Variational Calculus and its Applications in Control Theory and Nano mechanics

Professor Sarthok Sircar
Department of Mathematics
Indraprastha Institute of Information Technology, Delhi

Lecture 31

Broken Extremals / Hamiltonian Formulation Part 1

In this lecture we are going to continue our discussion for optimization of functional with variable end points and in this lecture, we are going to look at the general situation namely, when both the x and y coordinate at the end points are allowed to vary.

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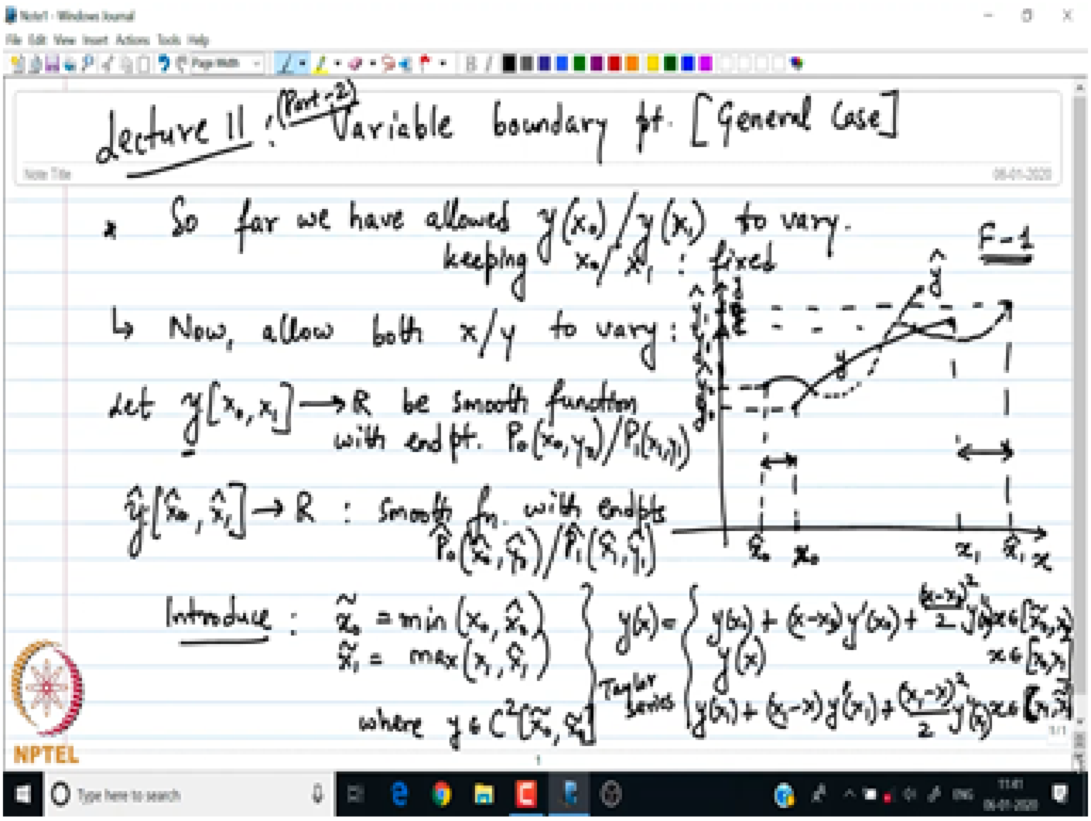

In this lecture we will look at the general variable boundary points, this is part 2 of our discussion, and this is the most general case that we can help for variable boundary point case discussion. I start the topic saying that so far what we have done, we have allowed the y coordinate to vary, we have allowed $y(x_1)$ and $y(x_0)$ to vary, We have allowed these points to vary keeping x_0, x_1 fixed .

Now we are going to do is that we allow both x and y to vary, this is the scenario we are going to look at, when the boundary points are such that both the coordinates are allowed to vary. let us see what is the situation through a diagram. So, what we have is the following. So, let me try to draw this diagram a bit carefully, this is in 2D Cartesian framework. So, suppose we are given an extremal. The extremal is described by the boundary points (x_0, y_0) and the second boundary point is (x_1, y_1) and we allow the perturbation to vary, let me call this perturbation as $\hat{y}(x)$.

This second curve is $\hat{y}$, notice the way how I have drawn is both x and the y coordinate can vary. So, let us say that this particular point is $\hat{x}_0$ and this particular point is $\hat{y}_0$. Similarly, we could have that this particular point is $\hat{x}_1$ and of course we can continue our discussion by saying that this particular point is $\hat{y}_1$, which means further we are going to refer this figure as F-1 and we are going to refer it again and again when we look at the derivation.

What we have is a following, let the curve $y\,[x_0, x_1] \to \mathbb{R}$ be a smooth function with end points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ So, these are the end points of the smooth curve. Similarly, we describe the perturbed curve $\hat{y}\,[\hat{x_0}, \hat{x_1}] \to \mathbb{R}$ be again smooth function with end points $\hat{P_0}(\hat{x_0}, \hat{y_0})$ and $\hat{P_1}(\hat{x_1}, \hat{y_1})$ and lets now generalize this end points (x_0, y_0) and introduce the notation $\tilde{x_0}$ and let us say that $\tilde{x_0}$ is the minimum of $(x_0, \hat{x_0})$ and $\tilde{x_1}$ is the maximum of $(x_1, \hat{x_1})$

I I am going to describe both this function y and its perturbed quantity $\hat{y}$ in the extended interval $[\tilde{x_0}, \tilde{x_1}]$, we see that in the perturbed interval y(x) can be defined as follows, I am using Taylor series, we see that y(x) is nothing but the same curve y(x), if x $\in [x_0, x_1]$, the original curve is recovered.

Now from the point $x \in [x_0 \tilde{,} x_0)$, I am talking about in this particular range. In this particular range when x takes its value, I can always write y(x) in terms of x_o using Taylor series.

$y(x) = y(x_o) + (x - x_o)y^{'}(x_o) + \frac{(x-x_o)}{2}y^{''}(x_0)\ \ x \in [\tilde{x_o}, x_o)$ I am describing in terms of x_o,we have set up our $\tilde{x_o}$ such that $\tilde{x_o} < x_o$

$y(x) = y(x_1) + (x_1 - x)y^{'}(x_1) + \frac{(x_1-x)}{2}y^{''}(x_1), \text{when}\ \ x \in [x_1, \tilde{x_1})$

So, I am assuming x_1 is bigger than x, and x_o is smaller than x in the previous interval, where my y is a differentiable function now, between the new interval $\tilde{x_o}, \tilde{x_1}$, I have extended the function y(x). Now without loss of generality I am going to assume that $\hat{x_o} < x_o$ where the perturbed function is defined, We could assume the other inequality also.

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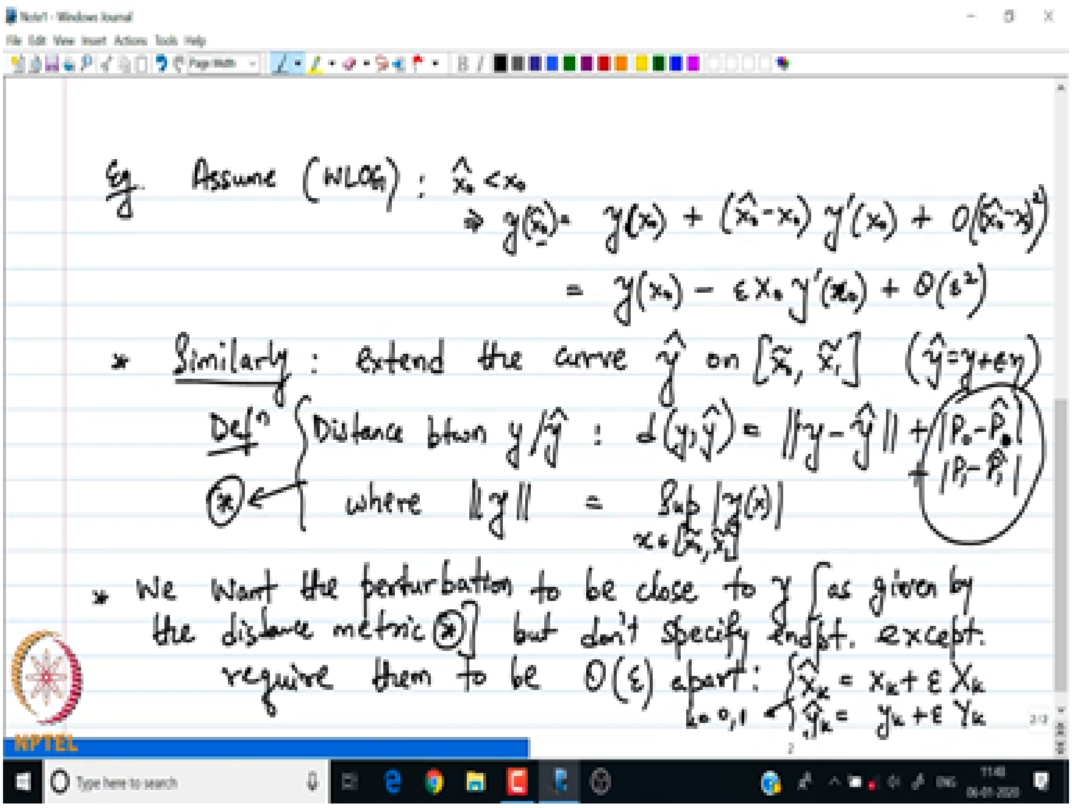

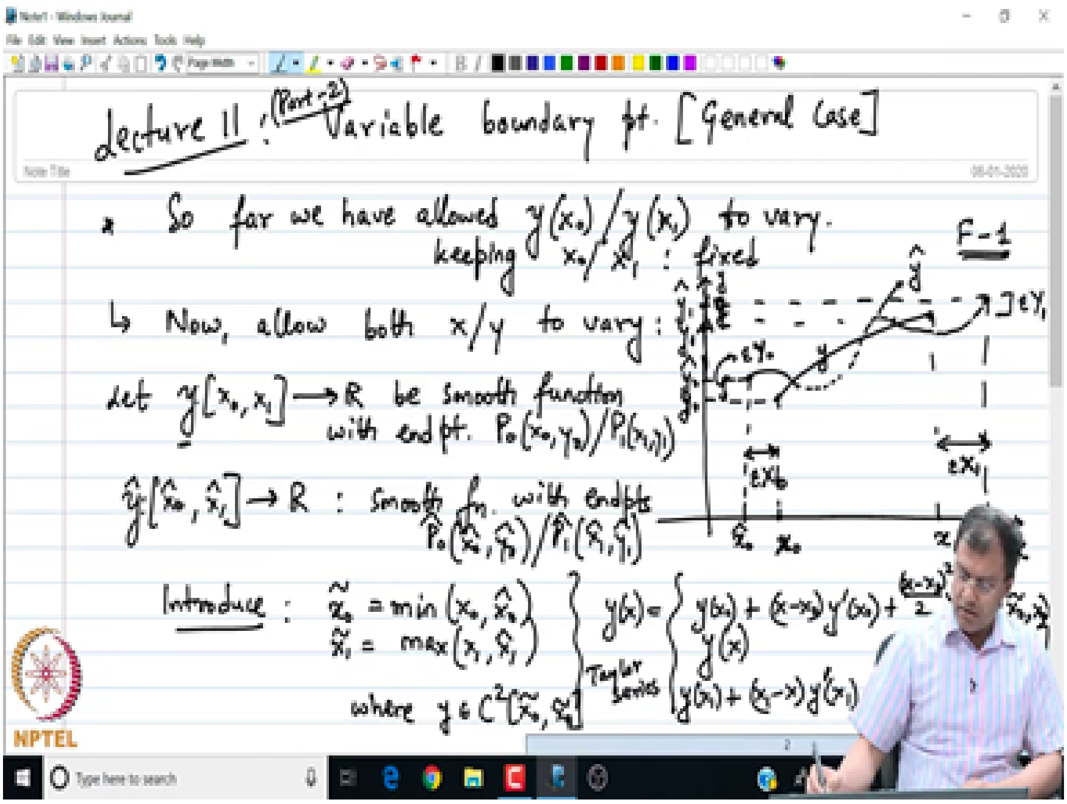

What I am saying, we can assume without loss of generality that $\hat{x_o} < x_o$ or so, if that is the case

$$\Rightarrow y(\hat{x_o}) = y(x_o) + (\hat{x_o} - x_o)y^{'}(x_o) + O\left((\hat{x_o} - x_o)^2\right)$$

Now let us say that the perturbation is such that it is only marginal or in the sense that the $\hat{x_o} - x_o = \epsilon X_o$ So, this is of the order of epsilon.

similarly, this particular distance is of the order of ϵY_o. So, the perturbation is such that the perturbation is linear order. So, similarly this is order ϵX_1, this particular perturbation is ϵY_o , these are my assumptions.

So, the assumption is so, if that be the case, I can rewrite my function $y(\hat{x_o}) = y(x_O) - \epsilon X_o y^{'}(x_o) + O(\epsilon^2)$ So, I can always rewrite my perturbed y at the perturbed value in terms of y at the original value of the x at the original interval.
So, the idea is, the similar exercise can be done to extend my perturbed function so, similarly I am going to extend I just say that I extend this perturbed function without writing the entire formula. So, I have shown the exercise for y, I assume the students will be able to do the similar exercise for $\hat{y}$.

We extend the curve $\hat{y}$ on the interval $[\tilde{x_o}, \tilde{x_1}]$, where $\hat{y} = y + \epsilon\eta$, I need a further definition, I define the distance between y and $\hat{y}$

$$d(y, \hat{y}) = \|y - \hat{y}\| + \left|P_o - \hat{P_o}\right| + \left|P_1 - \hat{P_1}\right| \qquad *$$

where I define my norm of $y - \hat{y}$ to be the supremum norm. I say that this is the supremum of $|y|$, So, $\|y\| = \sup_{x \in [\tilde{x_o}, \tilde{x_1}]} |y|$, we want allowed perturbations to be as close to y as possible. So, we want the perturbations are not arbitrarily choosen but it is chosen arbitrarily close, so, that it is close enough at the order of ϵ.

So, what I said is the following, we want the perturbation to be close to y, y is given by the distance metric $*$ but do not specify the end points. So, our end point is a variable of the problem here. We do not know what is x_o, x_1, y_o, y_1 except that they are at the order of they are close enough of the order of ϵ. So, the end points are not specified except the fact that we require them to be order ϵ apart.

So, what I mean to say is that $\hat{x}_k = x_k + \epsilon X_k$ and $\hat{y}_k = y_k + \epsilon Y_k$, where k = 0, 1 So, if we make this following assumption I can immediately find these norms.

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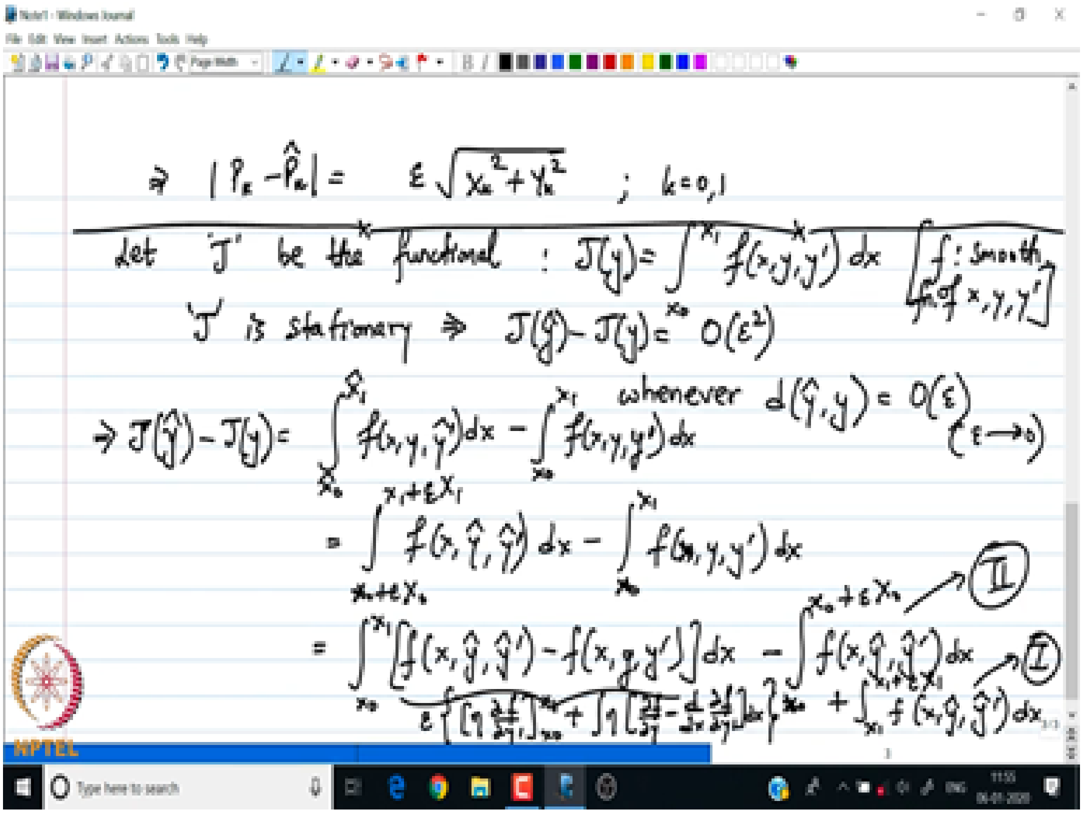

So, from here I can see that $\left|P_k - \hat{P}_k\right| = \epsilon\sqrt{X_k^2 + Y_k^2}$, k = 0, 1, Now I am ready to describe the extremal for the class of functional with variable end points. So, let us start our background. let J be the functional such that J(y) = $\int_{x_o}^{x_1} f(x, y, y^{'})dx$ such that f is a smooth function of x, y and $\hat{y}$.

I need J to be stationary, J is stationary to find the extremal. I need J stationary so, that the difference between $J(\hat{y}) - J(y) = O(\epsilon^2)$, that is whenever $d(\hat{y}, y) = O(\epsilon) \quad (\epsilon \to 0)$.

Variational at the functional J,

$$\Rightarrow J(\hat{y}) - J(y) = \int_{\hat{x_o}}^{\hat{x_1}} f(x, y, \hat{y^{'}})dx - \int_{x_o}^{x_1} f(x, y, y^{'})dx$$

$$J(\hat{y}) - J(y) = \int_{x_o + \epsilon X_o}^{\hat{x_1} + \epsilon X_1} f(x, y, \hat{y^{'}})dx - \int_{x_o}^{x_1} f(x, y, y^{'})dx$$

We can simplify this further, notice that I can break down this first integral in to set of 3 integrals, in the first integral let me look at the interval from x_o to x_1 and in this interval I have the common f as well, So, in this interval I have the standard difference. Both the functions are defined in this interval x_o and x_1.

$$= \int_{x_o}^{x_1} f(x,y,\hat{y}^{'})dx - f(x,y,y^{'}) - dx - \int_{x_o}^{\hat{x_o}+\epsilon X_o} f(x,y,\hat{y}^{'})dx + \int_{x_1}^{\hat{x_1}+\epsilon X_1} f(x,y,\hat{y}^{'})dx$$

So, now I have this 3 integrals and setting we need to find the stationary point or the stationary function we need to set this the sum of these 3 integrals equal to 0 and simplify. Notice that I can find the difference in the first integral and I see that this is nothing but this is of order in the order epsilon up to order epsilon terms this becomes the following quantity via standard Euler-Lagrange argument.

$$\mathbf{I}: \qquad \int_{x_1}^{x_1+\epsilon X_1} f(x,\hat{y},\hat{y}^{'})dx = \epsilon X_1 f(x,y,y^{'})|_{x=x_1} + O(\epsilon^2)$$

$$\mathbf{II}: \qquad \int_{x_o}^{x_o+\epsilon X_o} f(x,\hat{y},\hat{y}^{'})dx = \epsilon X_1 f(x,y,y^{'})|_{x=x_o} + O(\epsilon^2)$$

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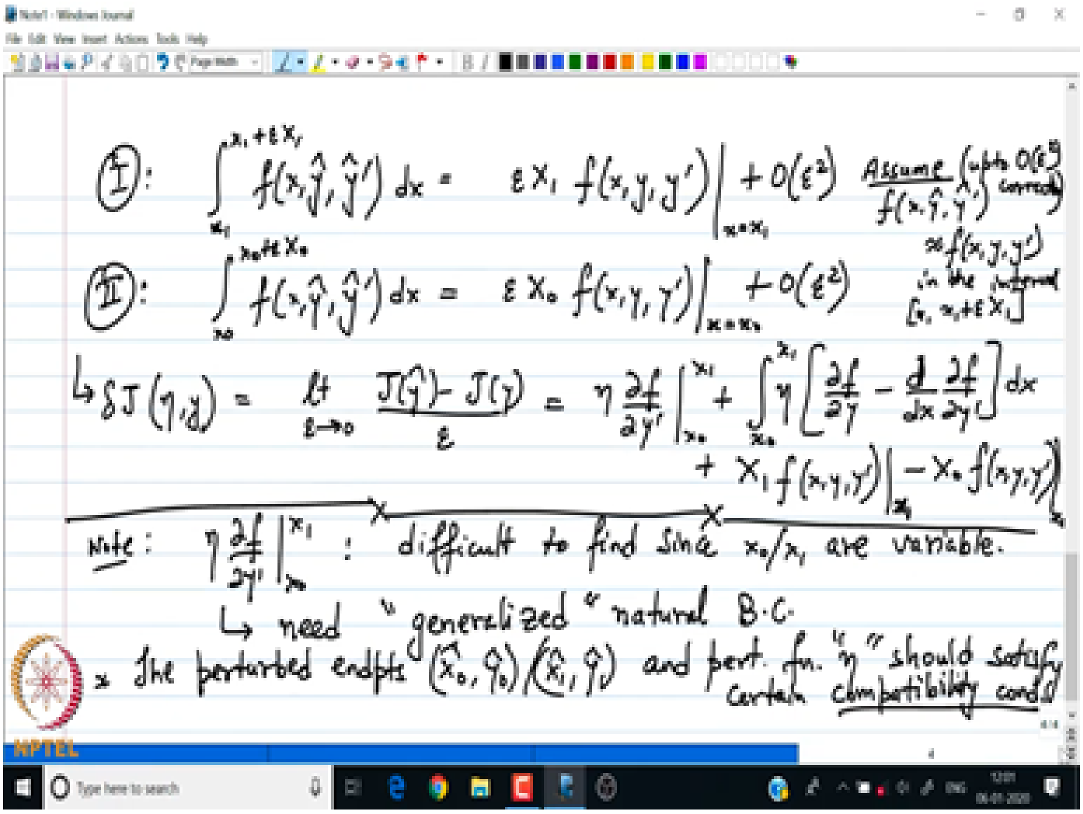

$$\delta J(\eta, y) = \lim_{\epsilon \to 0} \frac{J(\hat{y}) - J(y)}{\epsilon} = \eta \frac{\partial f}{\partial y'}|_{x_o}^{x_1} + \int_{x_o}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right] dx + X_1 f(x,y,y^{'})|_{x_1} - X_o f(x,y,y^{'})|_{x_o} \mathbf{III}$$

Well, although we have written this variation we know that these points small x_o and small x_1 are still variables. So, this is now still not in the form we are ready to evaluate. So, let us now do a little bit more simplification. So, far we express the variation in a more convenient form. So, what I just said is following.

Note that this quantity $\eta \frac{\partial f}{\partial y'}|_{x_o}^{x_1}$ is difficult to calculate, since my points x_o and x_1 are variable and we need to find a generalized natural boundary condition, which means that the perturbed end points and the perturbation η so, which means that we have to write down these, this expression on the right hand side such that certain consistency criteria's are also satisfied. So, what are these certain consistency criteria's? So, what I just said is the following.

The perturbed end points $(\hat{X}_o, \hat{Y}_o)/(\hat{X}_1, \hat{Y}_1)$ the two perturbed end points and the perturbation function η, certainly they will not in the more simplifying case will not vanish the end points where end points are not always well defined. They are also, varying. But certainly η is going to satisfy some constants which we are going to outline right now should satisfy certain compatibility condition. So, what are these compatibility condition? So, they must satisfy this compatibility condition.

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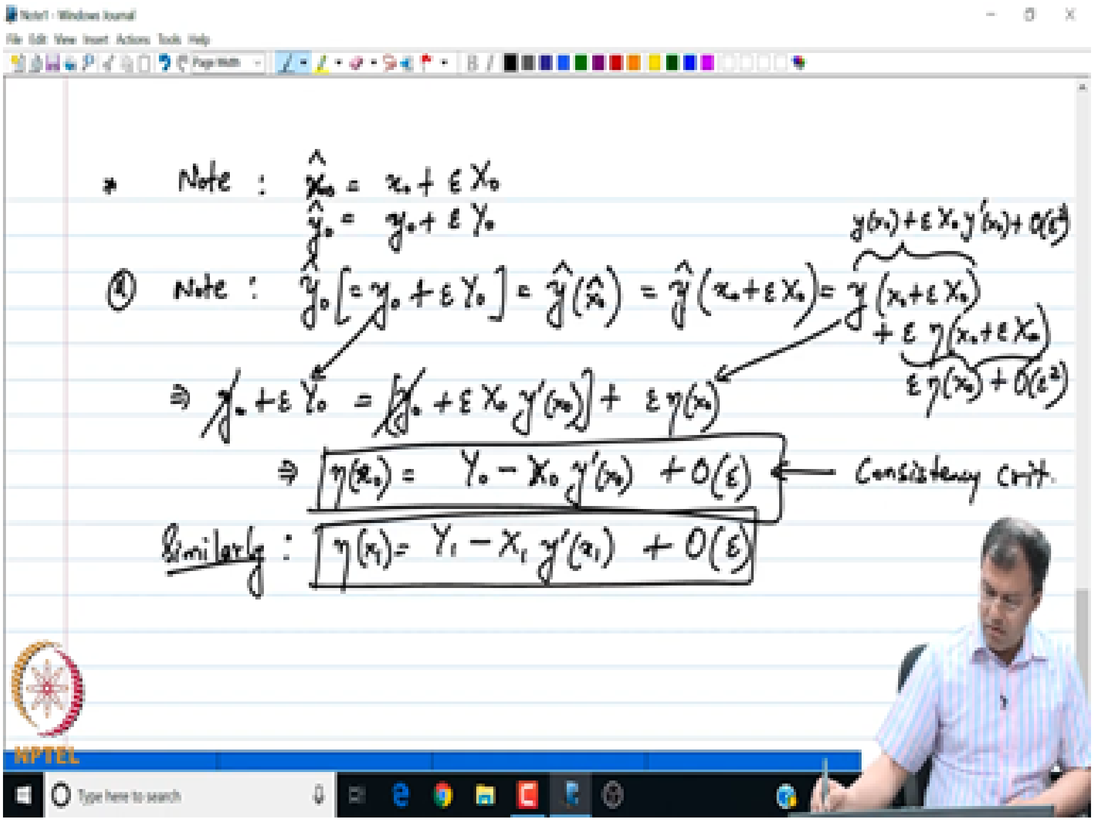
Note : x̂₀ = x₀ + ε X₀
ŷ₀ = y₀ + ε Y₀
Note : ŷ₀ [= y₀ + ε Y₀] = ŷ(x̂₀) = ŷ(x₀ + ε X₀) = y(x₀ + ε X₀) + ε η(x₀ + ε X₀)
y(x₀) + ε X₀ y′(x₀) + O(ε²)
ε η(x₀) + O(ε²)
⇒ y₀ + ε Y₀ = [y₀ + ε X₀ y′(x₀)] + ε η(x₀)
⇒ η(x₀) = Y₀ − X₀ y′(x₀) + O(ε)
Consistency crit.
Similarly : η(x₁) = Y₁ − X₁ y′(x₁) + O(ε)
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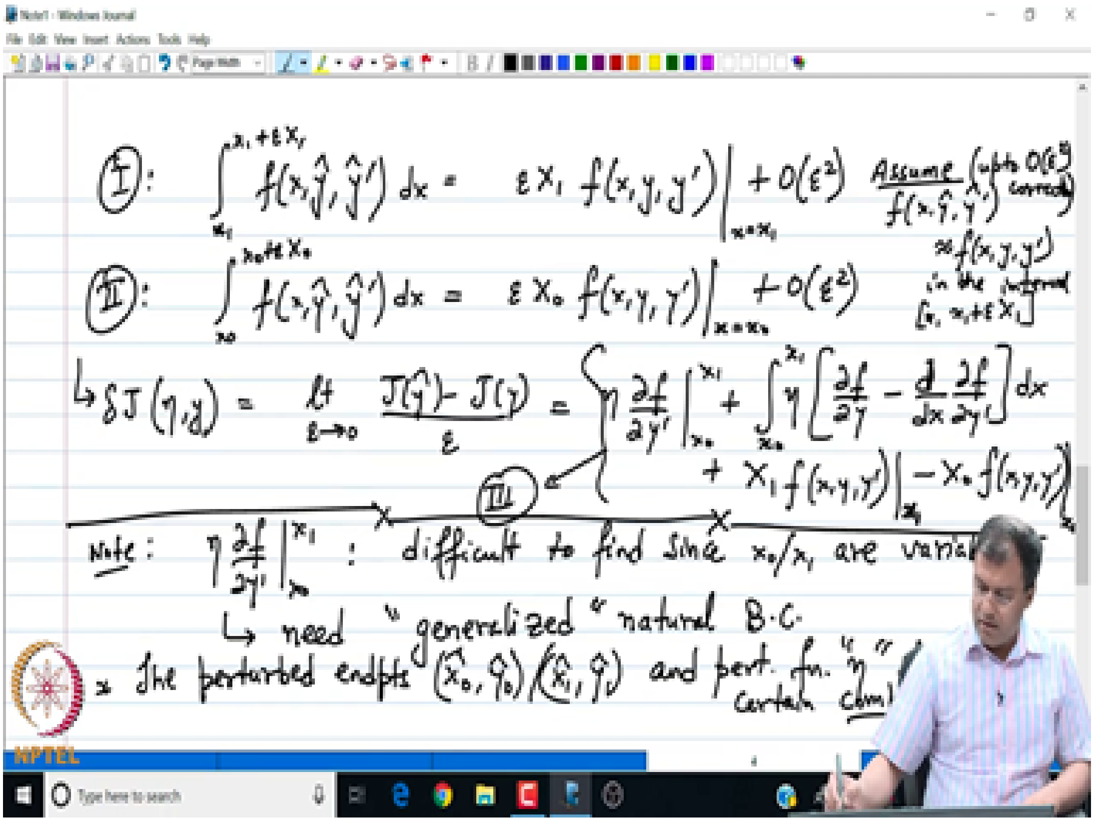
(I): ∫ from x₁ to x₁+εX₁ f(x, ŷ, ŷ′) dx = ε X₁ f(x, y, y′)|x=x₁ + O(ε²)
Assume (upto O(ε²) correct) f(x, ŷ, ŷ′) ≈ f(x, y, y′) in the interval [x₁, x₁+εX₁]
(II): ∫ from x₀ to x₀+εX₀ f(x, ŷ, ŷ′) dx = ε X₀ f(x, y, y′)|x=x₀ + O(ε²)
δJ(η, y) = lt ε→0 (J(ŷ) − J(y))/ε = η ∂f/∂y′ |x₀ to x₁ + ∫ from x₀ to x₁ η [∂f/∂y − d/dx ∂f/∂y′] dx + X₁ f(x, y, y′)|x₁ − X₀ f(x, y, y′)|x₀
Note : η ∂f/∂y′ |x₀ to x₁ : difficult to find since x₀/x₁ are variable
need "generalized" natural B.C.
The perturbed endpts (x̂₀, ŷ₀)/(x̂₁, ŷ₁) and pert. fn. "η" certain cond.
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Note that I am going to take my x_o so that $\hat{x_o} = x_o + \epsilon X_o$ and $\hat{y_o} = y_o + \epsilon Y_o$. Now

$$\text{Note :} \qquad \hat{y_o}\,[= y_o + \epsilon Y_o] = \hat{y}(\hat{x}_o) = \hat{y}(x_o + \epsilon X_o) = y(x_o + \epsilon X_o) + \epsilon\eta(x_o + \eta X_o)$$

$$\Rightarrow \qquad y_o + \epsilon Y_o = y_o + \epsilon X_o y^{'}(x_o) + \epsilon\eta(x_o)$$

$$\Rightarrow \qquad \eta(x_o) = Y_o - X_o y^{'}(x_o) + O(\epsilon)$$

$$\text{Similarly} \qquad \eta(x_1) = Y_1 - X_1 y^{'}(x_1) + O(\epsilon)$$

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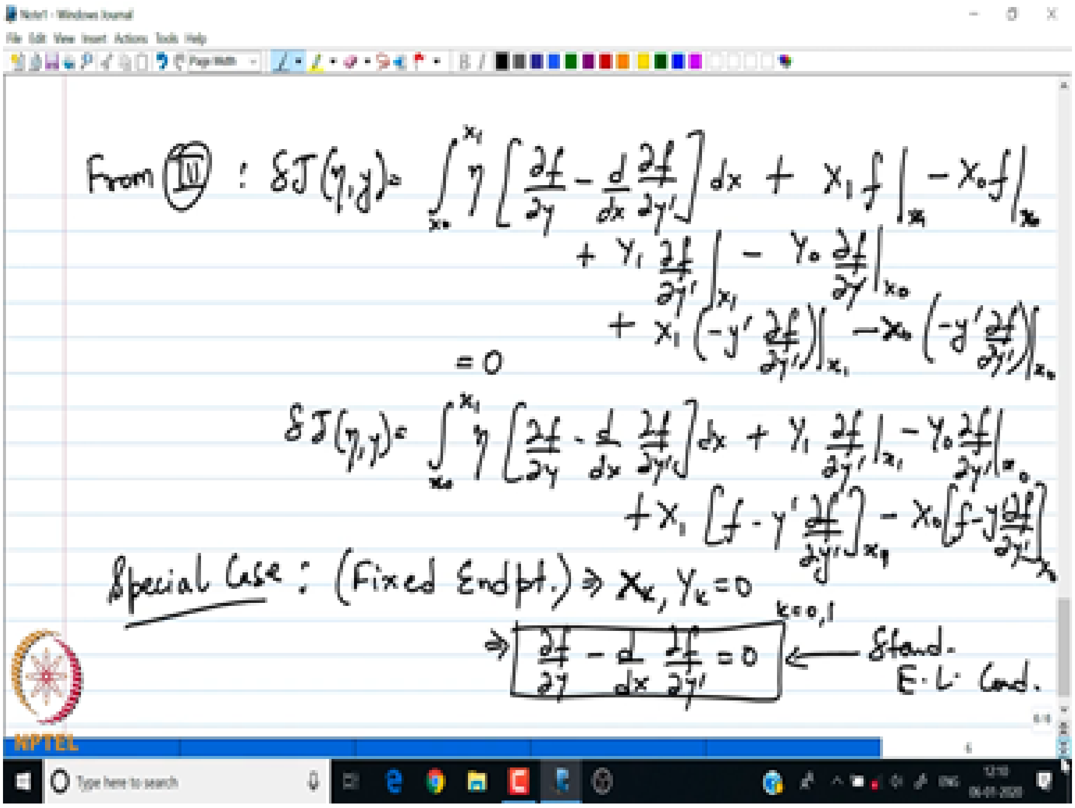
From (IV): $\delta J(\eta, y) = \int_{x_0}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] dx + X_1 f\Big|_{x_1} - X_0 f\Big|_{x_0}$
$+ Y_1 \frac{\partial f}{\partial y'}\Big|_{x_1} - Y_0 \frac{\partial f}{\partial y'}\Big|_{x_0}$
$+ X_1 \left(-y' \frac{\partial f}{\partial y'}\right)\Big|_{x_1} - X_0 \left(-y' \frac{\partial f}{\partial y'}\right)\Big|_{x_0}$
$= 0$
$\delta J(\eta, y) = \int_{x_0}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] dx + Y_1 \frac{\partial f}{\partial y'}\Big|_{x_1} - Y_0 \frac{\partial f}{\partial y'}\Big|_{x_0}$
$+ X_1 \left[f - y' \frac{\partial f}{\partial y'}\right]_{x_1} - X_0 \left[f - y' \frac{\partial f}{\partial y'}\right]_{x_0}$
Special Case: (Fixed End pt.) ⇒ $X_k, Y_k = 0$, $k=0,1$
⇒ $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$ ← Stand. E.L. Cond.
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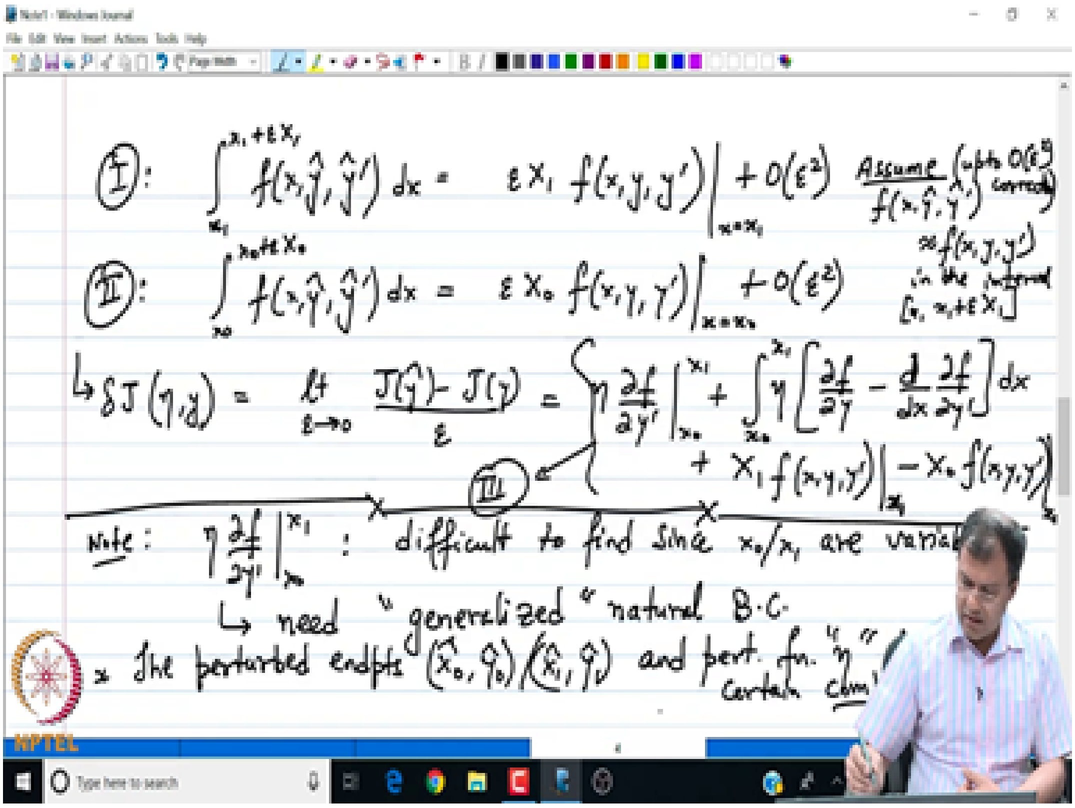
(I): $\int_{x_1}^{x_1+\epsilon X_1} f(x, \hat{y}, \hat{y}')\, dx = \epsilon X_1 f(x, y, y')\Big|_{x=x_1} + O(\epsilon^2)$
Assume (upto $O(\epsilon^2)$ correct) $f(x, \hat{y}, \hat{y}') \approx f(x, y, y')$ in the interval $[x_1, x_1+\epsilon X_1]$
(II): $\int_{x_0}^{x_0+\epsilon X_0} f(x, \hat{y}, \hat{y}')\, dx = \epsilon X_0 f(x, y, y')\Big|_{x=x_0} + O(\epsilon^2)$
↳ $\delta J(\eta, y) = \lim_{\epsilon \to 0} \frac{J(\hat{y}) - J(y)}{\epsilon} = \eta \frac{\partial f}{\partial y'}\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] dx + X_1 f(x,y,y')\Big|_{x_1} - X_0 f(x,y,y')\Big|_{x_0}$
(III)
Note: $\eta \frac{\partial f}{\partial y'}\Big|_{x_0}^{x_1}$: difficult to find since x_0/x_1 are variab
↳ need "generalized" natural B.C.
* The perturbed endpts $(\hat{x}_0, \hat{y}_0)/(\hat{x}_1, \hat{y}_1)$ and pert. fn. "η" certain com
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From **III** $\delta J(\eta, y) = \int_{x_o}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] dx + X_1 f|_{x_1} - X_o f|_{x_o} + Y_1 \frac{\partial f}{\partial y'}|_{x_1} - Y_o \frac{\partial f}{\partial y'}|_{x_o} +$

$$X_1\left(-y'\frac{\partial f}{\partial y'}\right)|_{x_1} - X_o\left(-y'\frac{\partial f}{\partial y'}\right)|_{x_o} = 0$$

$$\delta J(\eta, y) = \int_{x_o}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] dx + Y_1 \frac{\partial f}{\partial y'}|_{x_1} - Y_o \frac{\partial f}{\partial y'}|_{x_o} + X_1\left(f - y'\frac{\partial f}{\partial y'}\right)|_{x_1} - X_o\left(f - y'\frac{\partial f}{\partial y'}\right)|_{x_o}$$

let us look at the special case because I need to conclude something more. In the special case if we go back to our fixed end point criteria. $X_k, Y_k = 0$, there is no variation, k is 0 and 1. So, all these extra 4 terms will vanish and then my variation reduces to this integral constants and from there I get from lemma 2, lecture 2 that my Euler-Lagrange equation are recovered. So, in this special case scenario we must have that this must hold

$$\Rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \qquad \text{(Standard Euler-Lagrange condition)}$$

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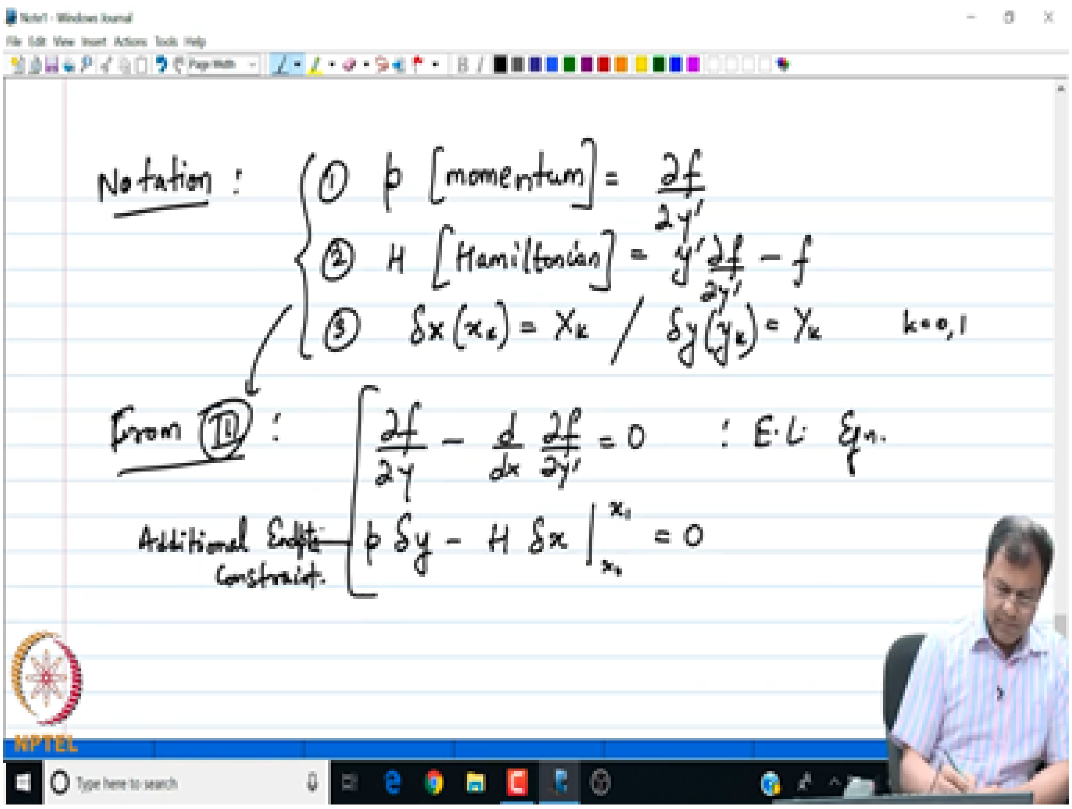

let me just introduce 2 new notations.

1 p[momentum] $= \frac{\partial f}{\partial y'}$

2 H[Hamiltonian] $= y'\frac{\partial f}{\partial y'} - f$

3 $\delta x(x_k) = X_k/\delta y(y_k) = Y_k \qquad k = 0, 1$

From **3** we have the first satisfaction of the Euler-Lagrange equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$ **IV** and further by clubbing my set of 4 boundary condition, I have that $p\delta y - H\delta x|_{x_o}^{x_1}$, In the more concise notation I have now come up with the natural boundary condition for the general class of variable end point problems. So, this is my additional end point constraint