suppose I have that f is the standard arch length integrand of the arch length functional $f = \sqrt{1 + (y^{'})^2}$, Then

$$p = \frac{\partial f}{\partial y^{'}} = \frac{y^{'}}{\sqrt{1 + (y^{'})^2}} \text{ or } y^{'} = \frac{p}{\sqrt{1 - p^2}}$$

$$H(x, y, p) = -\sqrt{1 + (y^{'})^2} + y^{'}p = -\sqrt{1 - p^2}$$

This is the way we approach normally when we have to perform the Legendary transformation.

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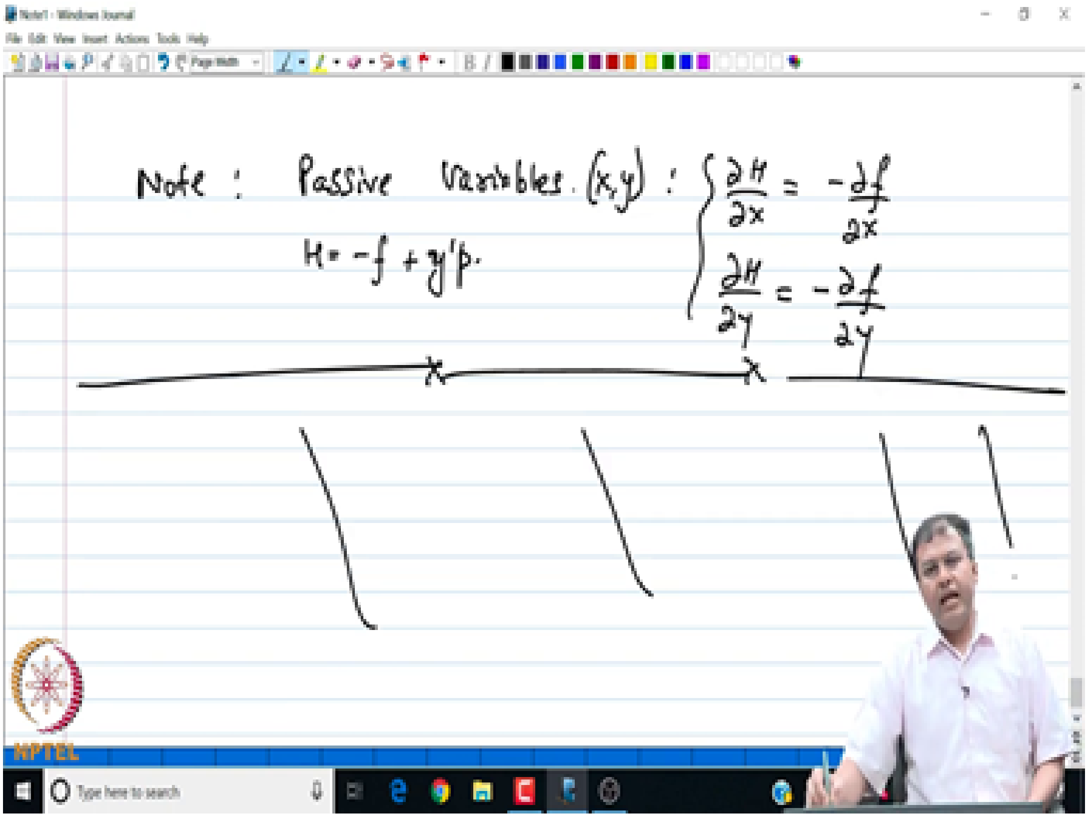

Finally I wrap up my discussion by noting that, note for the passive variables x and y the following relation holds

Note that since $H = -f + y^{'}p$, I have that

$$\frac{\partial H}{\partial x} = -\frac{\partial f}{\partial x} \text{ and } \frac{\partial H}{\partial y} = -\frac{\partial f}{\partial y}$$

I have found all the necessary derivative in the Cartesian frame or in the Euler Lagrange frame to the necessary derivative in the (p, H) frame or the new frame. So, that wraps up my discussion in this lecture and in the next lecture I am going to continue our topic of Hamiltonian Formulation namely how to derive the famous Hamilton-Jacobi equation and how is it useful to solve and give us the extremal in another formulation. Thank you for listening.