Variational Calculus and its Applications in Control Theory and Nano mechanics
Professor Sarthok Sircar
Department of Mathematics
Indraprastha Institute of Information Technology, Delhi
Lecture 33
Broken extremals/ Hamiltonian Formulation Part 3

let us look at another example, but before that let me also introduce an further generalized scenario of variable end point criteria.

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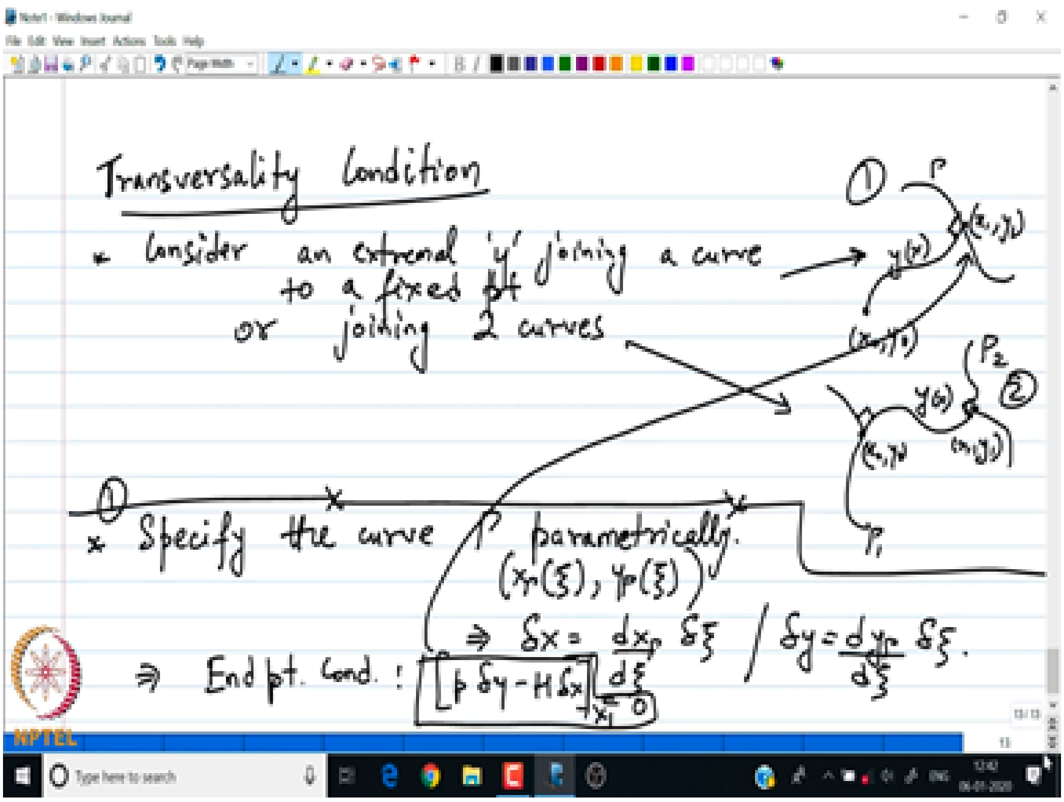

I am going to talk about a topic, that is known as the Transversality condition. So what exactly is this condition and when is it going to be applicable Transversality condition. So what exactly is this. Suppose we consider an extremal which is y (x) is an extremal. It joints the point let us say x (x_o, y_o) to a curve, let us say this is a curve which is denoted by Γ. We will show that the joining point is such this point (x_1, y_1). It joins the extremal such that joins the point to a curve and the joining point is such that the point of intersection is always at right angles. This is something we will show very soon.

Thus the question is what is the condition, what is the equivalent boundary condition for this set up where we have to extremize a functional, extremize subject to one of the end points on a given curve, on a specified curve Or we could also have a case where the extremal y(x) with two end points (x_o, y_o) and (x_1, y_1) are on two specified curves Γ_1, Γ_2 Well we will show that in both these points of intersection, this curve intersects extremals always at right angle. So again we would like to figure out what is the end point criteria for this case as well or in general what happens if the end points lie on a specified curve. So what I said is the following.

Consider an extremal y(x) joining a curve to a fixed point, that is my case 1, for example finding the shortest distance between 2 curves or the shortest distance between a point in a curve or a shortest distance between 2 curves, for such a case how to approach such a scenario, what we do is in this situation we are going to specify, first let the curve we defined in the form of a parameter. So we specify the curve Γ parametric, I am talking about case 1 here.

So specify my curve Γ parametrically, let me call this as the curve with components $(x_\Gamma(\zeta), y_\Gamma(\zeta))$
$\delta x = \frac{dx_\Gamma}{d\zeta}\delta\zeta / \delta y = \frac{dy_\Gamma}{d\zeta}\delta\zeta$
End point criteria condition $p\delta y - H\delta x|_{x_1} = 0$, this is the condition I am going to talk about. So I am going to replace δx and δy since they lie on the curve, so which means I get the following.

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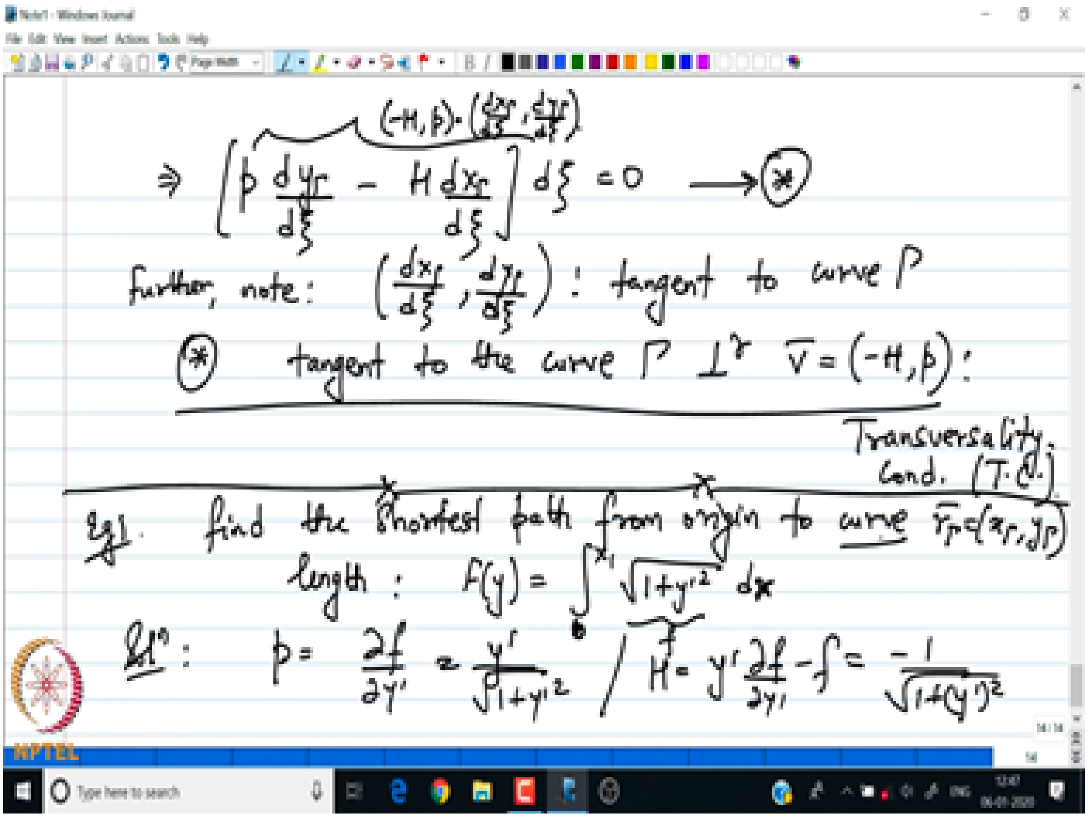

$$\Rightarrow \left[p\frac{dy_\Gamma}{d\zeta} - H\frac{dx_\Gamma}{d\zeta}\right]d\zeta = 0 \qquad *$$

These are tangent vector to the curve Γ, so this is nothing but the tangent vector* says that my tangent to the curve Γ is perpendicular to the $\bar{v}$ which is prescribed by (-H, p). Why, because notice that this is nothing but the dot product of $(-H, p).(\frac{dy_\Gamma}{d\zeta}, \frac{dx_\Gamma}{d\zeta})$ and we show that the dot product is 0.

Now we have shown that the transversality condition, what it says is that whenever the end points they meet under curve they will always do so such that they meet at right angles. I call this in short notation T.C, Let us look at a quick example in this class of function. So we have to find the shortest path from origin to the curve $\bar{\gamma} = (x_\Gamma, y_\Gamma)$ with the path length given by F(y) = $\int_0^{x_1} \sqrt{1 + {y'}^2}dx$, we see that of course the solution to this integral is the straight line.

Solution : $p = \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+{y'}^2}}, H = y'\frac{\partial f}{\partial y'} - f = -\frac{1}{\sqrt{1+{y'}^2}}$.

My Transversality condition let us plug in, well I do not need to solve the Euler-Lagrange because I already know the extremal which will be a straight line. I need to figure out the family of the specific family of the straight line. So I am going to directly use the end point criteria which is now going to be the transversality condition because one of the end point lie on the curve.

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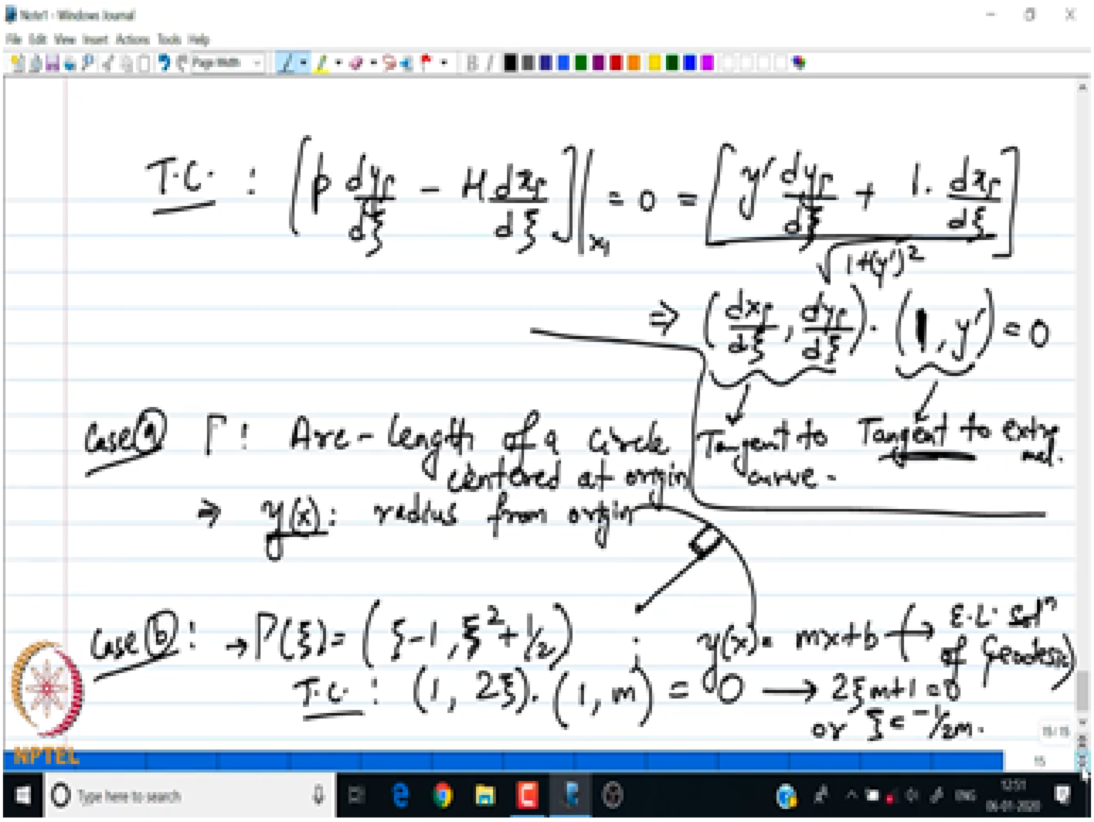

$$T.C: \qquad \left[p\frac{dy_\Gamma}{d\zeta} - H\frac{dx_\Gamma}{d\zeta}\right]|_{x_1} = 0 = \frac{\left[y'\frac{dy_\Gamma}{d\zeta} + 1.\frac{dx_\Gamma}{d\zeta}\right]}{\sqrt{1+y'^2}}$$

$$\Rightarrow \left(\frac{dx_\Gamma}{d\zeta}, \frac{dy_\Gamma}{d\zeta}\right).(1, y') = 0$$

Note that this is nothing but the tangent to the extremal curve, that we need to find. And this is nothing but the tangent to the curve on which the end point is defined. So the dot product is 0. So I know that, the further solution to this equation is possible once we know the exact form of Γ.

let me look at some specific cases, so suppose look at a simple case where my Γ is the Arc length of a unit circle centered at origin, I separate this centered at origin and then I expect that my extremal y(x) will be the radius because in that case for an arc length of a circle only the radius is going to meet the arc length at distance at perpendicular to the arc length, so that is my extremal, that can be found from directly from a Transversality condition. Let us look at slightly more involved case, suppose $\gamma(\zeta) = (\zeta - 1, \zeta^2 + \frac{1}{2})$

I know that my extremal y(x) = mx + b , this is directly from Euler-Lagrange solution of Geodesic. So we are solving the Geodesic problem, from here what I see is the following, the Transversality condition gives me, if it differentiates, I see that $(1, 2\zeta).(1, m) = 0 \;\; \Rightarrow \;\; 2\zeta m + 1 = 0$ or $\zeta = -\frac{1}{2m}$ Now further we know that there is a common point of meeting of the curve and the extremal.

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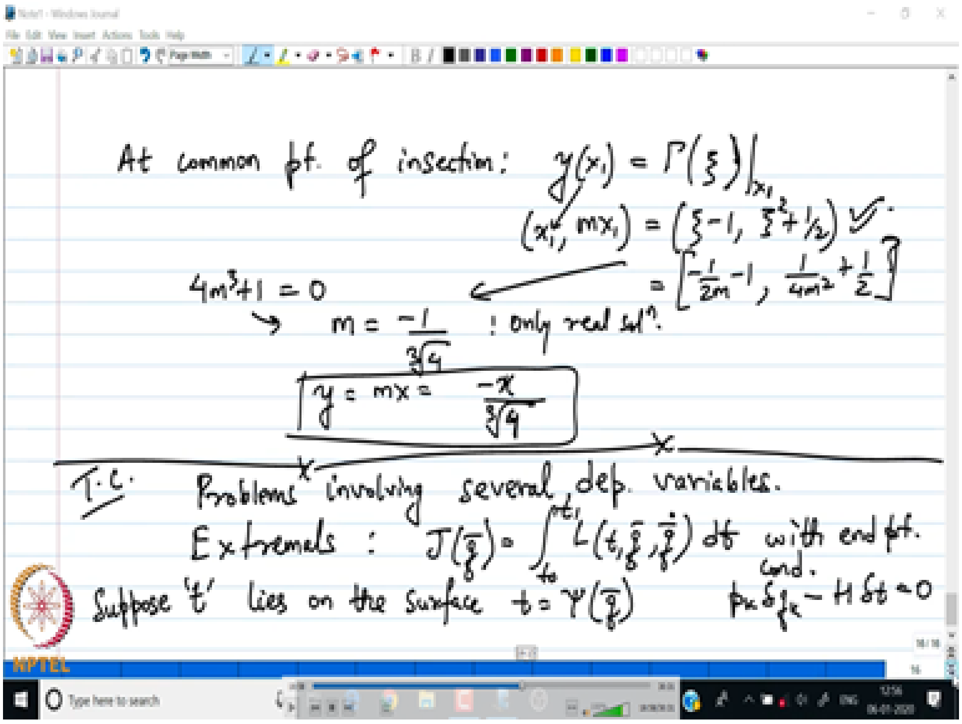

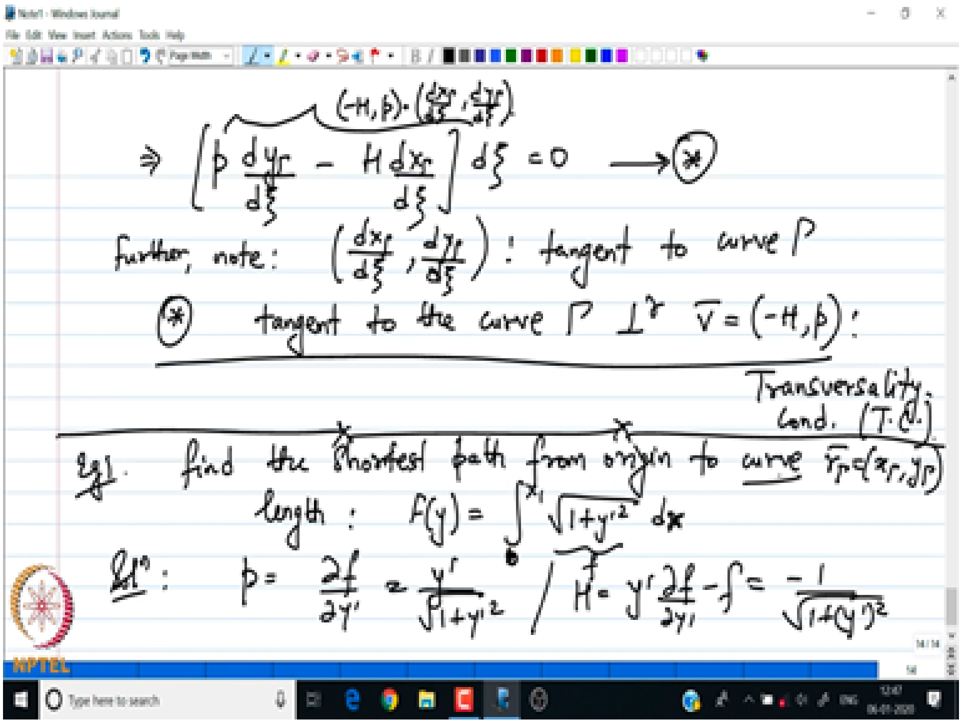

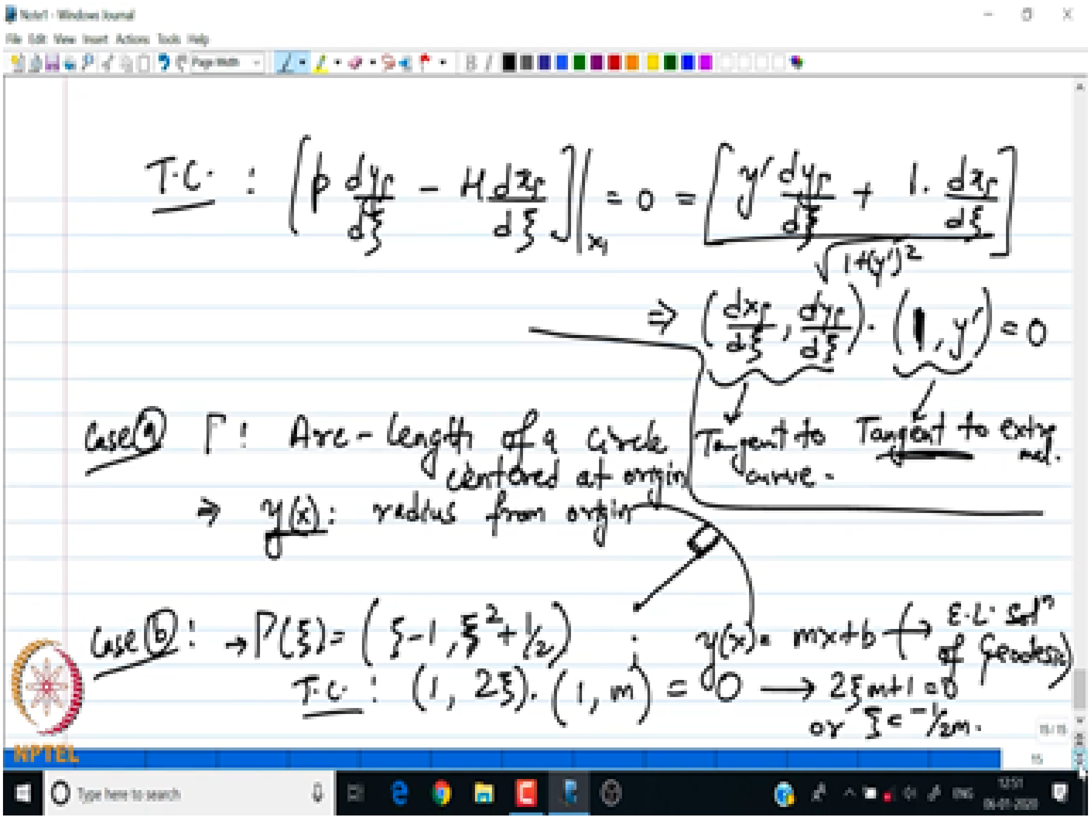

So at common point of intersection, I see that $y(x_1) = \Gamma(\zeta)|_{x_1}$ So what have we got is $y(x_1)$ will have the following form, the xy coordinates are (x_1, mx_1), we have one more condition that we have the original problems is that the starting point is from the origin. So my extremal will be such that my p is 0, so my extremal is y = mx because only this straight line passes through the origin. So which means $(x_1, mx_1) = (\zeta - 1, \zeta^2 + \frac{1}{2}) = \left[-\frac{1}{2m} - 1, \frac{1}{4m^2} + \frac{1}{2}\right] \Rightarrow 4m^3 + 1 = 0 \Rightarrow m = \frac{1}{\sqrt[3]{4}}$: Only real solution $y = mx = -\frac{x}{\sqrt[3]{4}}$ which is my specific class of function which satisfies the Transversality condition given this curve. So that completes the description of the problem.

let us now look at another generalize this Transversality problem for class of functional having several dependent variable. So what I just said is the following, the problems involving several dependent variables, we are talking about extremals of the form $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ of with end point condition given by $p_k \partial q_k - H\partial t = 0$, now further to impose the Transversality condition we also have to impose a curve on which one of these or both of these end points are lies, so suppose free variable t independent variable lies on the surface because now it is multiple dependent variables so we have surface given by $t = \psi(\bar{q})$ which means that I need to find the variation δt and $\delta\bar{q}$ or $\delta q_1, \delta q_2$. Now we look at that variation one by one.

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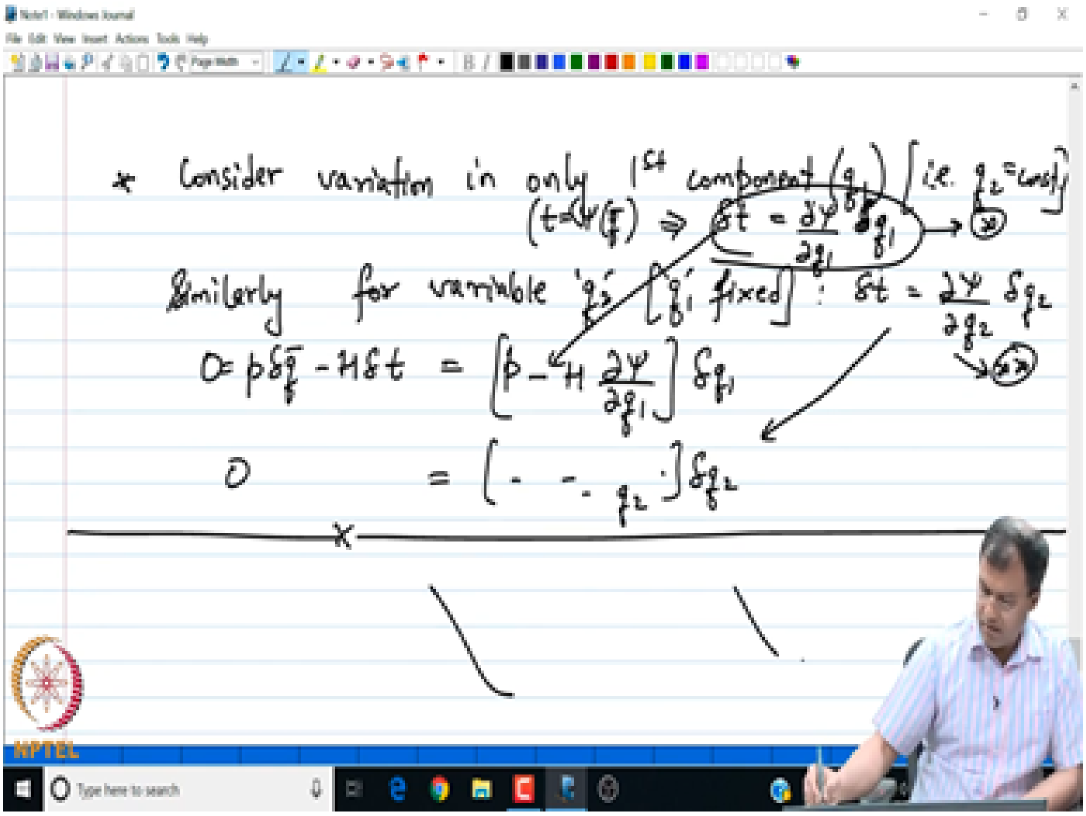

So let me just say that we look at variation only in the first component of q assuming the variation in the second component is absent, so consider the variation in only first component, let us say q_1, which means we are holding q_2 to be constant, the second component as constant and what I get is from here is that since t is in on the surface I see that $\partial t = \frac{\partial \psi}{\partial q_1} dq_1$ $*$
So δt can be express in terms of δq_1 as follows because t lies on the surface. Similarly, for variable q_2 for the other component q_2 with q_1 fixed, I can derive similar relation between δt and δq_2.

All these extremals are such that they satisfy the system of Euler-Lagrange equation. So now notice let me call this as star and call this as double star. So from here star my fixed end point condition $0 = p\partial\bar{q} - H\partial t = \left[p - H\frac{\partial \psi}{\partial q_1}\right]\partial q_1$, Similarly, $0 = p\partial\bar{q} - H\partial t = \left[p - H\frac{\partial \psi}{\partial q_2}\right]\partial q_2$.

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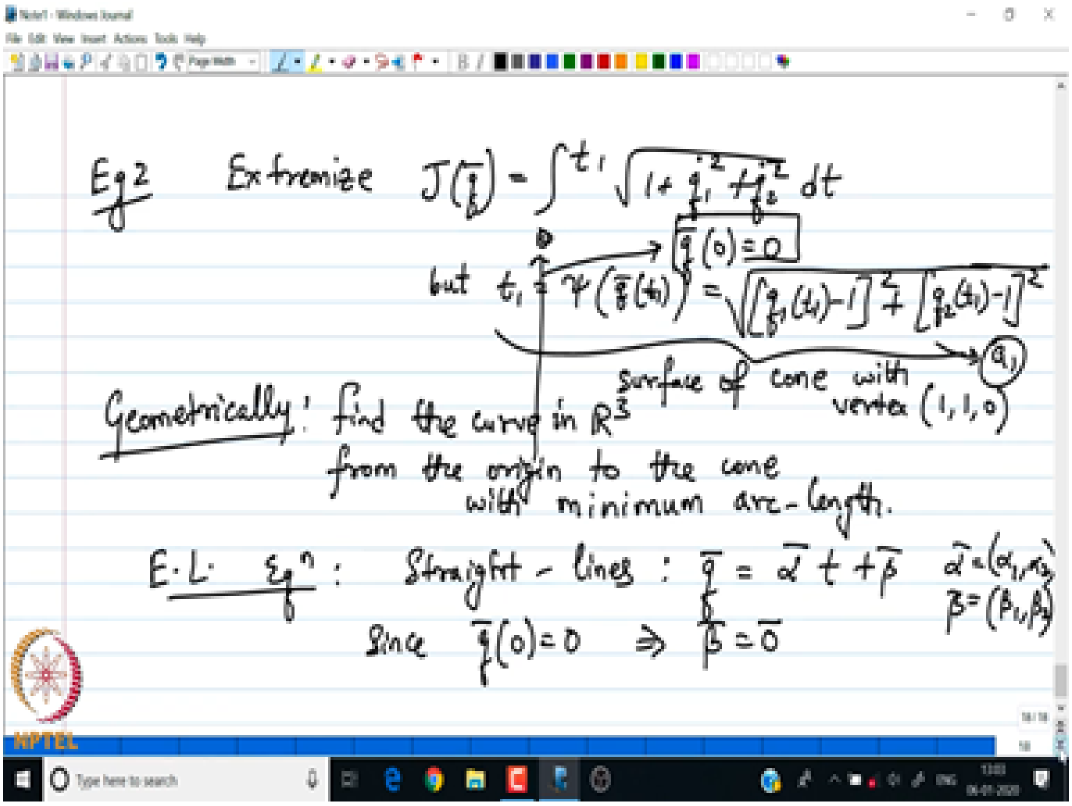

let us look at an example in this class of problem. the example I have is as follows. I need to extremize $J(\bar{q}) = \int_0^{t_1} \sqrt{1 + \dot{q}_1^2 + \dot{q}_2^2} dt$ I need to extremize this where $\bar{q}(0) = 0$ and t_1 is variable, but t_1 lies on the surface $\psi(\bar{q})$ which is given by $t_1 = \psi(\bar{q}(t_1)) = \sqrt{[q_1(t_1) - 1]^2 + [q_2(t_1) - 1]^2}$ $\mathbf{a_1}$
Essentially this curve is the surface of the cone, so in this particular surface students can draw that surface of cone with vertex (1, 1, 0) in 3D.

Essentially the problem is asking in the geometric sense find the curve in R^3 from the origin, we have taken t = 0, so from the origin to the cone with minimum Arc length. What is that disc, quantity inside this integral is the arc length functional. So we are trying to minimize the arc length. Now, well of course we need to satisfy the Euler- Lagrange, the system of Euler- Lagrange for this is a geodesic problem, our Euler- Lagrange equation will directly give me the solution as straight lines. So I need not solve that.

I see that the solution $\bar{q} = \bar{\alpha}t + \bar{\beta}$ where $\bar{\alpha}$ and $\bar{\beta}$ are vectors and $\bar{\alpha} = (\alpha_1, \alpha_2), \bar{\beta} = (\beta_1, \beta_2)$ and we see that further we are given 1 boundary condition $\bar{q}(0) = 0 \Rightarrow \bar{\beta} = \bar{0}$, Now we have already utilized all our boundary condition what is remaining is the Transversality condition.

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$$p_k = \frac{\partial f}{\partial \dot q_k} = \frac{\dot q_k}{\sqrt{1+\dot q_1^2+\dot q_2^2}}, \quad H = \sum_{k=1}^{2} \dot q_k p_k - L = \frac{-1}{\sqrt{1+\dot q_1^2+\dot q_2^2}}$$

T.C.: $\left.\frac{\partial L}{\partial \dot q_k} - H \frac{\partial \psi}{\partial q_k}\right|_{t=t_1} = 0 \longrightarrow \left.\dot q_k + \frac{\partial \psi}{\partial q_k}\right|_{t=t_1} = 0$

$(k=1,2)$

$\Rightarrow \alpha_k = -\frac{[\alpha_k t_1 - 1]}{t_1}$

$\Rightarrow \boxed{2\alpha_k t_1 = 1} \rightarrow (a_3) \quad k=1,2$

$\Leftrightarrow \alpha_1 = \alpha_2$

From (a_1), (a_2): $\alpha_1^2 + \alpha_2^2 = 1 \longrightarrow \alpha_1 = \alpha_2 = \frac{1}{\sqrt 2}$

$\left(\frac12, \frac12, \frac{1}{\sqrt2}\right)$: point of intersection of $\psi(\bar q)$ & $\bar q$ $\Rightarrow \boxed{q_1 = q_2 = \frac{1}{\sqrt2} t}$

Eg 2 Extremize $J(\bar q) = \int_0^{t_1} \sqrt{1+\dot q_1^2+\dot q_2^2}\, dt$, $\boxed{\bar q(0) = 0}$

but t_1 free, $\psi(\bar q(t_1)) = \sqrt{[q_1(t_1)-1]^2 + [q_2(t_1)-1]^2} \rightarrow (a_1)$

surface of cone with vertex $(1,1,0)$

Geometrically: find the curve in $\mathbb{R}^3$ from the origin to the cone with minimum arc-length.

E.L. Eqn: Straight-lines: $\bar q = \bar\alpha t + \bar\beta$, $\quad \bar\alpha = (\alpha_1, \alpha_2)$, $\bar\beta = (\beta_1, \beta_2)$

Since $\bar q(0) = 0 \Rightarrow \boxed{\bar\beta = \bar 0} \rightarrow (a_2)$

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To find the Transversality condition let us find the momentum and the Hamiltonian.

$$p_k = \frac{\partial f}{\partial \dot{q}_k} = \frac{\dot{q}_k}{\sqrt{1+\dot{q}_1^2+\dot{q}_2^2}}, H = \sum_{k=1}^{2} \dot{q}_k p_k - L = -\frac{1}{\sqrt{1+\dot{q}_1^2+\dot{q}_2^2}}$$

$$T.C: \ \frac{\partial L}{\partial \dot{q}_k} - H\frac{\partial \psi}{\partial q_k}|_{t=t_1} = 0 \ \Rightarrow \ \dot{q}_k + \frac{\partial \psi}{\partial q_k}|_{t=t_1} = 0$$

$$\Rightarrow \quad \alpha_k = -\frac{[\alpha_k t_1 - 1]}{t_1} \ \Rightarrow 2\alpha_k t_1 = 1, \ k = 1,2 \Leftrightarrow \alpha_1 = \alpha_2 \qquad \mathbf{a_3}$$

$$\text{From } \mathbf{a_1}, \mathbf{a_2}: \qquad \alpha_1^2 + \alpha_2^2 = 1 \Rightarrow \alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$$

Extremal $q_1 = q_2 = \frac{1}{\sqrt{2}}t$, so I have found the extremal and students can check that for this extremal the point of intersection of the curve $\psi(\bar{q})$ with the extremal $\bar{q}$ is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$ This needs to be checked and I leave it to the students to check, since everything is given to us. I end my discussion in this lecture and in the next lecture we are going to see another formulation of Euler-Lagrange which is going to cover a broader class of problems, specially arising in continuum mechanics. Mainly we are going to describe the Euler-Lagrange via the Hamiltonian formulation.