Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 34
Broken extremals / Hamiltonian Formulation Part 4

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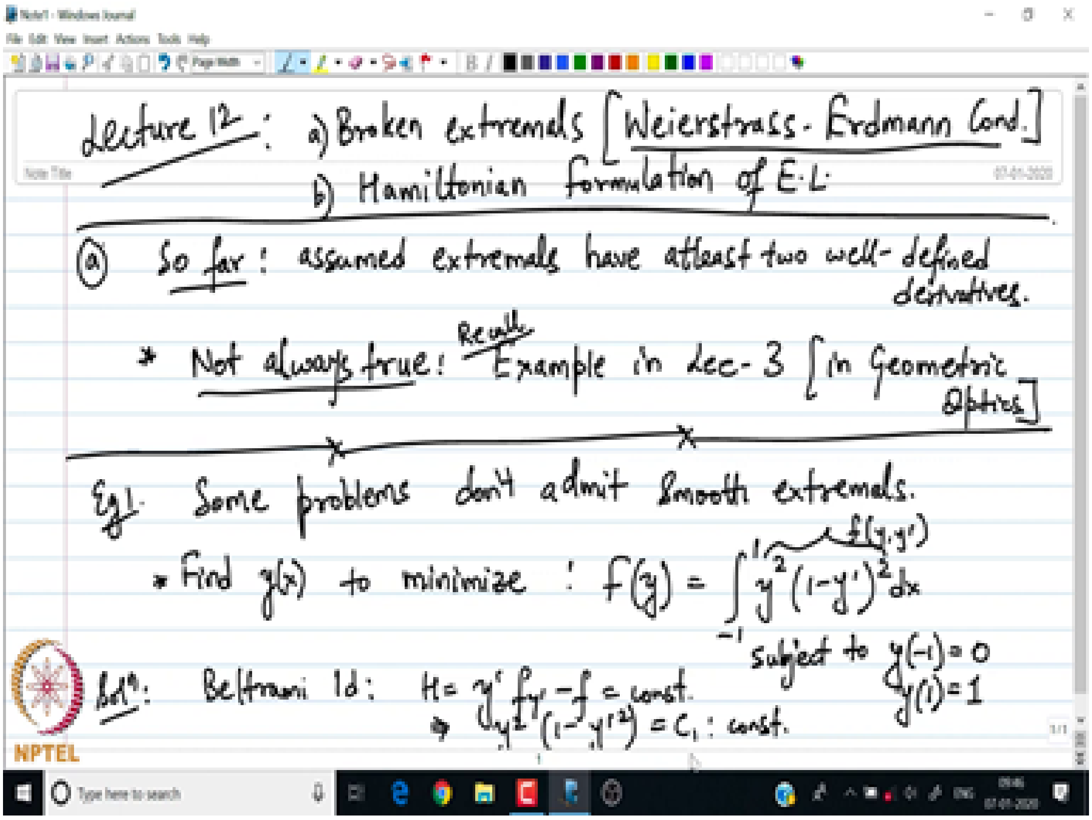

In today's lecture I am going to cover 2 different topics, namely the topics on broken extremals and in the later half of this lecture course I am going to talk about the Hamiltonian formulation of the necessary condition for extremals and I am going to introduce the famous Weierstrass-Erdmann Theorem.

Second half of this lecture will introduce the Hamiltonian formulation which is an alternative formulation of the Euler Lagrange Condition. Let me start the first topic. so far what we have found the extremals of several different types of functional but the underlying assumption of those extremals are that they are at least continuously differentiable up to order 2, if not then higher. So, today in this lecture we are going to relax that criteria and we will assume that the extremals are continuous but may not be continuously differentiable.

So, far the assumption is so we assumed extremals have at least two well defined derivatives, extremals have 2 well defined derivatives, we are going to relax that assumption today. So, this is not I just want to quickly give an example saying that there are several examples in this case of broken extremals or extremals which are not continuously differentiable, so not always true.

And the the the example that we have in mind is the example that we have already seen in our lecture 3, so students should recall the example in geometric optics where we found the optimal path that the light ray covered while following the formats principle, please recall that example of geometric optics.

That's an example that we found out a very easy way of finding a way which did not involve any particular result, a very easy way of finding an extremal which was not continuous in its derivatives but we also promised towards the end of that example that we are going to derive the general conditions. So, in this lecture we are going to do exactly that.

Before I derive the general conditions for broken extremal or the Weierstrass-Erdmann condition let us look at what is the significance of finding these conditions or what sort of extremals belong to this broken extremal category. Well, one quick example is there are several innocent looking functionals in which the extremals are not continuous at all, forget about this discontinuity in derivatives they are not even continuous functions.

Some problems do not admit smooth extremals, the problem that I have is we have to find y(x) to minimize the functional $F(y) = \int_{-1}^{1} y^2(1-y^{'})^2 dx$ subject to y(-1) = 0 and y(1) = 1. So, it seems that we can quickly find the extremal through the Euler Lagrange equation, we can see that for this integrand F is purely a function of y and $y^{'}$ and there is no explicit dependence on x.

To find the solution we can directly use our Beltrami identity saying that $H = y^{'} f_{y^{'}} - f$ = constant and when we plug the value of f, I get $y^2(1-y^{'^2}) = C_1$ which is our constant. So, so let us try to solve this problem, we could, to show the solution to this problem let us first assume that the constant on the hand side is 0, we will look at the case, general case later on.

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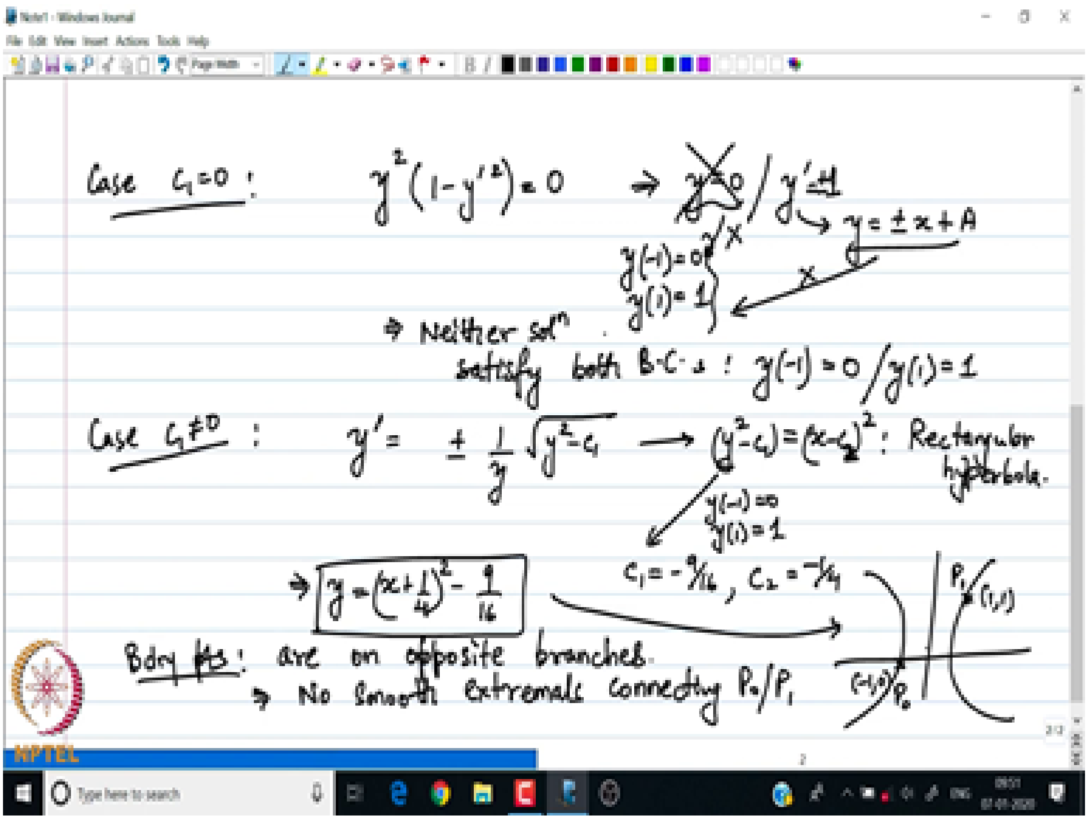

So, case when $C_1 = 0$, then I have the solution to the Euler Lagrange equation as the following $y^2(1-y^{'^2}) = 0 \Rightarrow y = 0$ or $y^{'} = \pm 1$ and this gives us $y = \pm x + A$ So, either the solution is a constant which is 0 or the solution is a straight line.

Further I have 2 conditions, note that y(-1) = 0 and y(1) = 1 , note that the first solution does not satisfy both these conditions. So, I am going to discard the first solution but notice the second solution y(-1) = 0, let me just assume, students can check that even this solution does not satisfy both the conditions.

So, the conclusion here is is neither neither solutions satisfy satisfy both boundary conditions, both boundary conditions are not satisfied simultaneously by neither of the solutions.
Let us now look at the case $C_1 \neq 0$, in that case I am going to rewrite the expression $y^2(1 - y'^2) = C_1$ as $y' = \pm\frac{1}{y}\sqrt{y^2 - C_1}$

When we solve that we get the solution which is $(y - C_1)^2 = (x - C_1)^2$, if we draw the curve will be a rectangular hyperbola, we have a two constant family of solutions, we have C_1 and C_2.

Now, the expectation is the constant C_1 and C_2 can be found through the boundary conditions, note the following, so first of all let me find C_1 and C_2, if we plug y(-1) = 0 and y(1) = 1, I immediately get the constants $C_1 = \frac{9}{16}$ and $C_2 = -\frac{1}{4} \Rightarrow y = \left(x + \frac{1}{4}\right)^2 - \frac{9}{16}$

Now if I were to draw this figure, let us see what this figure looks like, we will have a rectangular hyperbola it has two branches. Notice that (-1 , 0) the first boundary point and (1, 1) lies on the second branch and there is no way we will have an extremal which joins these two points because they are lying on two separate branches.

So, the conclusion here is that the boundary points are on opposite branches and it implies that there are no smooth extremals connecting P_o and P_1, it is quite clear from the diagram, which means that this problem is not going to admit any continuous solution, let alone continuously differentiable solution, so that is why we look for the broken extremals or the condition through which we can at least find extremals which can connect the boundary points which are at most piece wise continuously differentiable.

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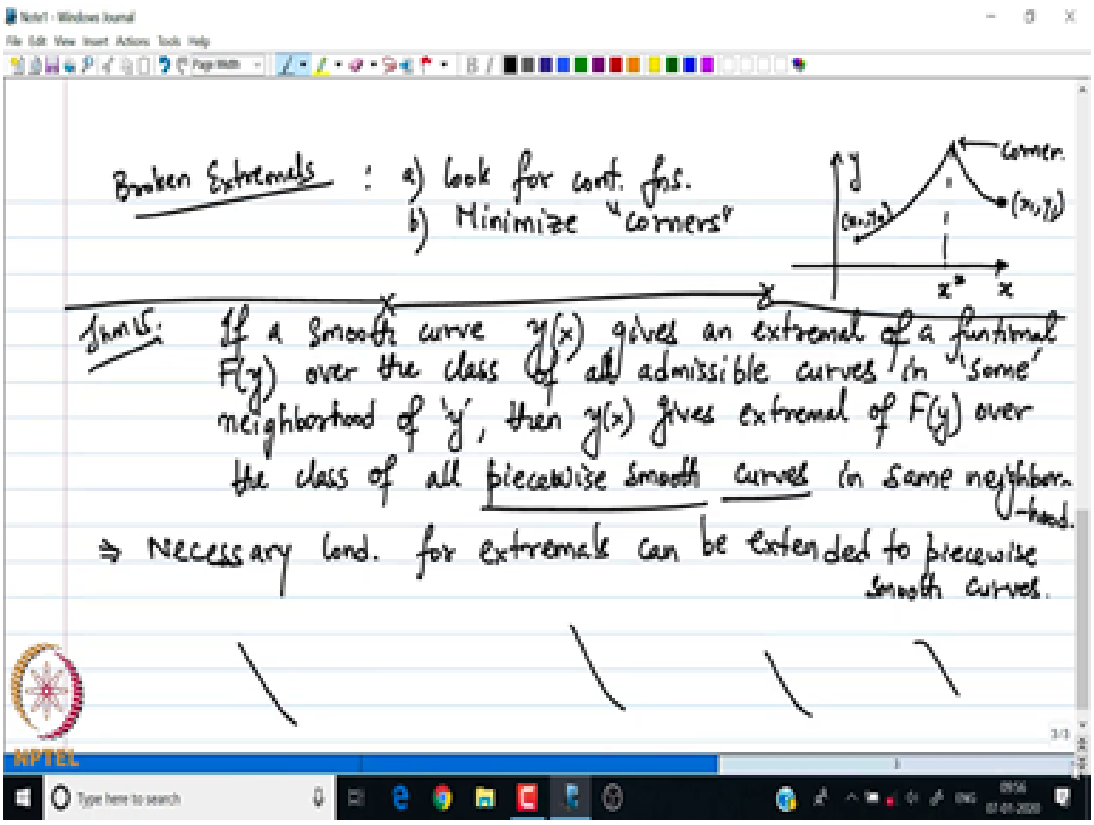

So typically I will have the following figure for broken extremals. So, let us say I am drawing the extremal y(x) and starting point is (x_o, y_o) and (x_1, y_1) are end point is my broken extremal could be the following like this. So, here at x^* I have the so called corner, although we are going to deal with continuous broken extremals but they will not have derivatives or the derivatives are not defined at least at some finite number of points, we call those points as corners.

In broken extremals we look for continuous functions and we try to minimize corners, the less the number of corners the better it is for us. So, thus question is notice that Euler Lagrange equation gives us the extremals which are continuously differentiable the way how Euler Lagrange equations are derived.

Thus question is, are those class of continuous or continuously differentiable functions also extremals over a class of functions which are piecewise continuously differentiable and the answer is yes and it is given in the form of a result which we stated in the form of a theorem.

Theorem 15: If a smooth curve y(x) gives an extremal of a functional F(y) over the class of all admissible curves in 'some' neighborhood of y, then function y(x) gives extremals of F(y) over the class of all smooth piecewise smooth curves in the same neighbourhood.

In short I can say that the necessary condition for extremals can be extended to the class of piecewise smooth curves. So, our Euler Lagrange equations will still work. Now, the second question that we ask is if that is the case the Euler Lagrange equation works how are we going to find these broken extremals? So, let me show you how we are going to find these broken extremals.

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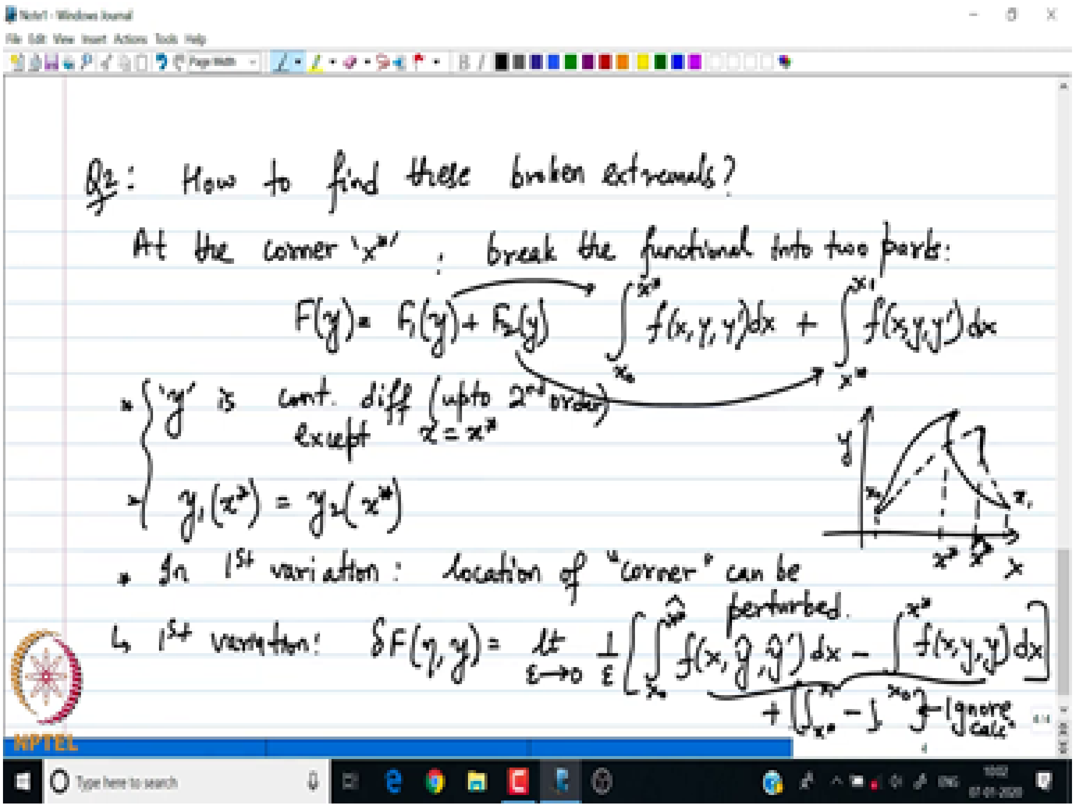

The second question that we have asked is how to find these broken extremals? Now, to do that let us say at the corner point x^* we break the integral into two parts, our functional f(y) $= F_1(y) + F_2(y)$, where $F_1(y) = \int_{x_o}^{x^*} f(x, y, y^{'})dx$ and $F_2(y) = \int_{x^*}^{x_1} f(x, y, y^{'})dx$.

We have broken the entire integral at the corner point. So, let me draw this figure, so we are talking about let us say a functional of this form where this is my corner point.

So, then the requirement is that 'y' is continuously differentiable up to second order except at the corner point $x = x^*$ and further I also require that the solution is at least continuous which means that the solutions y_1 matches with the solution y_2 at x^*, so they are at least continuous over the entire interval.

Since we are following the similar strategy of finding Euler Lagrange we have to set up the first variation and that is we do by perturbing the function. Now, thus question is what happens to this corner points during perturbation, it turns out that we can also change the corner points while perturbing our extremal and finding the variation in the functional.

So, what I said is the following, in our first variation when we derive our Euler Lagrange equations the location of "corner" can be perturbed.
So, we can also have a similar situation of this form where new location of the corner is $\hat{x^*}$, so we can always perturb, this is also allowed in our derivation.

So, which means when we write down the first variation $\delta F(\eta, y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\int_{x_o}^{\hat{x^*}} f(x, \hat{y}, \hat{y}^{'}) dx - \int_{x_o}^{x^*} f(x, y, y^{'}) dx \right]$ plus there will be an integral, there will be two more integrals from the other interval from x^* to x_1.

So, at this point we just ignore these two integrals to just highlight how are we going to write this difference. So, whatever result we find out for this difference that I have shown will be also valid for the other difference, so that is why we ignore the calculation, we ignore the calculation of this setup because it follows the similar steps as in the first two, as in the processing of the first two quantities here.

Notice, in the last lecture we have derived the transversal criteria, as with the transversal criteria if we notice this difference of the integral quantity we expect that this will boil down to the Euler Lagrange criteria, this difference plus the other difference will boil down to Euler Lagrange criteria plus some extra terms which will not vanish.

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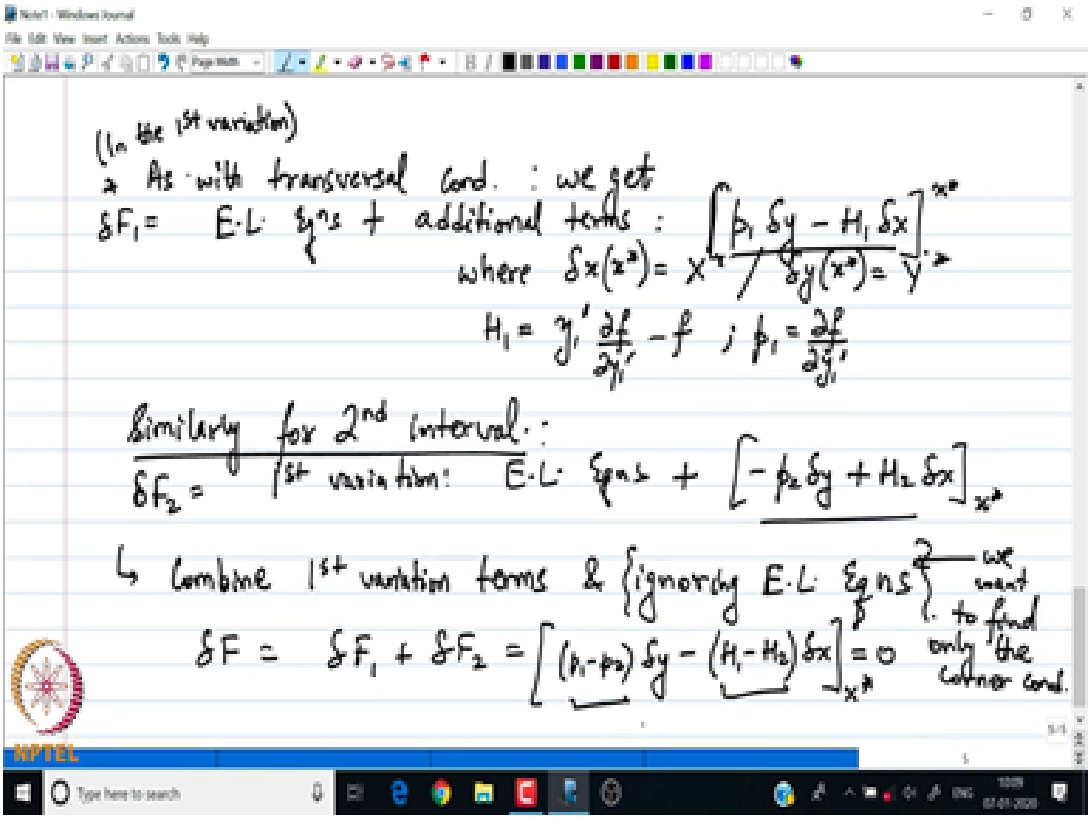

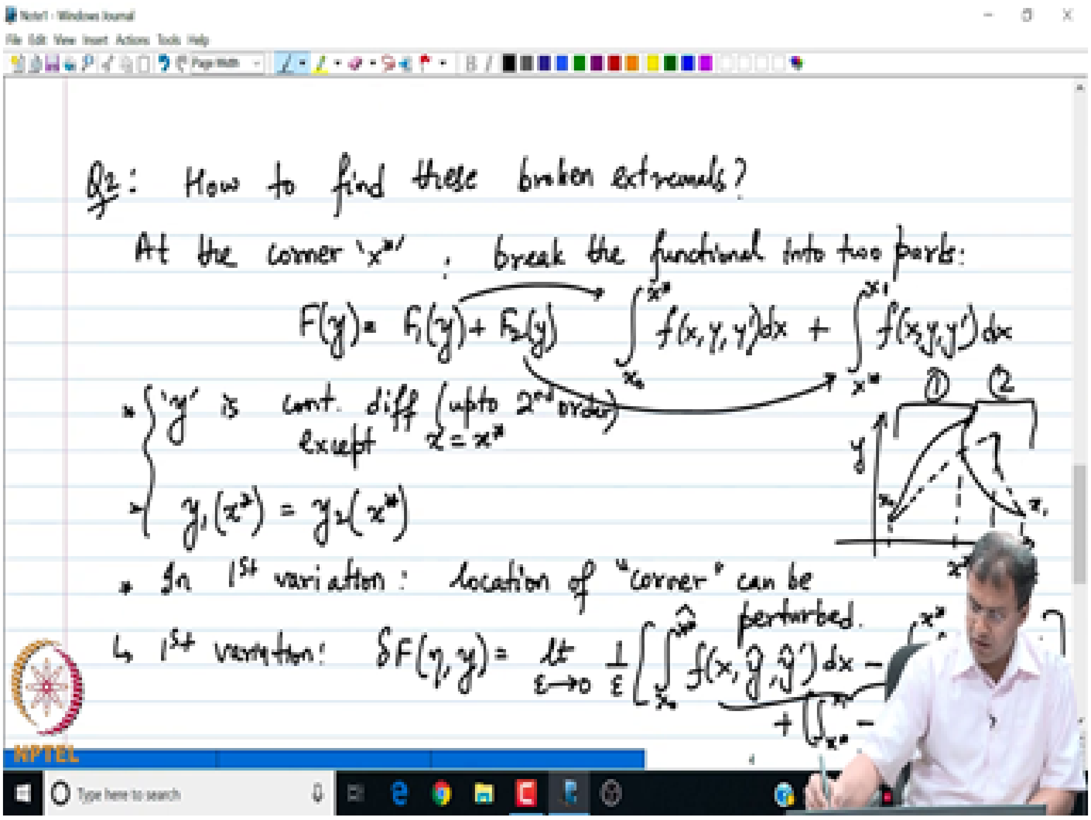

So, what I just said is that with the transversal condition, we get Euler Lagrange equations plus additional terms and those additional terms once we simplify will come out to be the following over the first interval the additional terms are p_1 the same transversal terms that we had found in transversal condition $p_1\delta y - H_1\delta x$ where this is over the first interval.

So, over this first interval my additional criteria is this quantity set equal to 0 and so the additional terms are these quantities in the first variation, what I am trying to say is we have not yet set the first variation to 0. So, I am just writing down the terms that we will obtain in the first variation, we will get Euler Lagrange in the integral as an integral constraint plus these certain other quantities.

Now, where my $\delta x(x^*)$ the corner point comes out let us say that this is X^* and $\delta y(x^*) = Y^*$, so, which means my Hamiltonian function is by definition $H_1 = y_1^{'}\frac{\partial f}{\partial y_1^{'}} - f; p_1 = \frac{\partial f}{\partial y_1^{'}}$.

Similarly when we take from x_1 to x^*, so x^* is the upper bound. So, this term is evaluated such that the evaluation at x^* is the one on the right hand side it is the upper bound. So, similarly for the second interval the first variation will again give Euler Lagrange equations plus the terms of the form $[-p_2\delta y + H_2\delta x]$.

When I say the second interval I am talking about this interval over which the perturbation is happening that is towards the right of x^*. Now, notice that I have put in a minus sign in this term as compared to the previous term because now x^* appears on the left. So, now x^* the value evaluated at x^* serves as the lower bound and hence the change in sign.

We combine the first variation terms so I am not deriving very rigorously for the case of broken extremal because we have done so for the case of continuous extremal and so I am just giving some basic ideas. So, variation, so we combine the first variation terms and right now we ignore the interior, we ignore the Euler Lagrange equations because that needs, well, that will be true so and ignoring the Euler Lagrange equations and that is done because we are, we want to find only only the corner conditions.

So, the Euler Lagrange equations are very well satisfied for smooth extremals but at the corner it is these two extra terms that will give us the necessary condition. So, once we ignore the Euler Lagrange equation we get that the first variation of f will be $\delta F = \delta F_1 + \delta F_2 = [(p_1 - p_2)\delta y - (H_1 - H_2)\delta x] = 0$ and this is evaluated at x^* on the lower bound.

Now the corner conditions will be such that this will also be 0, so the so the first variation becomes 0 when not only the Euler Lagrange expression becomes 0 but also the corner condition becomes 0, only then we are guaranteed to have extremals. So, from here, so let me call this as, so this is the combined condition that we have at the corner we could have one quantity varying independent of the other quantity to get two sets of corner conditions.

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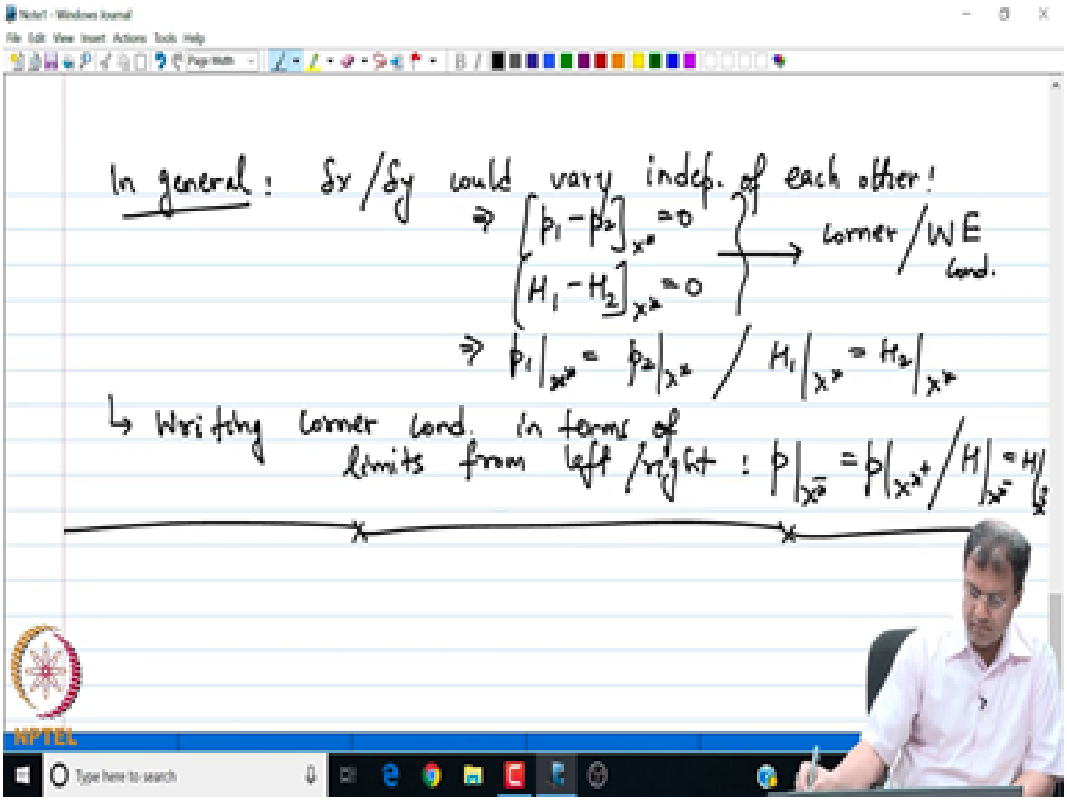

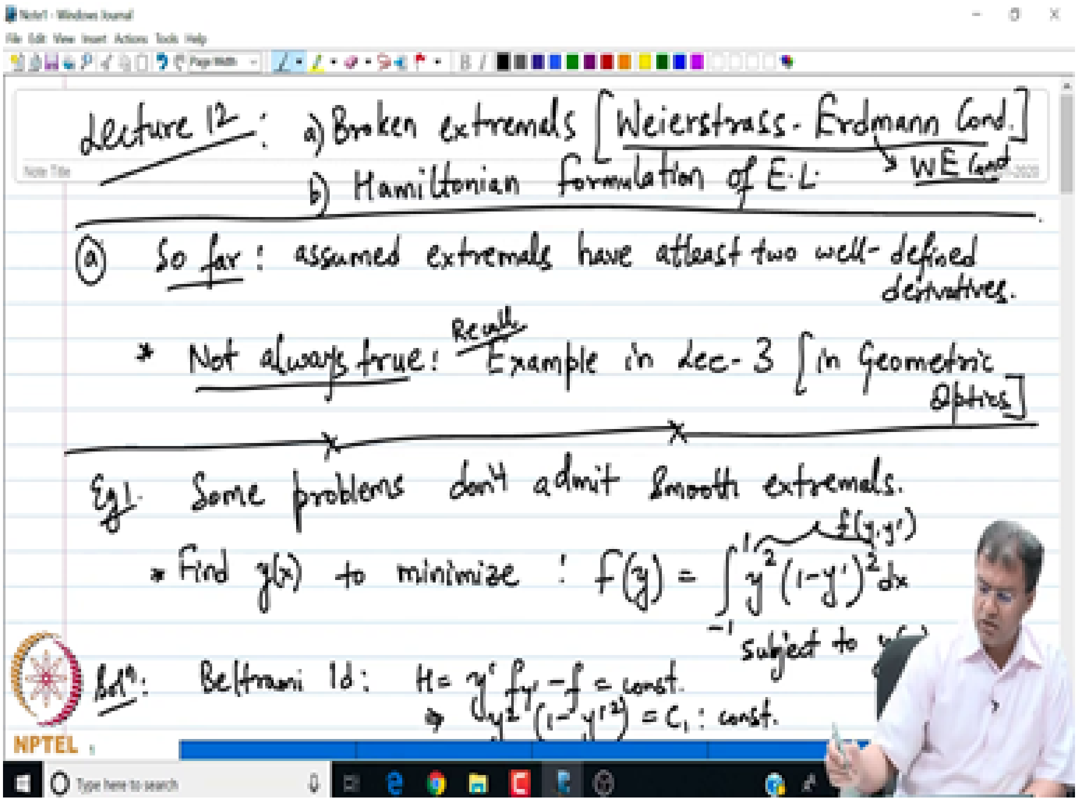

So, in general δx and δy could vary independently of each other, so not necessarily they are combined and then we will have which means that the coefficients at the corner conditions would be separately 0 and hence that gives me my corner conditions or the so called the Weierstrass-Erdmann condition.

let us say that my Weierstrass-Erdmann conditions are WE conditions, so I am using a short abbreviation in my subsequent discussion, so WE conditions. So, from here I get $p_1(x^*) = p_2(x^*)$ and similarly $H_1(x^*) = H_2(x^*)$, then we could also rewrite this corner conditions by approaching we could say that H_1 is the evaluation of the Hamiltonian for the function from the left of x^*.

Because y_1 is at the left of x^*, the extremal y_2 is at the right of x^* so this condition on the right hand side is evaluated on the right of x^* and similarly for the other equation, are $p_1(x^*) = p_2(x^*)$ and $H_1(x^*) = H_2(x^*)$

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Lecture 35
Broken extremals / Hamiltonian Formulation Part 5

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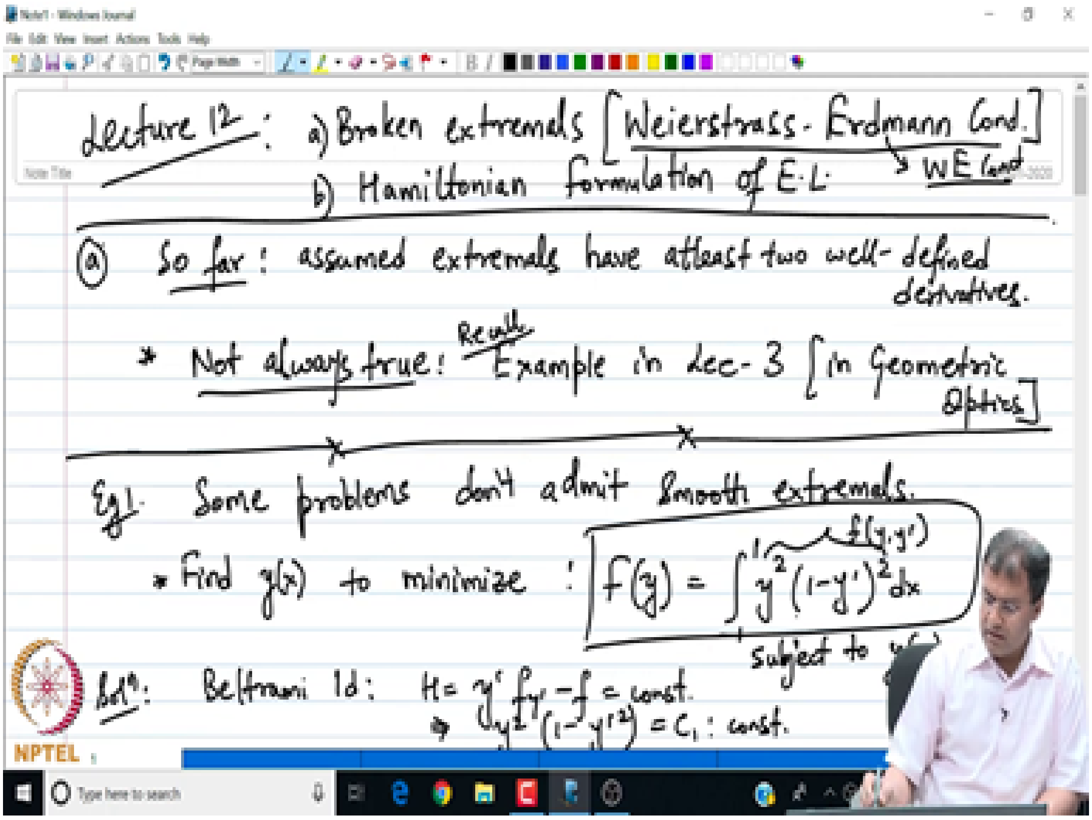

Let us now let us now go back to our example 1, we would like to see how to solve the same example for which we did not even get the extremal. So, I am talking about the example that we discussed right in the first slide with the extremal given as follows.

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