Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 32

Broken Extremals / Hamiltonian Formulation Part 2

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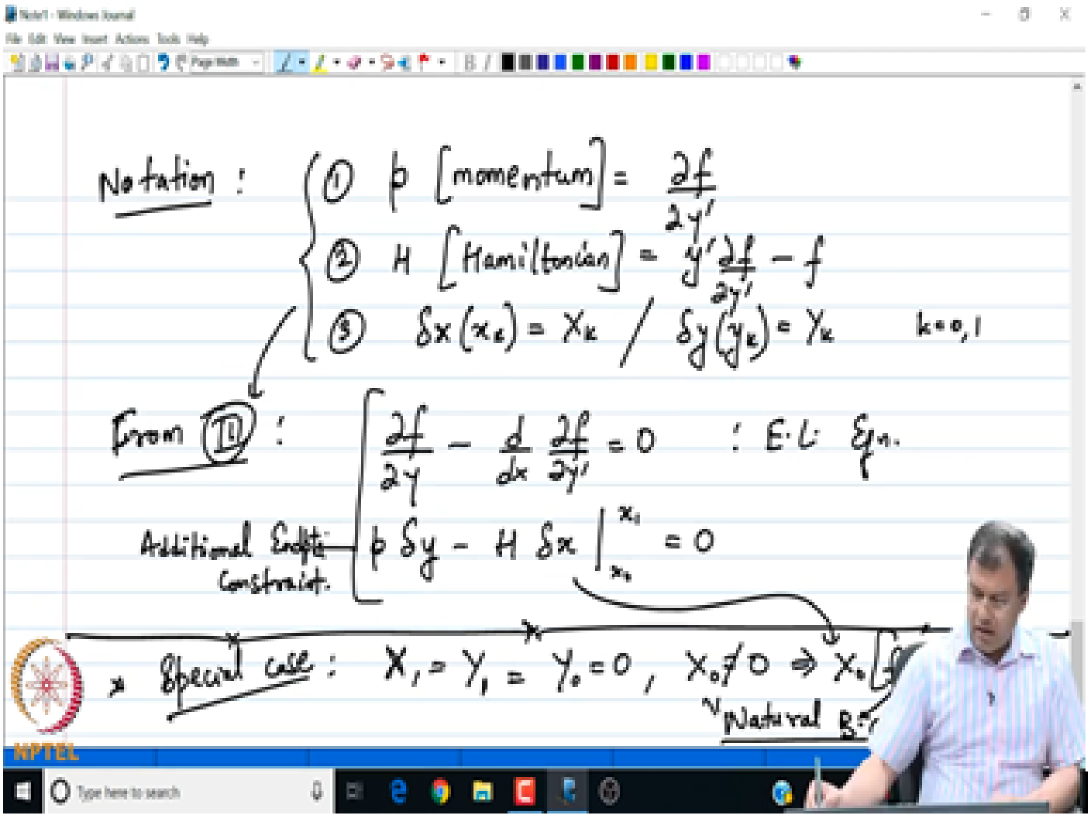

I can quickly see that this particular constraint can be reduced to our standard set of natural boundary condition. Notice that in the special case, if we allow our some of our variables to 0, i.e $X_1 = Y_1 = Y_o = 0, X_o \neq 0 \Rightarrow X_o \left[f - y^{'} \frac{\partial f}{\partial y^{'}}\right]_{x_o} = 0$

So, this is the value that we set and we see that this is nothing but our natural boundary condition if my end point at the left hand side the x-coordinate end point is not fix. So, we recover our natural boundary condition in this special case.

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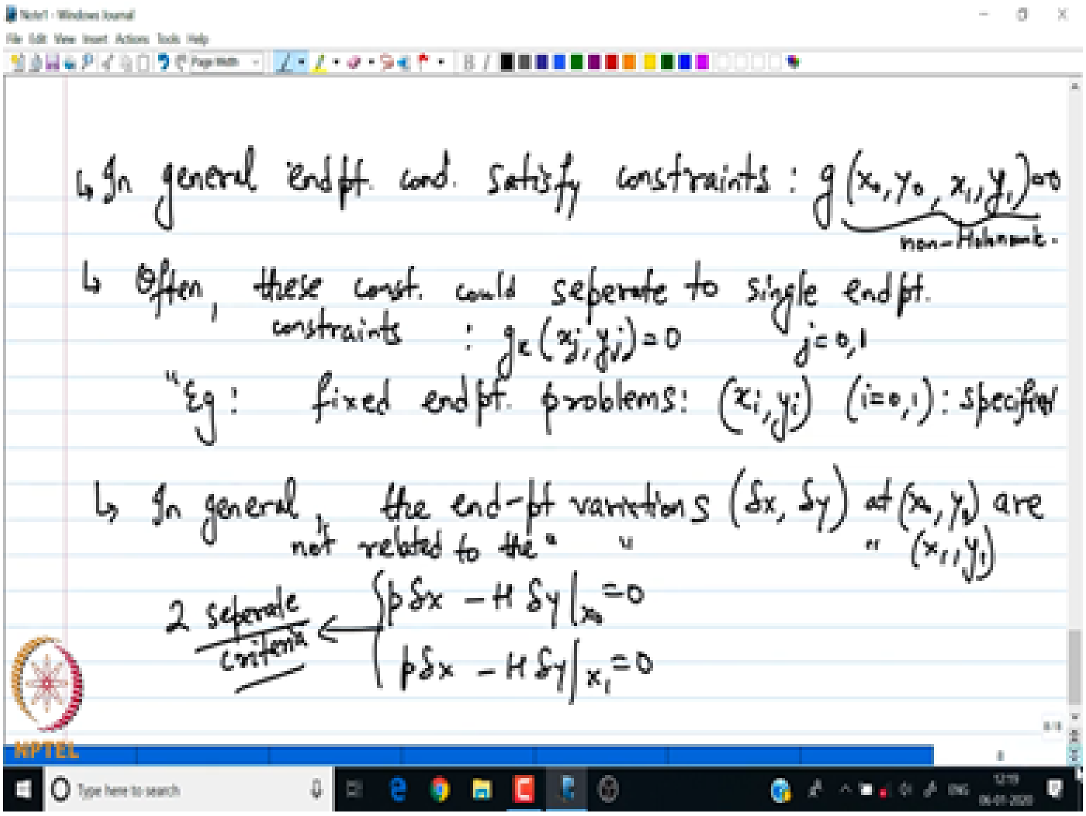

let me now discuss the general case a little bit more. So, the general variable boundary point condition is going to satisfy general end point condition satisfy constraints of the form which is non-holonomic constraint of the form $g(x_o, y_o, x_1, y_1) = 0$ and this is in general a non-holonomic constraint and often this constraints may reduce to a single end point constraints.

So, what I am saying is often they these constraints could reduce, could separate, we are going to look at this discussion with an example. These constraints separate to single end point constraints, to see that the constraints we get $g_k(x_j, y_j) = 0$, If j = 0 or 1, these are my single end point constraints.

Simplest example in this criteria is the fixed end point problem, for example, if I have fixed end points problems, let me say that (x_i, y_i) $(i = 0, 1)$ is specified, if the end point variations of (x_o, y_o) are not link to the end point variations of (x_1, y_1), In general, these are the different cases I am trying to highlight. We will look at the examples for each of this cases.

In general the end point variations, when I say end point variations I am talking about δx and δy, end point variations at x_o, y_o are not related to the end point variations at x_1, y_1. So, in that case we will have set of 2 end point criterias which are completely independent of each other. So, in that case we have the condition $p\delta x - H\delta y|_{x_o}$ and $p\delta x - H\delta y|_{x_1}$ and these are 2 separate criterias.

If that is the case since δx and δy are independent and are arbitrary at each end point we could certainly have that p and H, they are equal to 0 at each end point. let us look at some special cases of this criteria.

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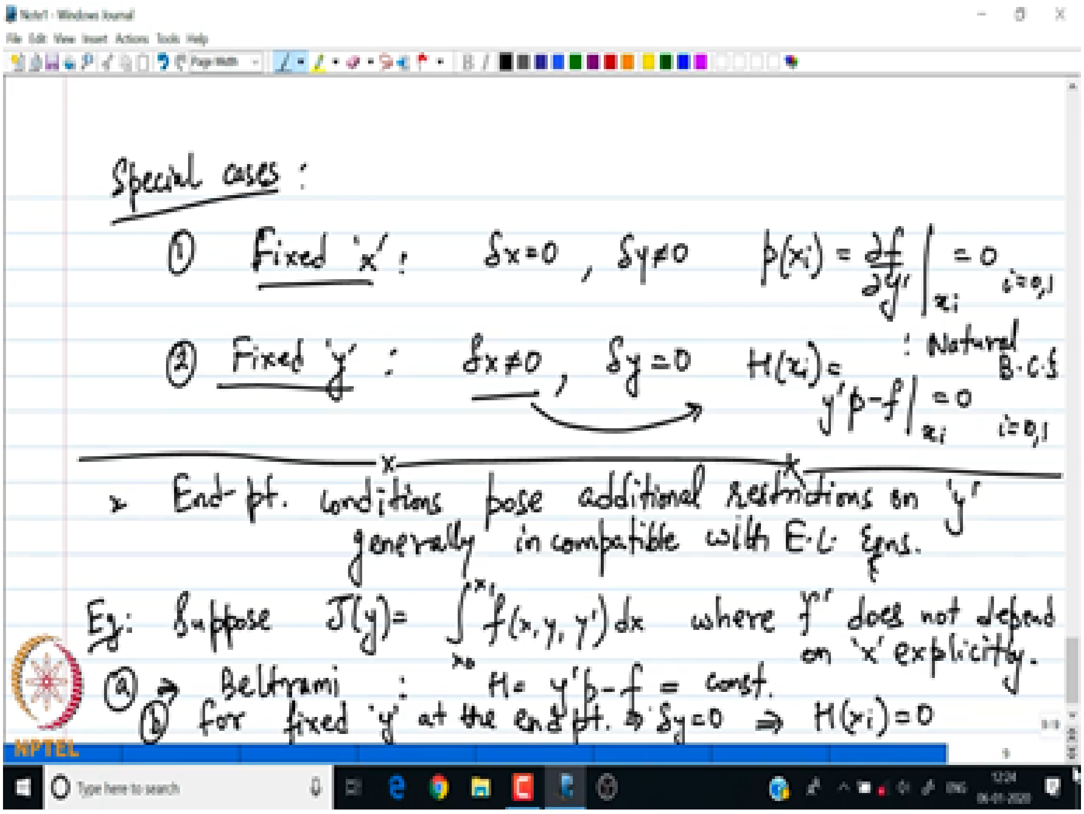

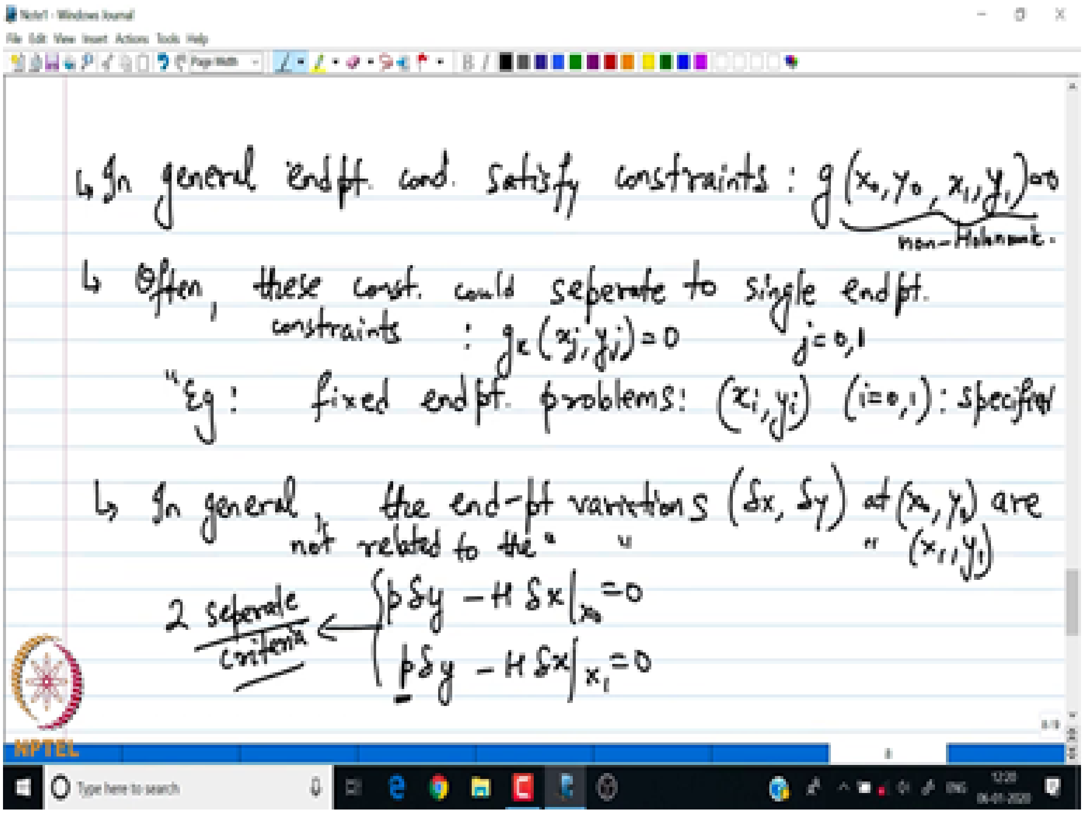

So, what I am trying to say is let us look at some special cases of the fact that, first case is we fix our x. We do not allow x-coordinate to vary. So, in that case $\delta x = 0$ and $\delta y \neq 0$, that is the case that my momentum defined at x_i which is $p(x_i) = \frac{\partial f}{\partial y'}|_{x_i} = 0$ $(i = 0, 1)$, these are natural boundary condition Similarly, if we can fix variable y, $\delta x \neq 0$ and $\delta y = 0$, $H(x_i) = y' p - f|_{x_i} = 0$ $(i = 0, 1)$.

We are going to see that although we have completely described these variable end points functionals the extremal of these variable end point functionals but in many cases the end point criteria pose severe restriction on the extremal that is generated as a solution to the Euler-Lagrange equation. So, what I just said is that our end point criterias that we have found end point conditions the pose additional sometimes very limiting restrictions, additional restrictions on y. Generally, in compatible with my Euler-Lagrange equations.

let us look at an example, Suppose $J(y) = \int_{x_o}^{x_1} f(x, y, y')dx$, where my f does not depend on x explicitly, I am looking at a specific class of functional. I allow my f such that f does not depend on x explicitly and students are familiar that the optimization of this class of functional will lead to the Beltrami identity setting that identity to a constant. So, which means let me call this relation again using the Beltrami identity that I get $H = y' p - f$= Constant, that is the reduced Euler-Lagrange equation.

Now, also our end point criteria says that we have a fixed x, we have suppose we also impose that we have a fixed y. So, what have we got is, for the case if we have fixed y at the end points that is $\delta y = 0$, then these leads to our reduced bound end point condition that $H(x_i) = 0$, So, which means from a and b I see that the solution to the Euler-Lagrange is such that the Beltrami function is identically equal to 0 i.e $H \equiv 0$ $\forall x \in [x_1, x_2]$

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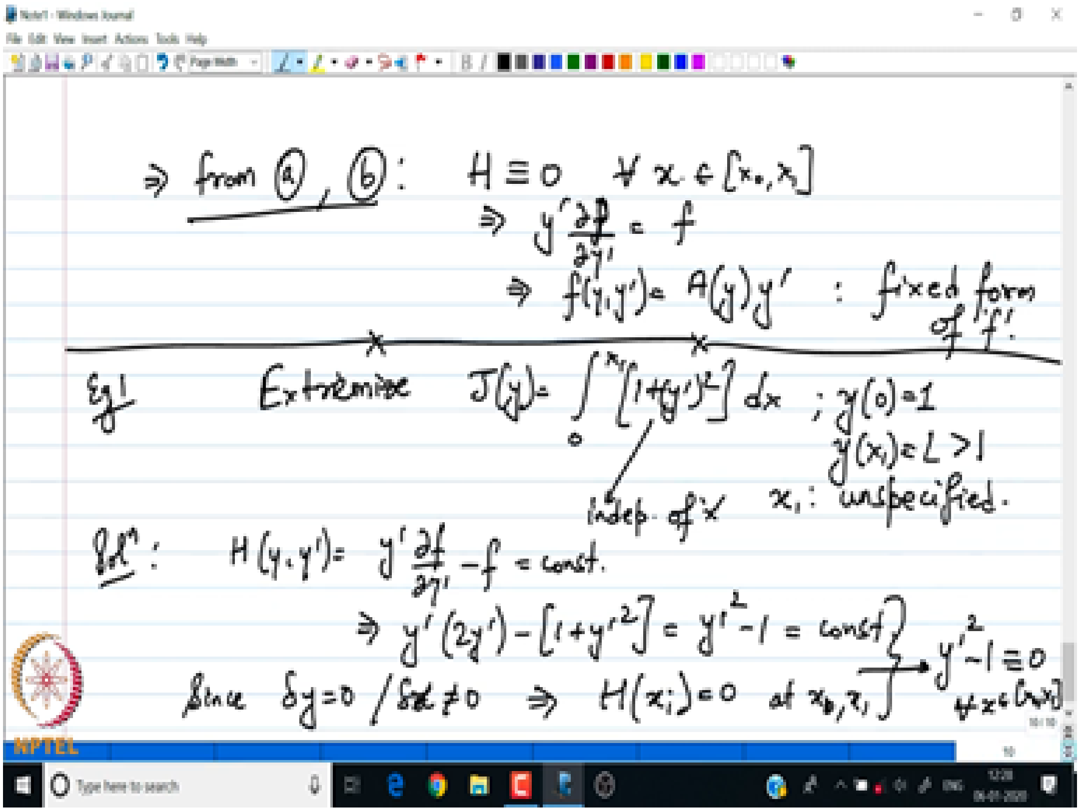

I see that the Euler-Lagrange equation reduces $y' \frac{\partial f}{\partial y'} = f \Rightarrow f(y, y') = A(y)y'$ which means that this class of functional with variable end points such that the y-coordinate is fixed. Can only be solved if the integrand of this functional is linear in y'.

So, we could only get a solution for this fixed form of f. Otherwise we will not get the solution. So, let us look at a proper example in this discussion. We want to extremize our functional
$J(y) = \int_0^{x_1} \left[1 + (y^{'})^2\right] dx; y(0 = 1), y(x_1) = L > 1$ So, x_1 is unspecified and again we see that this function is independent of x. So, to find the extremal we can write away use some Beltrami identity.

which means that my Beltrami function $H(y, y^{'}) = y^{'} \frac{\partial f}{\partial y^{'}} - f =$ Constant
$\Rightarrow y^{'}(2y^{'}) - \left[1 + (y^{'})^2\right] = (y^{'})^2 - 1 =$ Constant
Since $\partial y = 0/\partial x \neq 0 \Rightarrow H(x_i) = 0$at x_o, x_1
From these 2 conclusions, I can conclude that, the joint conclusion is that the Beltrami function $(y^{'})^2 - 1 \equiv 0 \forall x \in [x_o, x_1]$ So, then all I have to do is to solve the equation that I have found.

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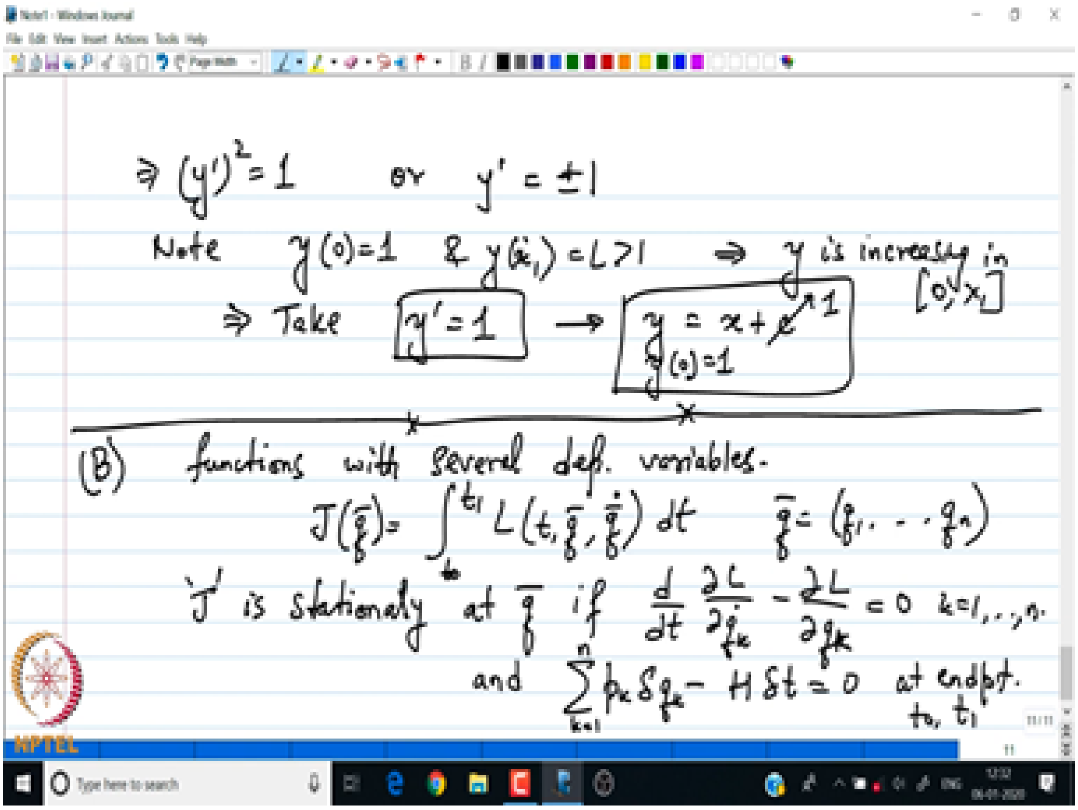

$\Rightarrow (y^{'})^2 = 1$ or $y^{'} = \pm 1$, it seems that we have 2 solutions but note the boundary conditions, y(0) = 1 and $y(x_i) = L > 1$ which means y is increasing in $[0, x_1]$ which means that we take the solution $y^{'} = 1$ That is the derivative must be positive for the solution to increase or from here I get $y = x + c$ and also y(0) = 1 that implies c = 1 and hence I get my extremal.

let us look at an example of this variable end point problem with integrands of the functionals containing multiple dependant variables, I see that our functional $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}}) dt \;\; \bar{q} = (q_1, ...q_n)$, We have seen that J is stationary at $\bar{q}$ if first of all the system of Euler-Lagrange are satisfied, if I have this following ODE being satisfying $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0, \;\; k = 1, ...n$
and $\sum_{k=1}^{n} p_k \partial q_k - H \partial t = 0$, t is my independent variable, , that is my at end pointsx_o and x_1 or in terms of free independent variables t_o and t_1. Let us now look at an example our in this class of functionals.

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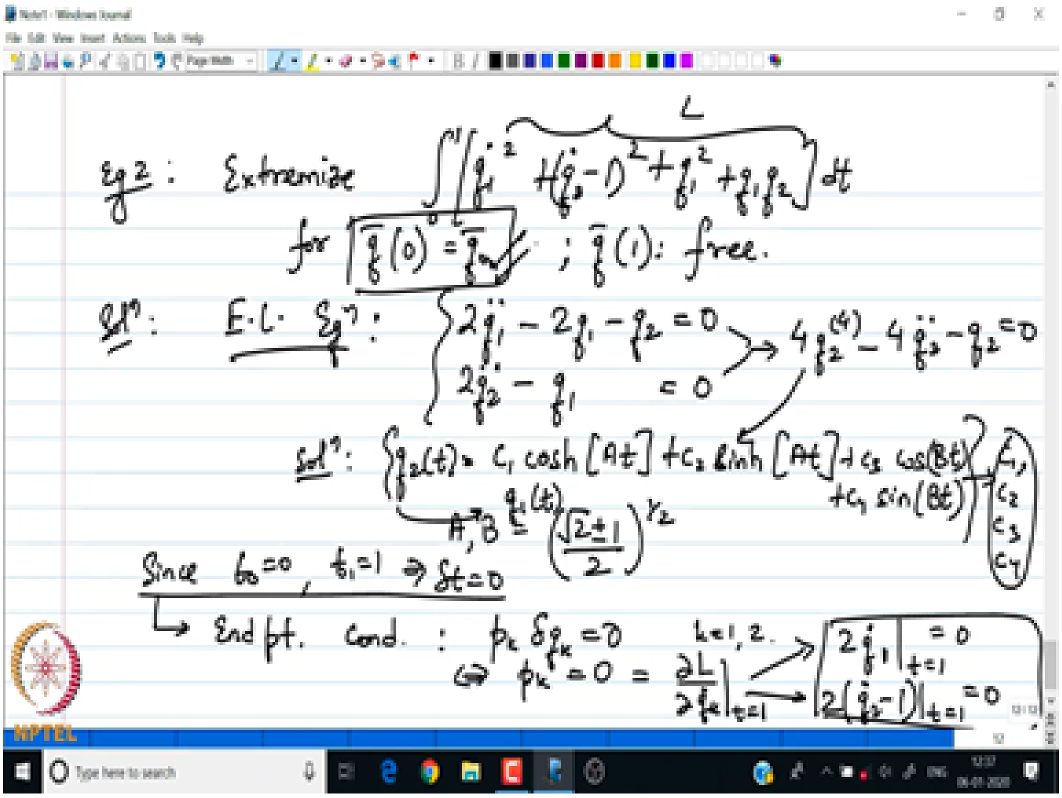

The example that I have is that follows. We want to extremize $\int_0^1 \left[\dot{q}_1^2 + (\dot{q}_o - 1)^2 + q_1^2 + q_1 q_2\right] dt$ for the boundary condition $\bar{q}(0) = \bar{q}_o; \bar{q}(1)$; free,So, we have fixed the x-coordinate of the boundary points but we have kept one of the y-coordinate to be variable. So, again we impose the Euler-Lagrange constraint, Euler-Lagrange equation for this system is as follows.

$2\ddot{q}_1 - 2q_1 - q_2 = 0$ and $2\ddot{q}_2 - q_1 = 0$ and from here I see that what I am trying to do is to solve or first eliminate one of the variables by taking the second derivative of the second equation and eliminating q_1. I get $4q_2^{(4)} - 4\ddot{q}_2 - q_2 = 0$ and from here I get a set of solutions
$q_2(t) = C_1 \cosh{(At)} + C_2 \sinh{(At)} + C_3 \cosh{(Bt)} + C_4 \sinh{(Bt)}$, where A and B are constants equals to $\sqrt{\frac{\sqrt{2} \pm 1}{2}}$.

Now there is no variation in the free variable t since $t_o = 0, t_1 = 1 \Rightarrow \partial t = 0$, No variation in the free variable. so, this statement is implying that my end point condition reduces to $p_k \partial q_k = 0, k = 1, 2 \Leftrightarrow p_k = 0 = \frac{\partial L}{\partial \dot{q}_k}|_{t=1}$

From here I get 2 equations satisfying at the boundary, so, this is at t = 1 at the other boundary which is variable. So, first boundary is fixed. So, at the variable boundary condition we need to differentiate we get 2 conditions. We differentiate L.
We get $2\ \dot{q}_1|_{t=1} = 0$ and $2(\dot{q}_2 - 1)_{t=1} = 0$

So, these are my 2 conditions and now notice that my q_2 from here I can always find $q-1$, But note from here in this discussion I have 4 constants, C_1, C_2, C_3 and C_4 but I have 2 end point condition and I also have 2 boundary condition which will fully determine these constants, So, I am not going to mention the value of these $C_i's$ and leave the problem for the students to find out the value of this constants in terms of $\bar{q}_o$