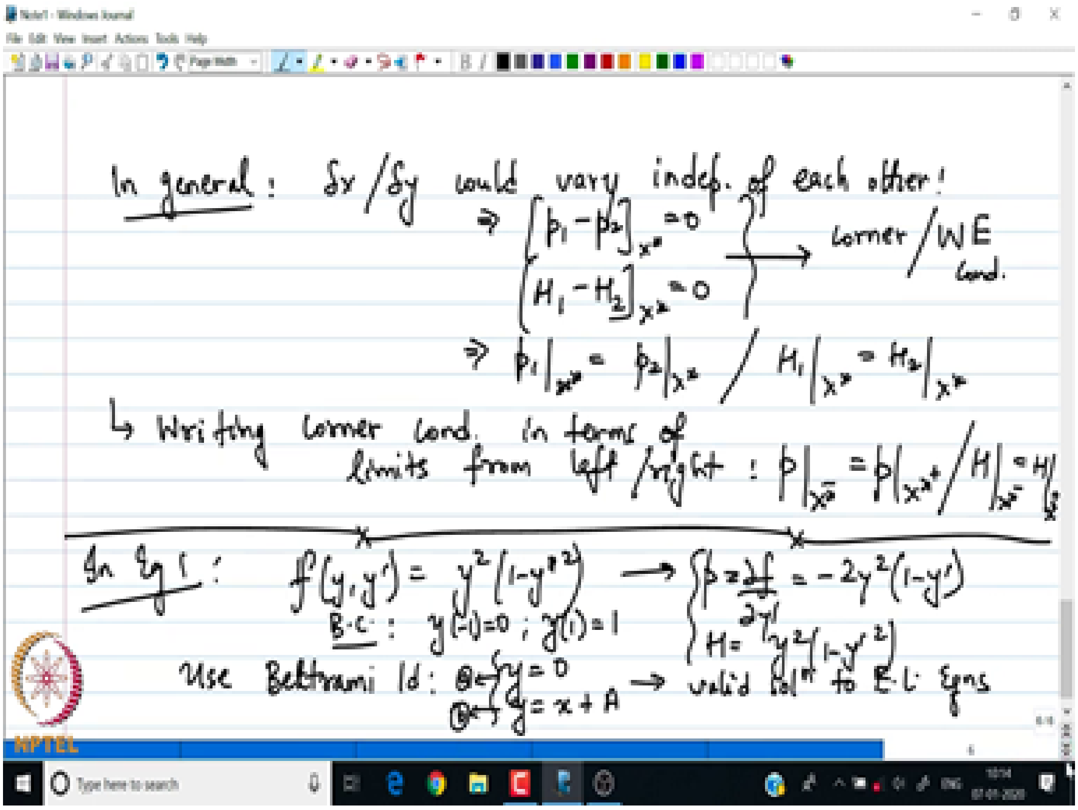

In Example 1, $f(y, y^{'}) = y^2(1-y^{'})^2$ and we saw that in this case $p = \frac{\partial f}{\partial y^{'}} = -2y^2(1-y^{'})$ and Hamiltonian H $= y^2(1-y^{'2})$.

Using Beltrami identity, I see that I get two equations either y = 0 or y = x + A. We have seen that in our discussion of the solution to this example, we are right now assuming one of the results y = x + A x and of course these are all valid solutions to Euler Lagrange equations.
But they do not simultaneously satisfied the boundary conditions y(-1) = 0, y(1) = 1.

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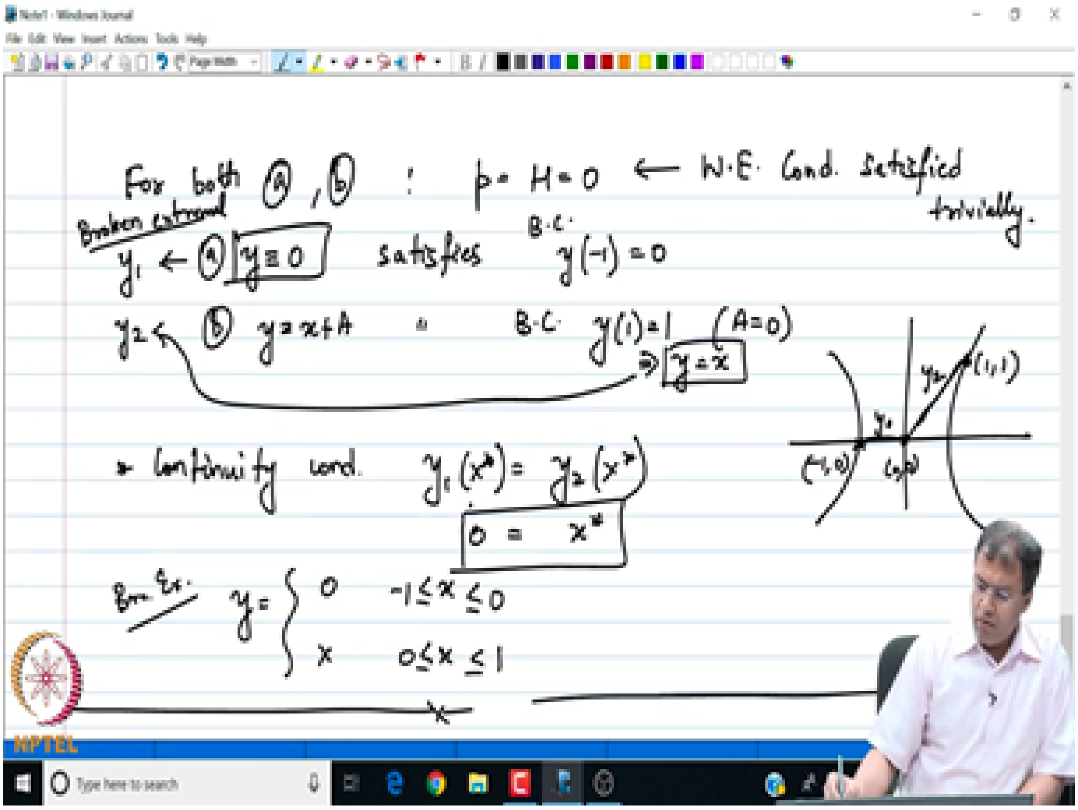

Now, notice that if we find momentum and Hamiltonian for both y = 0 or y = x + A , we can show that the the momentum and the Hamiltonian both are 0, which means that we do not have to worry about the Weierstrass-Erdmann conditions because they are satisfied from the left and from the right they are satisfied trivially.

We have got the following. So, now we have 2 functions, so note that, y is identically 0 satisfies the boundary condition y(-1) = 0 and solution y = x + A satisfies the boundary condition y(1) = 1 provided A = 0. So, extremal in this case is y = x.

Let us draw the figure, we have the rectangular hyperbola, this is my first boundary point and my second boundary point (1, 1). So, we have 2 solutions, one is y is identically 0, the other is y = x and this we have to construct a broken extremal such that the extremal joins the 2 points under consideration.

So broken extremal is such that solution from the left is $y_1 = 0$ and solution from the right $y_2 = x$. Now, my continuity condition of the broken extremals will give me that $y_1(x^*) = y_2(x^*)$, x^* the corner point, the solutions must match.

I get that $0 = x^*$ from here or the solutions match at 0. So, extremal is

$$y = \begin{cases} 0 & -1 \leq x \leq 0 \\ x & 0 \leq x \leq 1 \end{cases}$$

this is my broken extremal solution to this problem, we have at least found an extremal which satisfies the Euler Lagrange equation and also the Weierstrass-Erdmann condition.

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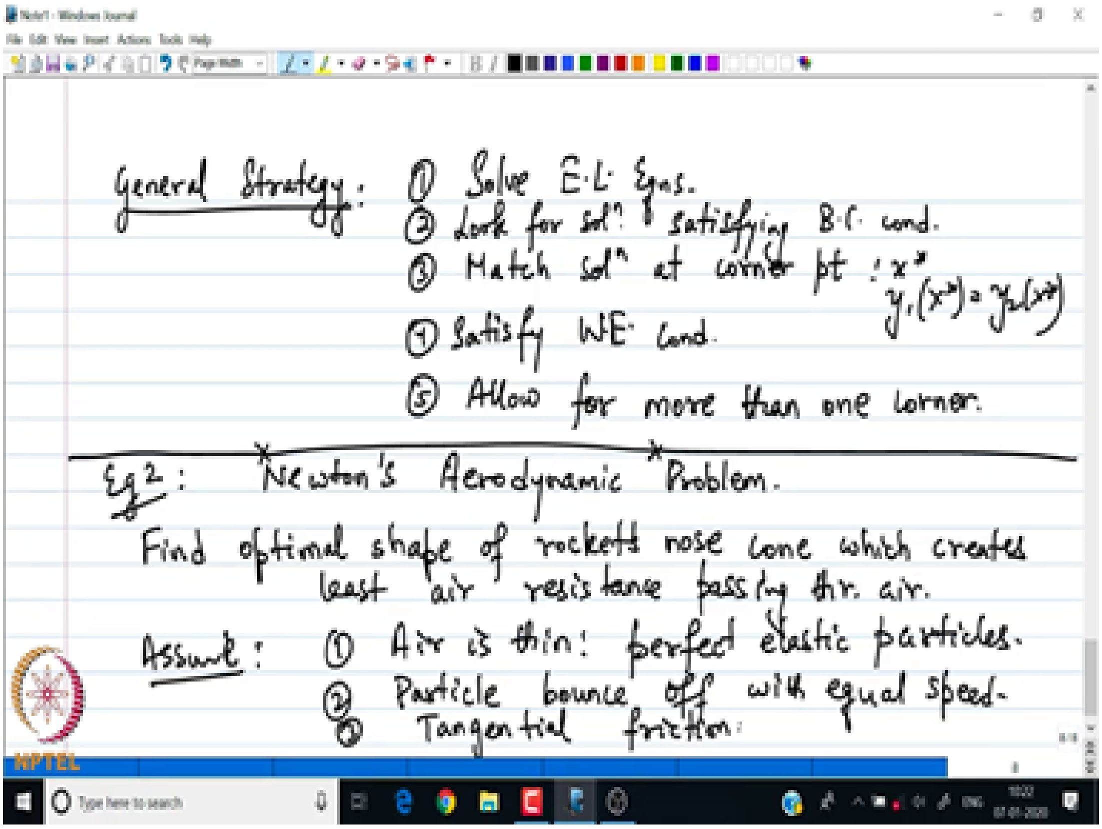

Now, let just summarize our discussion in this topic by saying that how to find this class of broken extremal. So, the general strategy is the following. We solve Euler Lagrange, we look for solutions satisfying satisfying the boundary conditions, though solutions could be different from each other.

And for different solutions we match the solution at corner points, let us say there is just 1 corner point x^*, so we match it by $y_1(x^*) = y_2(x^*)$ and then we also satisfy the Weierstrass-Erdmann condition and finally we allow the problem in general can allow for more than 1 corner point.

Now let us look at a very interesting example in this family of broken extremal namely the example of finding the extremal for the Newton's Aerodynamic Problem. The example I want to discuss is regarding Newton's Aerodynamic Problem. So, what exactly is this problem? Now, physically we have seen when we drive cars, we have seen that the cars are designed in such a way that their front shape almost 90 percent of the design is such that the front shape of the car is all slanted.

Also when we see look at some of the other vehicles which move through the air like the air craft, like the rocket, we see that almost all of these vehicles they have a slanted body at the front and that is because so as to reduce the frictional drag or the resistance of the air. So, we are in this lecture we are going to derive the shape of these objects which move through the air by looking at the frictional drag which acts on the surface of these moving objects.

So, the Newton's Aerodynamic Problem says to find the optimal shape let us say of rocket's nose, so this is just 1 object, rocket's nose cone which creates least air resistance passing through the air, assume we have couple of assumptions in this problem

First, air is thin which means that we have perfectly elastic particles, we will see what is the implications of these assumptions, perfectly elastic particles. Especially when we have the collision of the air particles with the object which is moving through the air, the collision is perfectly elastic.

So, then another assumption is that the particles bounce with equal speed, particles when they move, the air particles bounce through the surface with equal speed and we ignore the tangential friction.

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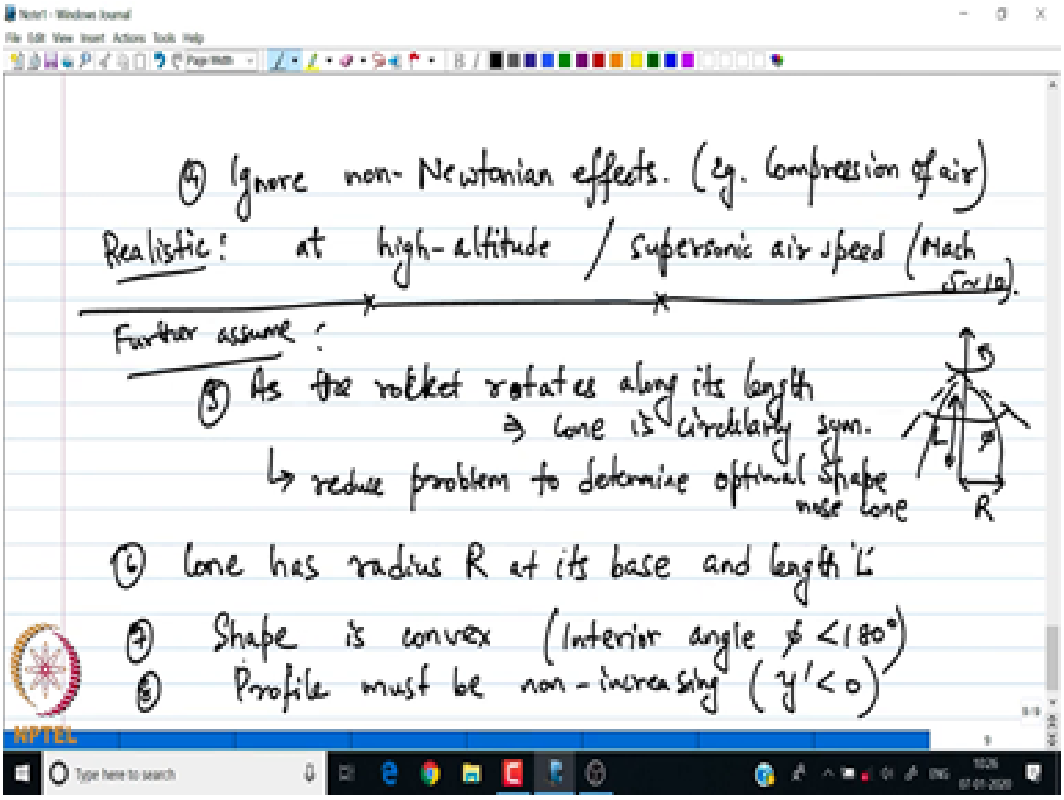

And then we also ignore non-Newtonian effects like the compression of air, rarefaction of air and so on, So, then with this assumption we have not simplified x too much, this assumption still holds for problems where the object is moving is a very thin atmosphere.

So, realistic cases under this assumption follows objects moving through thin atmosphere like high altitude motion where rockets move, motion so realistic at high altitude where and with supersonic air speed, typically of the order of mach number 5 to 10. So, again typically of the order of the speed where rockets move.

Then I have further assumptions in order to set up the problem, we assume that, when the rocket moves it also rotates about its vertical access. To provide stability the rocket while moves up through the air it rotates, which means that the shape will be such that it is symmetric about this length access. so as the rocket along its length implies that the cone is circularly symmetric, we reduce the problem to determine the profile of the nose's cone.

So, reduce problem to determine optimal shape of nose cone, the rest of the body we do not care because that will be more or less cylindrically, will be cylinder. So, then another observation is that the cone is such that it has a radius, it has a radius R at its base, so I am just specifying some dimensions of the object and length 'L'.

Which means that at the base the radius of the cone is R and the length of the cone is taken to be capital 'L' that will give us the boundary conditions. And also, we also assume that the shape of this cone is convex or it means that the interior angle the interior angle of this object is let us say we draw the interior angle, this angle is less than this angle $\phi < 180$ degree and finally we also have that the profile of the cone must be non-increasing. So, what I mean to say is that when we find out the slope of the shape is negative and that follows from assumption 7 that the shape is convex.

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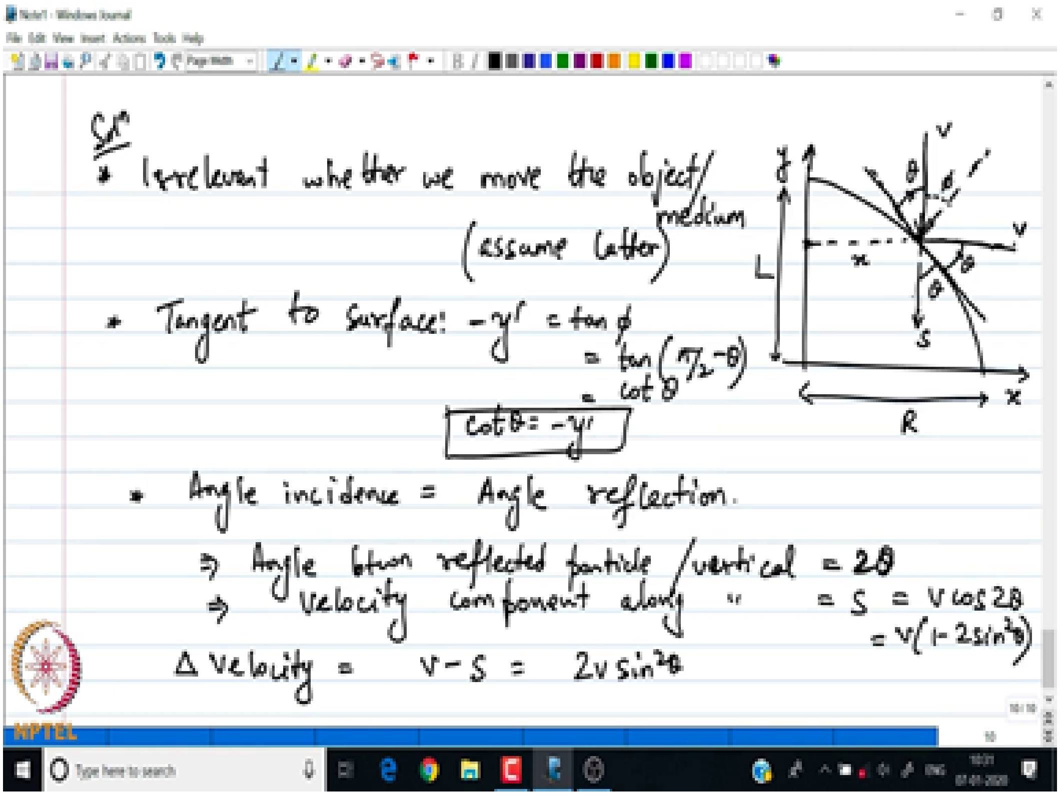

With this I am ready to set up the problem. Now, let me draw half the figure here, so this is my x axis, this is my y axis and let us say this is my length L of the cone and this is my base radius R and from here let me take any point which has a radius x, also let me draw the tangent to this point and let me let me do some geometry here, so this is the horizontal and vertical is as follows and also let us say if I have an air particle which is hitting the surface of the cone let us say it makes an angle θ.

Then by geometry, I have that this angle is also θ, but it bounces off with the same angle so this is also θ, so if this is my V then this will be also my V. So, I am just using all the assumptions that I have just stated in the problem and let us say the component of V along this direction is S. So, I am ready to write down the functional.

So, note that, so first of all it is irrelevant irrelevant whether we move the object or the medium, we can keep the object fixed medium moving or medium fixed object moving to look at the problem. So, we assume the latter that the medium is moving and then also we have to figure out what is the tangent to the surface.

The tangent to the surface is $y^{'}$, $y^{'}$ will be the tangent to the surface with respect to the vertical angle. So

$$-y^{'} = \tan\phi = \tan\frac{\pi}{2} - \theta = \cot\theta$$

let me say that the tangent has a negative sign because it is a decreasing slope. So, what I get is that is $y^{'} = -\cot\theta$

Since my angle of incidence is equal to angle of reflection and this is because of perfectly elastic collision, this means that the angle between the reflected particle and the vertical. So, it implies that the angle between reflected particle and the vertical is 2θ.

And we can see that also from the diagram the velocity component along the vertical is S and we can see that S $= V\cos 2\theta = V(1 - 2\sin^2\theta)$, which means that the change in the velocity is the initial velocity minus the final velocity and I get that this is also equal to $V - S = 2V\sin^2\theta$, which means we can write down the total force or the frictional drag which is acting on the surface.

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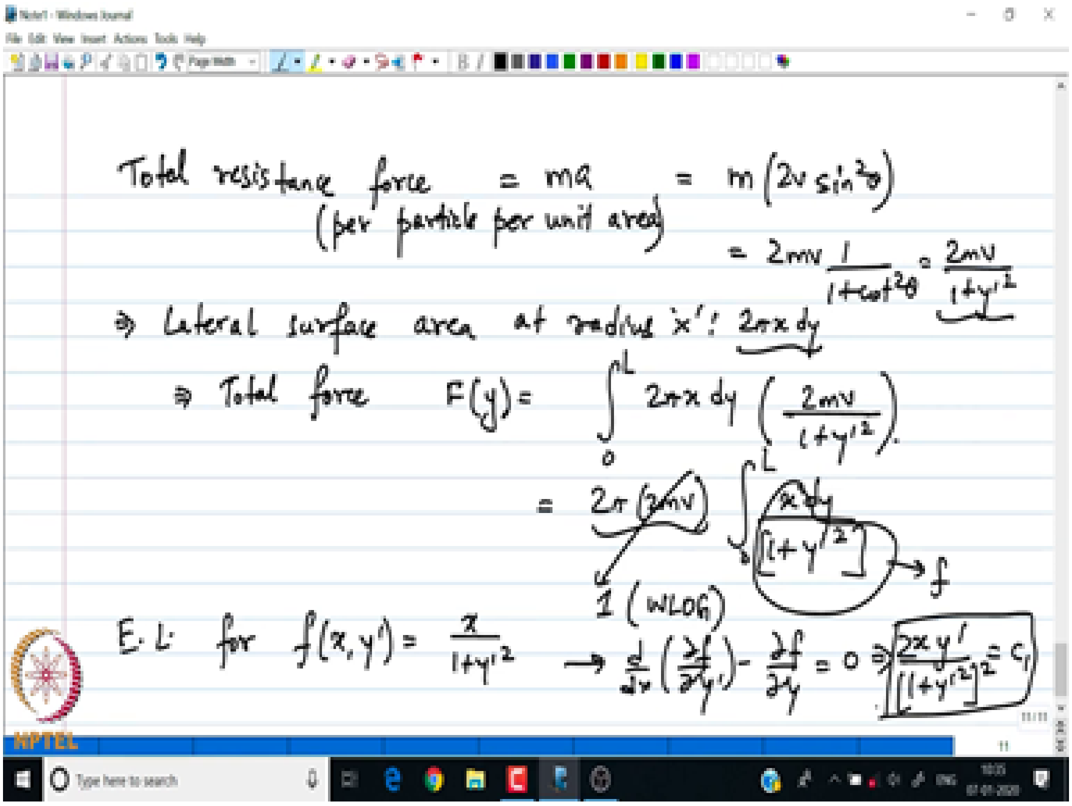

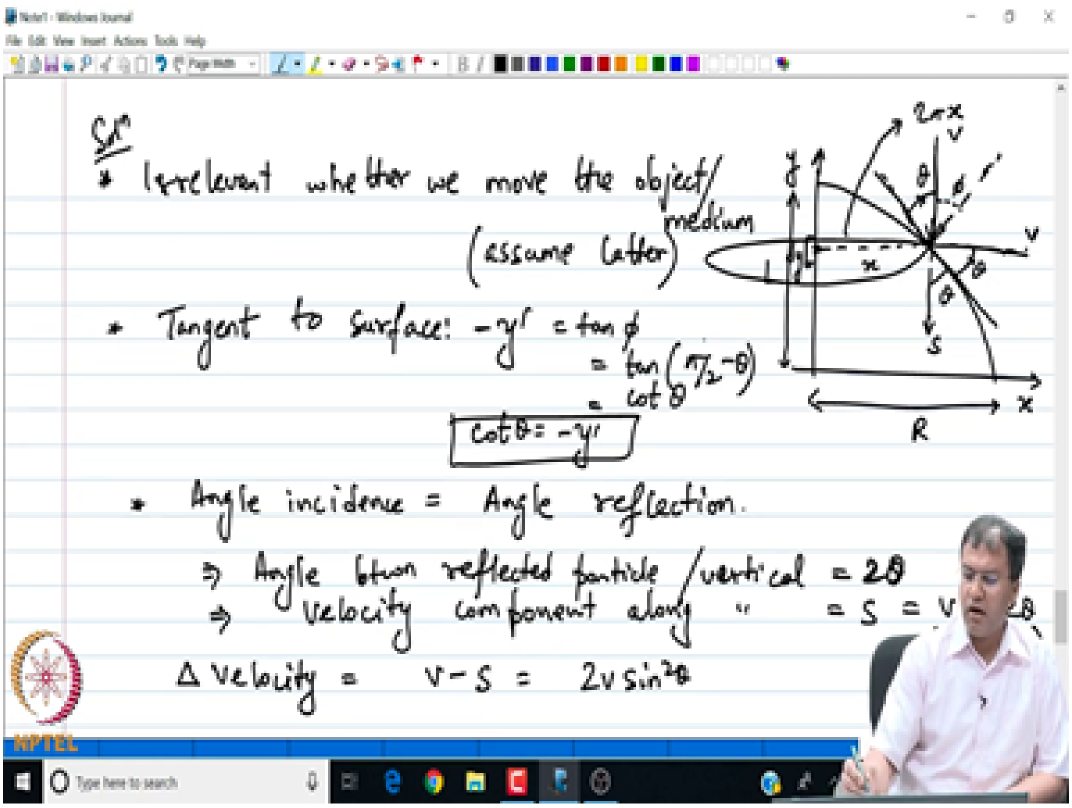

The total resistance force and this will be per particle per unit area. So, the total force is mass times acceleration Newton's second law and that is equal to

$$m(2V\sin^2\theta) = \frac{2mV}{1+\cot^2\theta} = \frac{2mV}{1+y'^2}$$

Then notice again the figure that we have a lateral surface area so we have to figure out the total force. The total force will be force per unit area that we have unit length that we have just found times the integration over the length. So, let us say one element dy has a perimeter or the lateral surface area along the element dy is $2\pi x$, this is a circle of radius x.

Which means my lateral surface area at radius x is $2\pi xdy$, which means total force

$$F(y) = \int_0^L 2\pi xdy\left(\frac{2mV}{1+y'^2}\right) = 2\pi(2mV)\int_0^L \frac{xdV}{1+y'^2}$$

Now, let us take this constant to be just 1 without loss of generality because they are not going to determine the optimal shape.

So, we have to optimize this particular functional, to find the optimal shape we need to write down the Euler Lagrange equation for

$$f(x, y') = \frac{x}{1+y'^2}$$

So, when we do that we have

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \Rightarrow \frac{xy'}{(1+y'^2)^2} = C_1$$

I have integrated once with respect to x, this is my equation that I have to solve to find the extremal shape.

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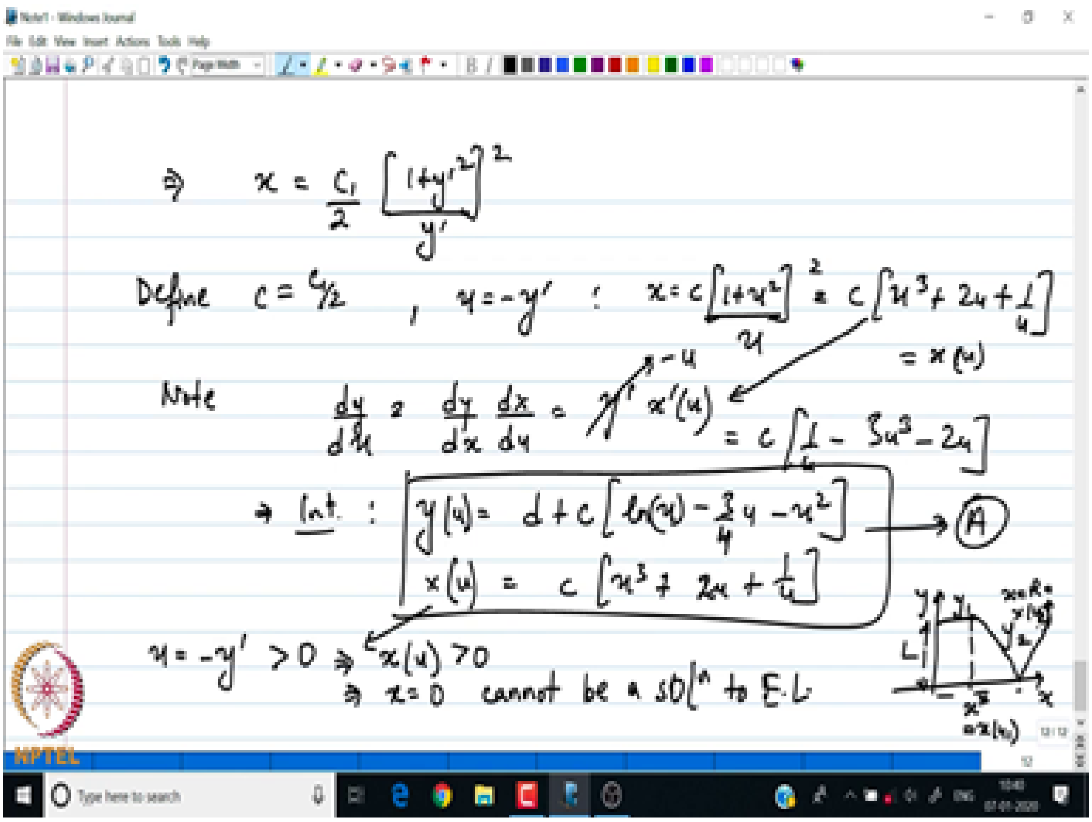

To solve that, what I have is the following, so this means that $x = \frac{C_1}{2}\frac{[1+y'^2]^2}{y'}$. So, let us now introduce new set of constants $C = \frac{C_1}{2}$ and function $u = -y'$.

$$\Rightarrow \quad x = \frac{C[1+u^2]^2}{u} = C[u^2 + 2u + \frac{1}{u}]$$

Similarly I can write my coordinate variable y as a parameter of u, note that

$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = y' x'(u)$$

$$y' = -u \;\text{ then }\; \frac{dy}{du} = C\left[\frac{1}{u} - 3u^3 - 2u\right]$$

$$\text{After integrate} \qquad y(u) = d + C\left[\ln(u) - \frac{3}{4}u - u^2\right] \text{ and } x(u) = C\left[u^3 + 2u + \frac{1}{u}\right] \qquad \mathbf{A}$$

this is my parametric representation of the extremal for both the components x and y.

Now, notice that $u = -y' > 0 \Rightarrow x(u) > 0$, which means that x = 0 cannot be a solution to my Euler Lagrange equation.

So, if we were to draw the shape of the cone, the cone will be such that the profile never hits x = 0 which means that the profile given by **A** will be such that it will never hit the y axis. So, which means we can take 2 profiles, one as y_1 the other as y_2. And the profiles they match at let us say the corner point x^* and let me call this as $x(u_1)$, we need to find u_1 and this is this is x = R which is at $x(u_2)$, some other parameter u_2. So, so what is the conclusion so far?

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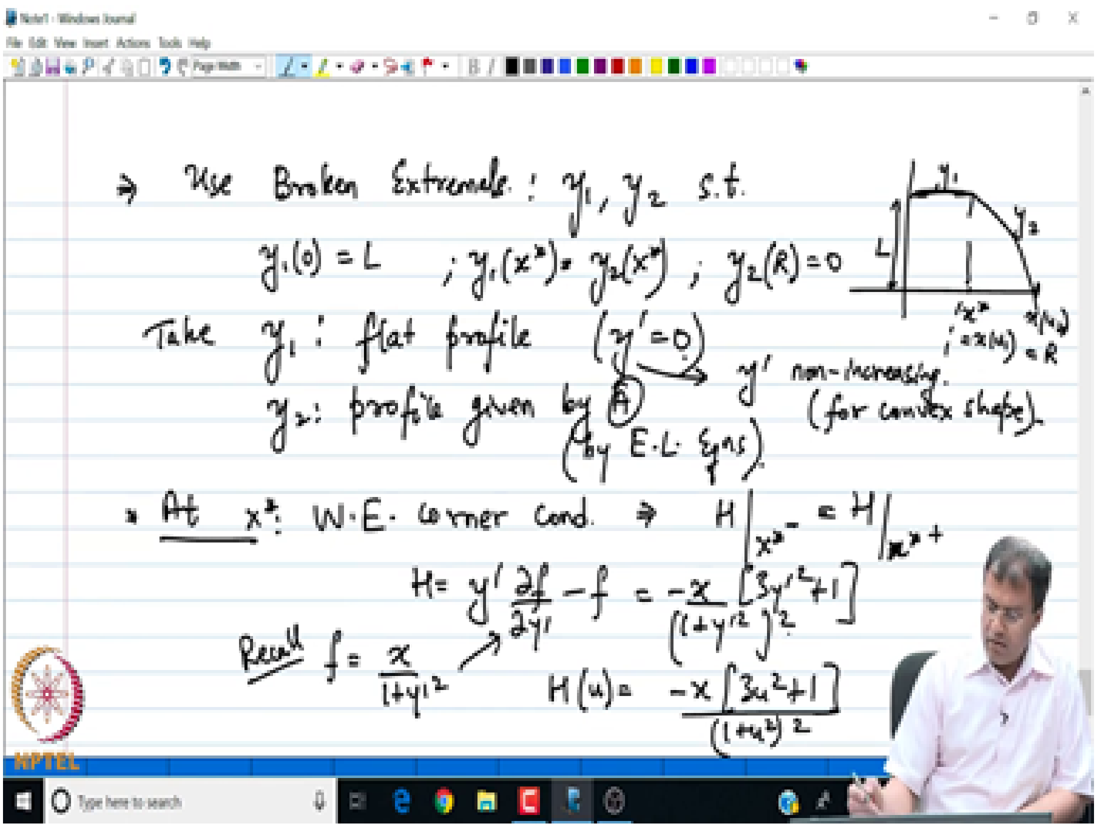

The conclusion is, since x = 0 is not a solution it implies we use the idea of broken extremals to find out the solutions, to find y_1 and y_2, let me again draw the figure. So, we have y_1 let us say of this profile and y_2 of this profile and they match at the point x^* and this is $x(u_2)$ which is equal to R and this is x^* which is also at x at u_1.

So, what have we got? We have broken extremals y_1 and y_2 such that $y_1(0) = L$ and $y_1(x^*) = y_2(x^*)$ and we also have $y_2(R) = 0$. So, we can take the profile as follows, we can take y_1 to be the flat profile i.e. $y^{'} = 0$

We have made our assumptions that $y^{'}$ has to be non-increasing. So, if $y^{'}$ is not negative, it cannot be positive, it can only be 0 that is what we are left with so that the shape is a convex shape.

And then we can take y_2 to be the profile given by my Euler Lagrange solution **A** or this is arrived from the Euler Lagrange equation, note that at x^* the point of continuity we also should satisfy the Weierstrass-Erdmann condition, which means that $H_{|^-_{x^*}} = H_{|^+_{x^*}}$, where my H is this Hamiltonian function

At $x^{*-} \Rightarrow y^{'} = 0 \Rightarrow u = 0$ which means that

$$H_{|_{x^*}} = \frac{-x^*(3u^2+1)}{(1+u^2)^2}$$

At $x^{*+} \Rightarrow y^{'} \neq 0$

$$\Rightarrow \quad H_{|^+_{x^*}} = \frac{-x^*(3u^2+1)}{(1+u^2)^2}$$

$$\Rightarrow \quad H_{|_-} = H_{|_+} \text{ at } x = x^*$$

$$\Rightarrow \quad \frac{3u^2+1}{(1+u^2)^2} = 1 \Rightarrow \quad u = 0, \pm 1 \quad (u > 0)$$

So, now thus question is what is the value of u, I know that u is positive because the profile is non-increasing because u is - $y^{'}$ and this must be positive. So, we take my u = 1, so $x^* = x(u_1)$ and then we plug the value of u_1 to find what is x^* and that will be my corner point.So, finally we have notice how many conditions we have in the problem, we have 4 unknowns u_1, u_2, c and d and we have 4 equations, we have 4 equations give by $y(u_1) = L, x(u_2) = R, y(u_2) = 0$

Well, I already know what is $u_1 = 1$ So, we have 3 conditions, 3 unknowns and 3 equations and hence the system is fully determinable. I want to end the discussion on this topic by mentioning that what have we got. The shape that we have got of the cone profile is a profile of the following, we have got a shape like the following were it is flat atthe top.

So the shape is like a frustum of a cone which is in between 2 parallel plates, this is also the famous shape of Meplats this sort of a profile is also knows as Meplats which is commonly found in bullets, which also move in the limit of high mach number range.