**Variational Calculus and its Applications in Control Theory and Nano mechanics**
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**Lecture 30**
**Problems with Holonomic and Non-Holonomic Constraints, Variable Endpoints Part 6**

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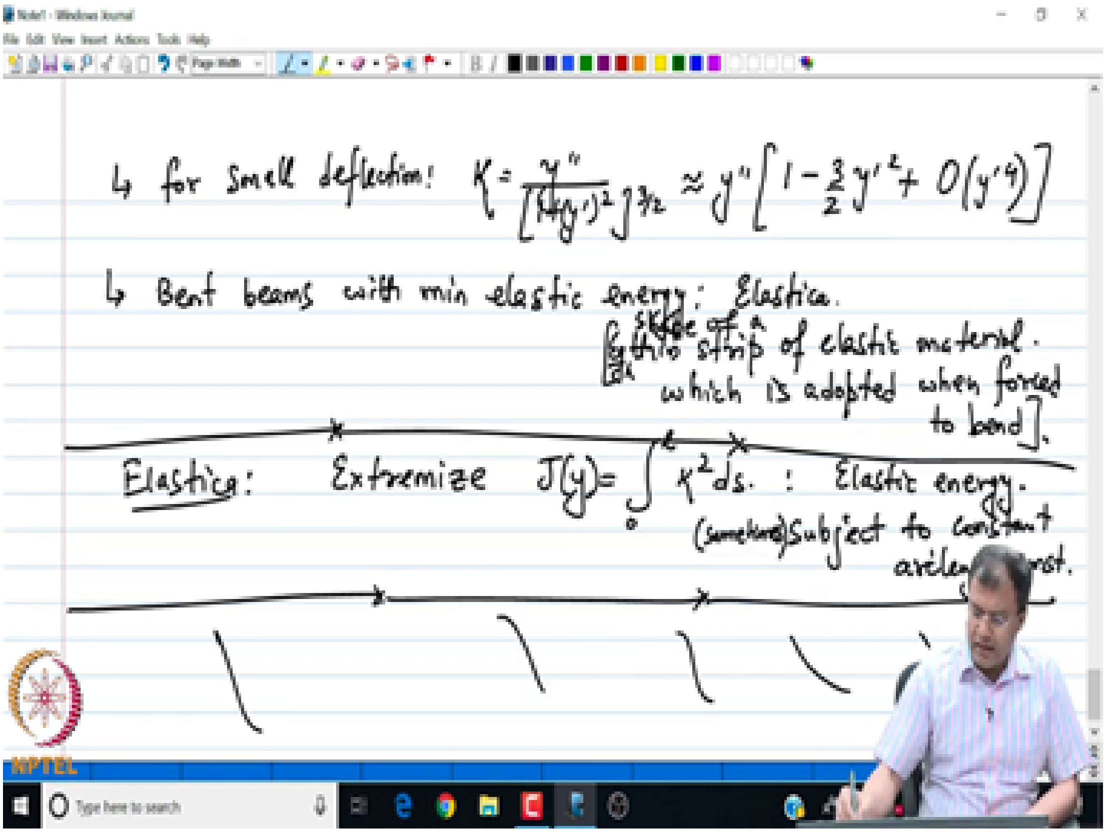


For elastica, We have the functional which we extremize or minimize, the following functional which is the elastic energy $J(y) = \int_0^1 \kappa^2 ds$, where $\kappa$ is my curvature with the most general formula that we have just derived. So, this is elastic energy, of the beam. Sometimes, we impose that the length of the elastica or the length of the material is also fixed.

Sometimes we also subject to the isoperimetric constraint, arc length, the constant arc length constraint. We are going to look at all these classes of problems. Now, in my first case let us look at a case of elastica with no arc length constraint or a case for unconstrained optimization of elastica.

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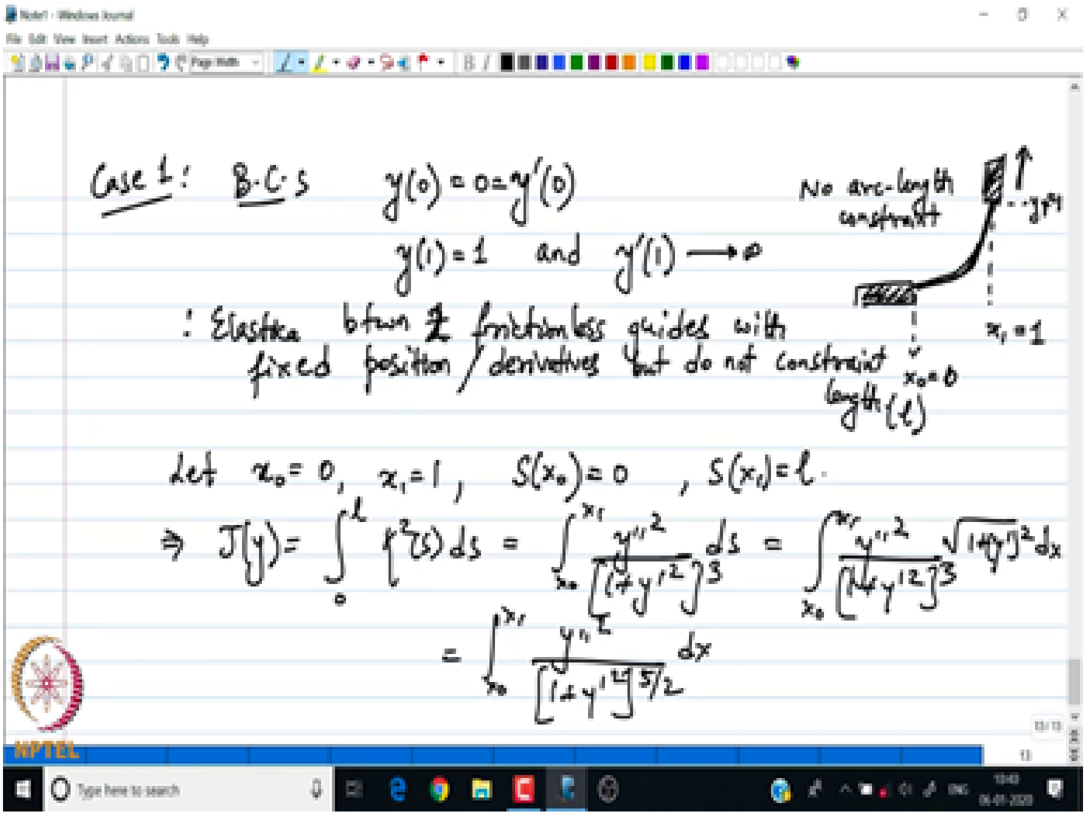


So, the setup of the problem is as follows.

**Case 1**, we have elastica, let me draw the figure and it might be clear. So, we have elastica which is clamped between two ends. So, we have elastica which is clamped here. let me call this end point as $x_0 = 0$ and I call this as $x_1 = 1$, we see that we are not imposing no arc length constraint at this point. So, which means that this is a case where we impose fixed boundary conditions $y(0) = 0 = y^{'}(0)$ y(1) = 1 and $y^{'}(1) \rightarrow \infty$ So, the slope becomes.

So, essentially this is a case of elastica between two frictionless guides with fixed position and derivatives but it does not constraint the length, we have the length l, which is also a fixed perimeter, but is an unknown of the problem.

let describe the point $x_0 = 0$, $x_1 = 1$ and Let me impose that the arc length at the point $x_0$ is 0 and of course the arc length at $x_1$ will be equal to l. So, we are imposing our functional, we are writing the functional in terms of its arc length. So,

$$\Rightarrow J(y) = \int_0^l \kappa^2(s)ds = \int_{x_0}^{x_1} \frac{y^{'2}}{\left[1 + y^{'2}\right]^3} ds = \int_{x_0}^{x_1} \frac{y^{'2}}{\left[1 + y^{'2}\right]^3} \sqrt{1 + y^{'2}} =$$

$$= \int_{x_0}^{x_1} \frac{y^{'2}}{\left[1 + y^{'2}\right]^{\frac{5}{2}}}$$

we have to extremize this functional. We see that this is a functional with second derivative so, we are going to use Euler-Poisson equation to find the extremal.

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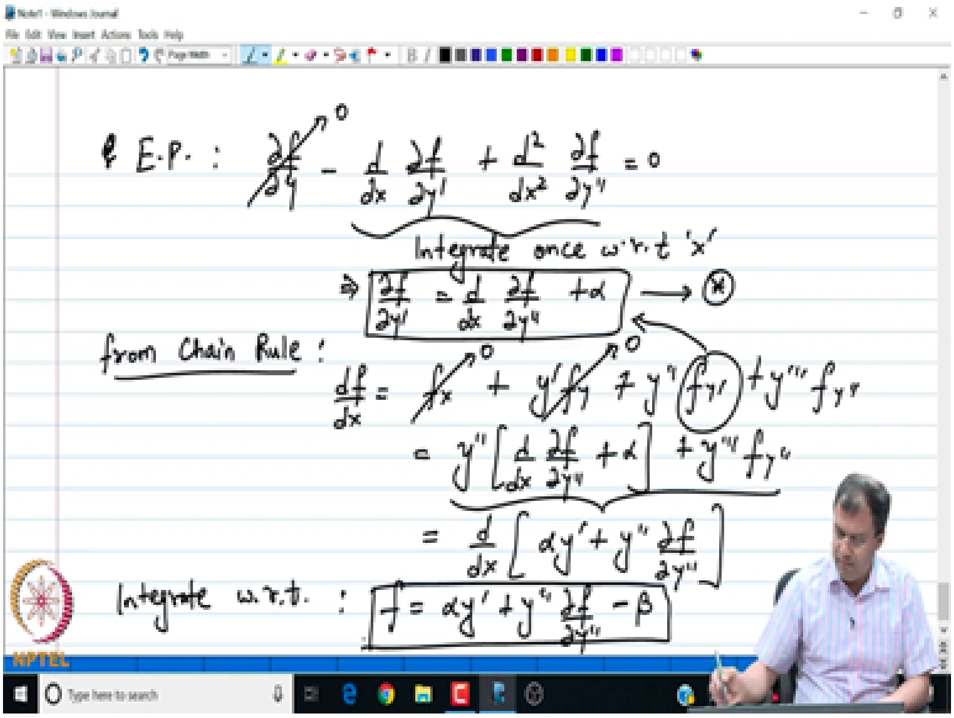


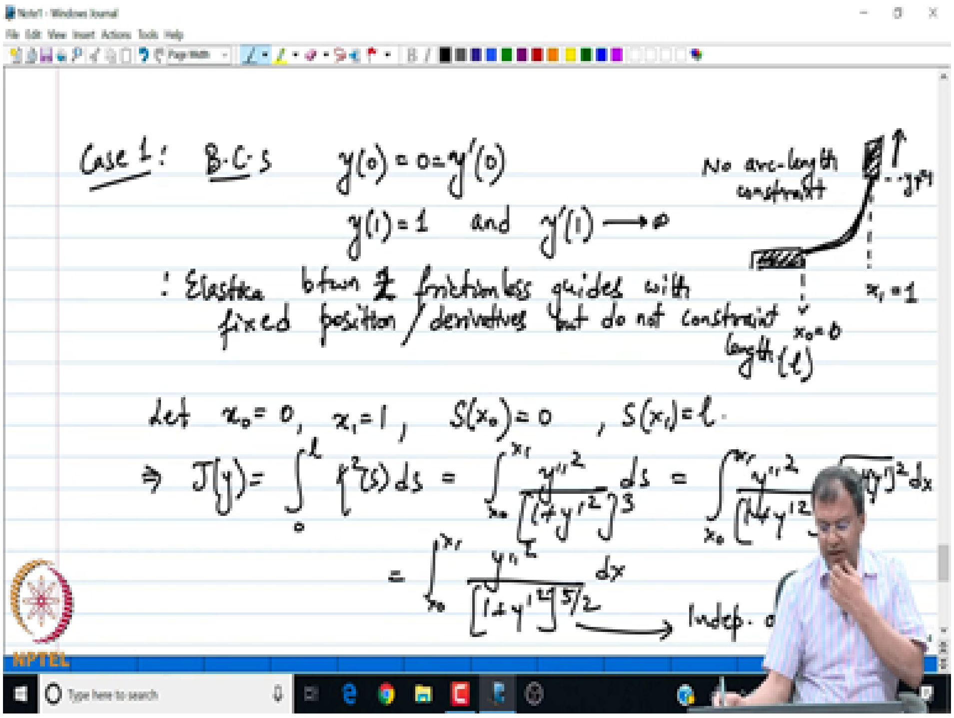


we use Euler-Poisson equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$. Notice that this particular integrand is independent of y. so, we have $\frac{\partial f}{\partial y} = 0$. and we can integrate this expression once with respect to x.

which is going to be give us $\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{\partial f}{\partial y''} + \alpha$ $\quad *$

where $\alpha$ is my constant of integration. Now, notice we have a standard result from chain rule. So, I can write

$$\frac{df}{dx} = f_x + y^{'} f_y + y^{''} f_{y'} + y^{'''} f_{y''} = y^{''} \left[ \frac{d}{dx} \frac{\partial f}{\partial y''} + \alpha \right] + y^{'''} f_{y''}$$

$$= \frac{d}{dx} \left[ \alpha y^{'} + y^{''} \frac{\partial f}{\partial y''} \right]$$

We can differentiate with respect to x. We are going to get back this expression in the previous underlined line , the total derivative of f with respect to x is the total derivative with respect to this quantity.

We can integrate with respect to x again to come to the fact that $f = \alpha y^{'} + y^{''} \frac{\partial f}{\partial y''} - \beta$ So, we have simplified our Euler-Poisson equation and now we are ready to substitute our expression for the integrand small f.

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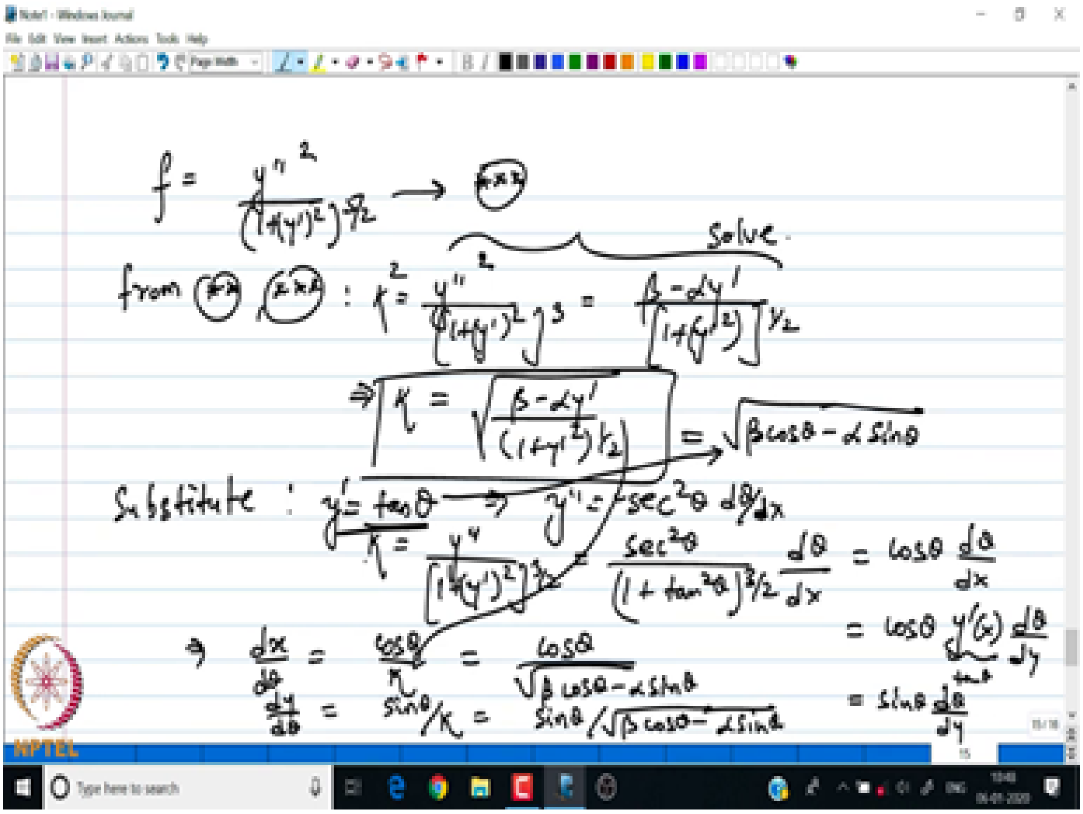
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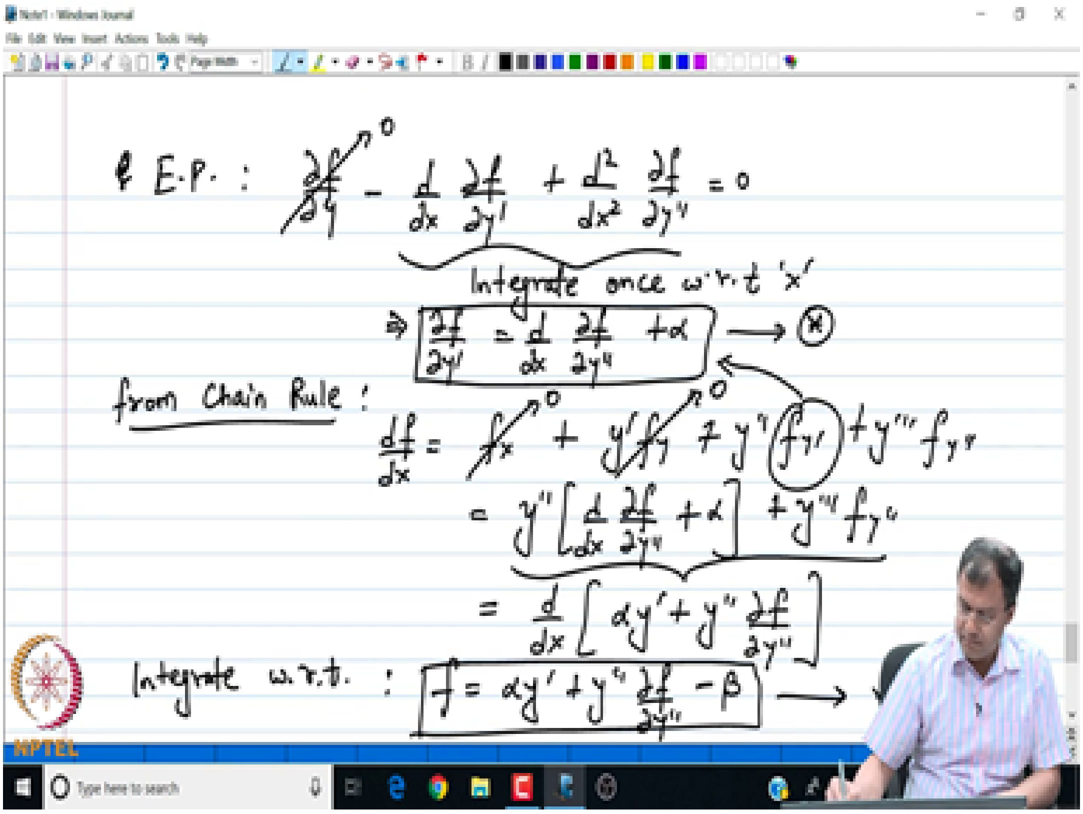
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$$f = \frac{y^{''2}}{(1+(y'))^{\frac{5}{2}}} \qquad\qquad ***$$

after simplification we get the following expression

$$\kappa^2 = \frac{y^{''2}}{\left(1+y^{'2}\right)^3} = \frac{\beta - \alpha y^{'}}{\left(1+(y^{'})^2\right)^{\frac{1}{2}}}$$

$$\Rightarrow \sqrt{\frac{\beta - \alpha y'}{\left(1+(y')^2\right)^{\frac{1}{2}}}} = \sqrt{\beta\cos\theta - \alpha\sin\theta}$$

Subsitute $y^{'} = \tan\theta \qquad \Rightarrow \qquad y^{''} = \sec^2\theta \frac{d\theta}{dx}$

$$\kappa^2 = \frac{y^{''2}}{\left(1+y^{'2}\right)^{\frac{3}{2}}} = \frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^{\frac{3}{2}}}\frac{d\theta}{dx} = \cos\theta\frac{d\theta}{dx} = \sin\theta\frac{d\theta}{dy}$$

So, from here I get set of two relations

$$\Rightarrow \frac{dx}{d\theta} = \frac{\cos\theta}{\kappa} = \frac{\cos\theta}{\sqrt{\beta\cos\theta - \alpha\sin\theta}}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\sin\theta}{\kappa} = \frac{\sin\theta}{\sqrt{\beta\cos\theta - \alpha\sin\theta}}$$

So, the crucial part is now, all we need to do is to integrate this quantity on the right hand side with respect to theta to get the parametric representation of our extremal. Now, I am going to quickly show the way to solve, before we do that we have used the substitution and we must also show what is the corresponding boundary condition for this substituted variable.

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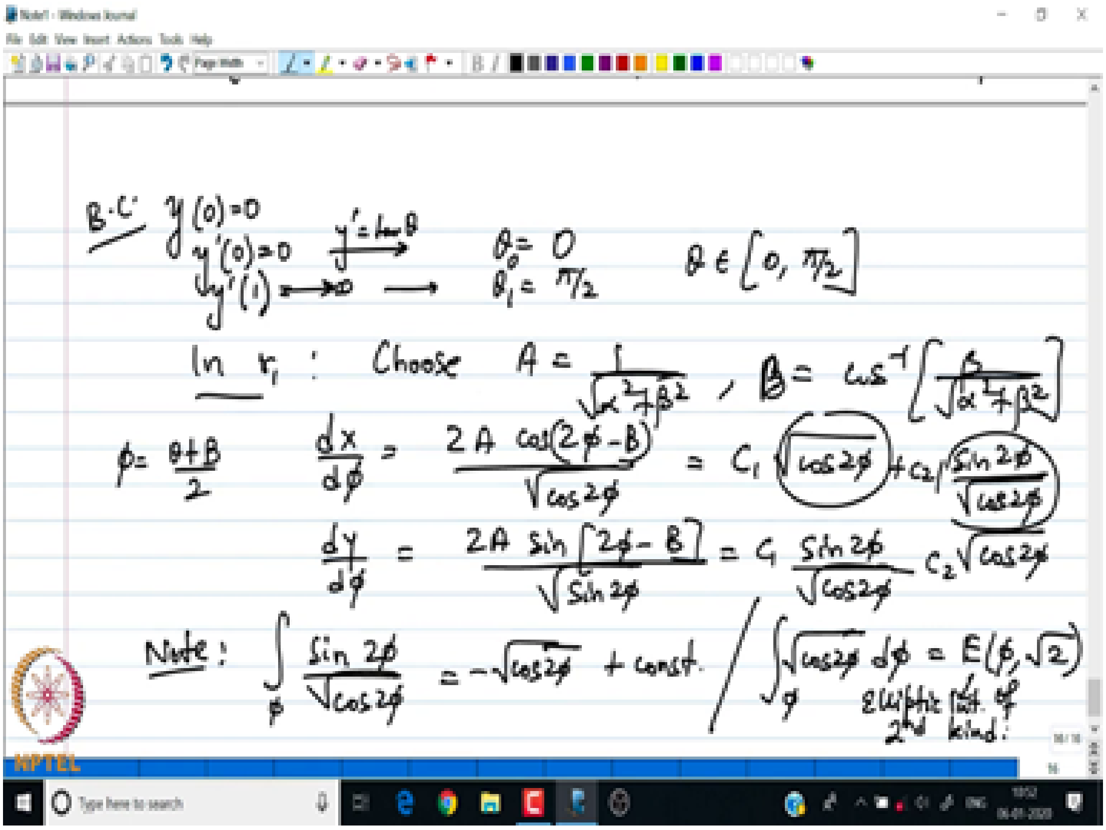


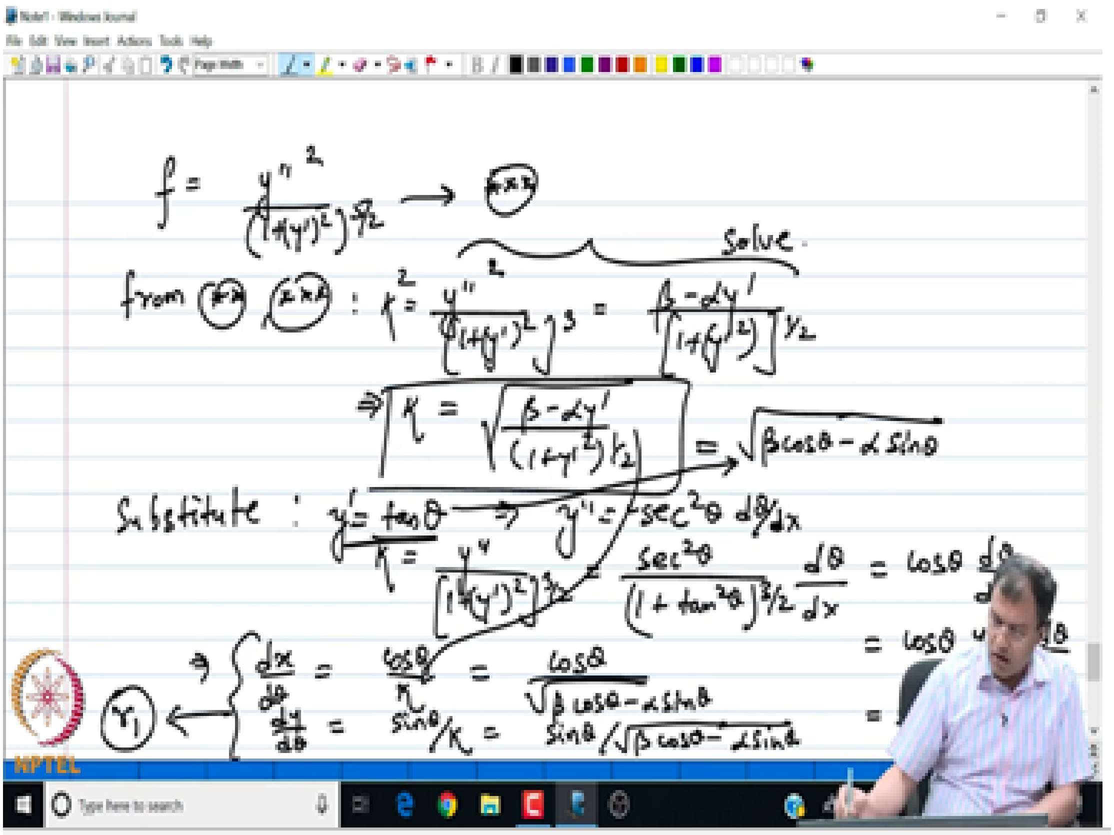


Notice that y(0) = 0, $y^{'}(0) = 0$ we have substituted $y^{'} = \tan\theta \ \Rightarrow \theta_o = 0$ and for $y^{'}(1) \rightarrow \infty \ \Rightarrow \theta_1 = \frac{\pi}{2}$, where $\theta \in \left[0, \frac{\pi}{2}\right]$ let me call this set of equation as $r_1$.

In order to solve $r_1$, we now start substituting, we choose our constant $A = \frac{1}{\sqrt{\alpha^2+\beta^2}}, B = \cos^{-1}\left[\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right]$ and once we use that, our set of equations reduces to the following new set. I am using another variable $\phi = \frac{\theta+\beta}{2}$ and new set of equations are as follows

$$\frac{dx}{d\phi} = \frac{2A\cos(2\phi - \beta}{\sqrt{\cos 2\phi}} = C_1\sqrt{\cos 2\phi} + C_2\frac{\sin 2\phi}{\sqrt{\cos 2\phi}}$$

$$\frac{dy}{d\phi} = \frac{2A\sin(2\phi - \beta}{\sqrt{\sin 2\phi}} = C_1\frac{\sin 2\phi}{\sqrt{\cos 2\phi}} - C_2\sqrt{\cos 2\phi}$$

Now all I need to do is to integrate this, note that, in order to integrate

$$\int_\phi \frac{\sin 2\phi}{\sqrt{\cos 2\phi}} = -\sqrt{\cos 2\phi} + \text{Constant}$$

and $\int_\phi \sqrt{\cos 2\phi} d\phi$ the answer can be written in the form of an elliptic integral. In fact, this is the definition of elliptic integral, So $\int_\phi \sqrt{\cos 2\phi} d\phi = E(\phi, \sqrt{2})$, this is elliptic integral of the second kind

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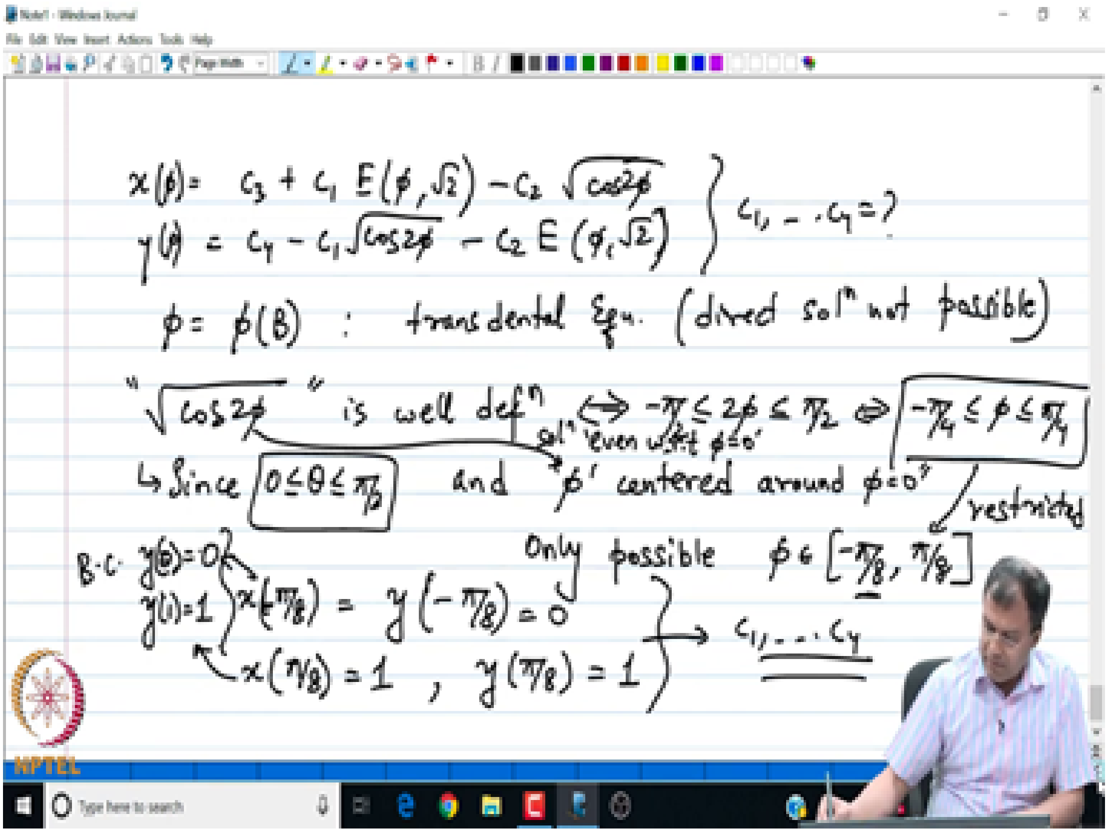


The final answer that I get $x(\phi) = C_3 + C_1E(\phi, \sqrt{2}) - C_2\sqrt{\cos 2\phi}$ and $y(\phi) = C_4 - C_1\sqrt{\cos 2\phi} - C_2E(\phi, \sqrt{2})$ Now, this final result is in the form of 4 constants $C_1$ to $C_4$ and the way to derive is as following.

It is not easy to find all this constants because we have done lots of substitutions, but note that the original $\phi$ was an angle with respect to the intermediate constant B and these are again this is a transdental equation in general, and we know that now, the direct solution is not possible, but what we do is the following.

What we do is look at this that we need to properly, so, we know that cos varies from -1 to + 1, but in order so that the square root is well defined, It is well defined if the argument inside the square is positive or I have that my angle $-\frac{\pi}{2} \leq 2\phi \leq \frac{\pi}{2} \Leftrightarrow -\frac{\pi}{4} \leq \phi \leq \frac{\pi}{4}$ So, that is my range of the angle that $\phi$ can take.

However, since I have already derived that $\leq \theta \leq \frac{\pi}{2}$, this is something that we had to begin with and so, this was the larger range of $\phi$. $\phi$ will be certainly within this range, now also we have that $\phi$ is centered around 0. We have a solution even with respect to $\phi = 0$. So, it is centered around 0 Well, centered around $\phi = 0$ which means that this situation is only possible when this range is further restricted to $\phi \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$.

Look we need a range which is centered around 0, but we also had to satisfy the constraint that the original angle $\theta$ is from 0 to $2pi$ and that is possible when $\phi$ has a much more restricted range. Now, all that is left is we have to evaluate $x(\frac{\pi}{8})$. So, my boundaries are from $-\frac{\pi}{8}$ to $\frac{\pi}{8}$ and note that we had the boundary conditions y(0) = 0 and y(1) = 1, $x(-\frac{\pi}{8}) = y(-\frac{\pi}{8}) = 0$ and $x(\frac{\pi}{8}) = y(\frac{\pi}{8}) = 1$ that is the second set of boundary condition that we have. So, we have 4 equations and we have 4 unknowns $C_1$ to $C_4$ and hence our problem is quite well defined

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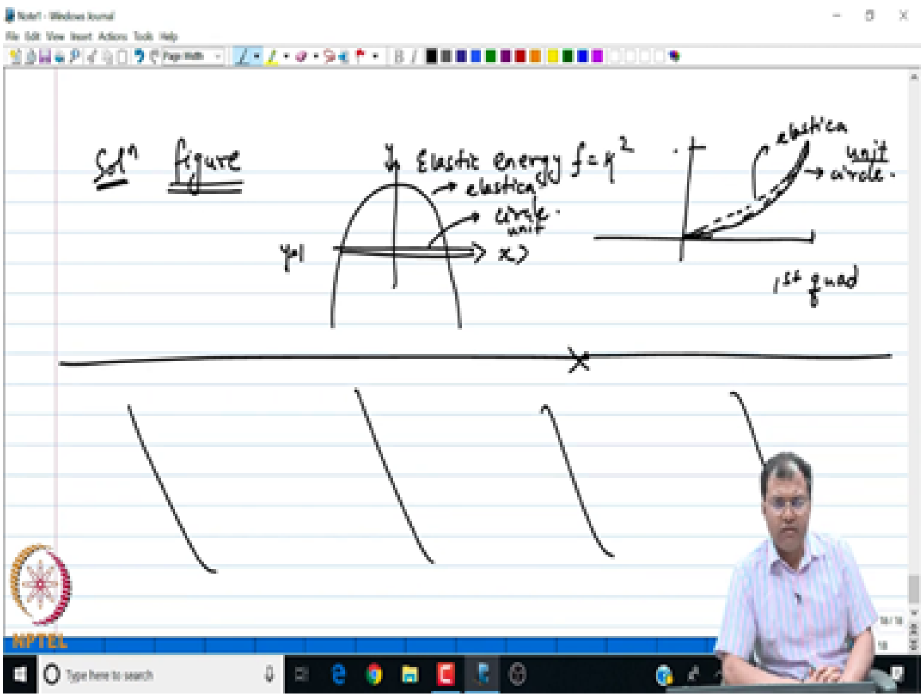


Now, before we wrap up this example, I just want to highlight how the solution figure, so, how does the shape of my elastica compares with the shape of a unit circle. We are talking about the solution in the first quadrant. So, in the first quadrant if we were to plot a unit circle, it will have the following shape and on the other hand if we were to plot the shape of elastica, it comes out to be quite close.

So, this question is why did not we just use the approximation in such a way that simplifies our problem and gives us a shape which is quite close to unit circle. It seems quite intuitive that the shape should have followed a functional which is very similar to the one that gives unit circle as the extremal. But, on the other hand if we plot our elastic energy which is $\kappa^2$ and Once, we know the solution we can definitely plot the elastic energy.

Now, for a circle, let this is my y = 1. So, for a circle the elastic energy lies parallel to the y axis. So, this is for a circle and for elastica this is purely a curve which is not parallel to the x axis. So, it seems that although the shape is very similar to the unit circle, my elastic energy tells or paints a very different story and hence, the solution to this problem is non-trivial.

So, we can extend our solution methodology to another case which we are going to look at in our next lecturing session. So, thank you very much for listening. So, in our next lecture I am going to talk about the case of the variable and points where both x and y are also varying. We will also complete this example of elastica by looking at another test case where we impose the length condition or the isoperimetric constraint. So, thank you very much for listening in this lecture. Thank you.

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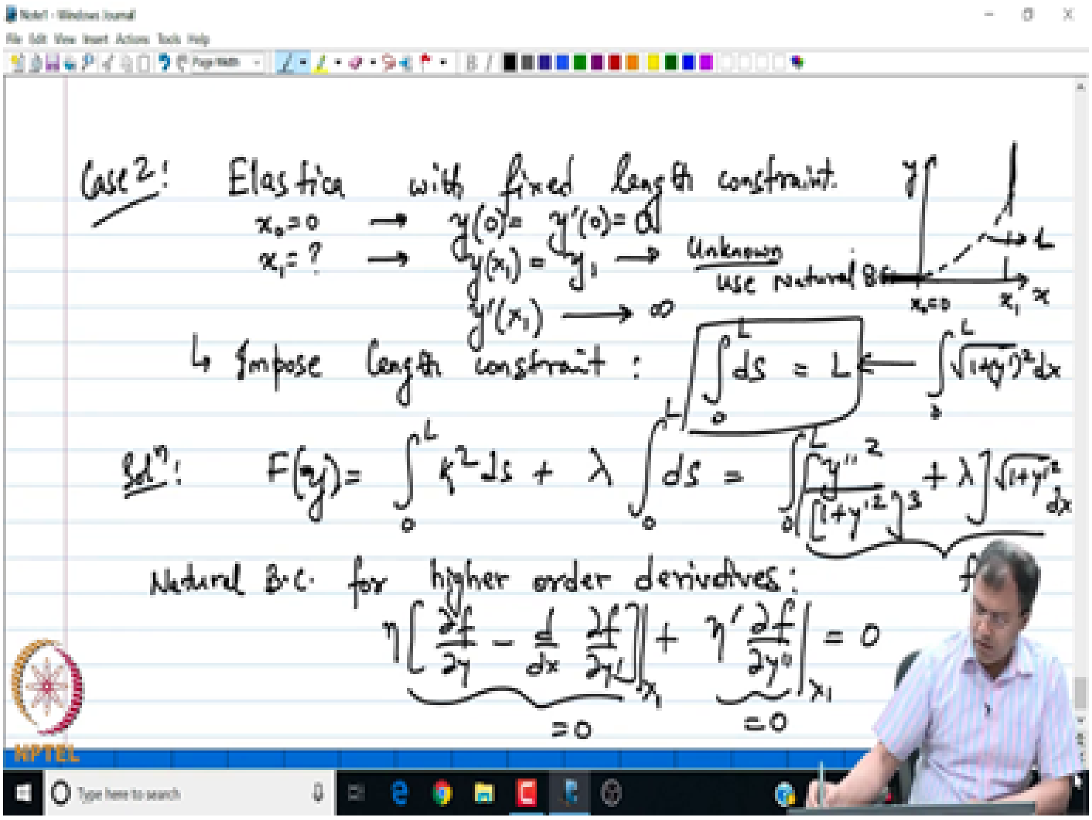


Well, the last topic of this lecture is, we will continue our discussion on the case of elastica. We are going to look at another case of elastica where we impose the length constraint, here we have a case of elastica where we are going to impose our length constraint. So, lets call this as $x_1$ and let the total length of the elastica is l. So, this is the case of elastica with fixed length constraint. So, now the set of conditions that we have is on the side, on one of the end points, $x_o = 0$ and second end point $x_1$ is not known.

But $x_o = 0$, I have the condition y(0) $= y^{'}(0) = 0$ and $y(x_1) = y_1$ is an unknown. So, we do not know what is the position, but we also have that the slope approaches the vertical or the y axis. So, the slope eventually becomes parallel to the y axis. So, we do have the slope information at the second point. Now, so this $y_1$ is an unknown of the problem, so, we definitely we have to use our natural boundary conditions.

Further we have the length constraint, we impose the length constraint which means that $\int_0^L ds = 1$. So, we fix our length and impose a length constraint. So, which means that the functional has to be optimized is the following.

$$F(y) = \int_0^L \kappa^2 ds + \lambda \int_0^L ds = \int_0^L \left[ \frac{{y''}^2}{\left(1+{y'}^2\right)^3} + \lambda \right] \sqrt{1+{y'}^2} dx$$

So, this is my integral that I am trying to optimize and my natural boundary condition are as follows.

We have to use the condition for higher order derivatives and we see that this imposes the following constraint $\eta \left[ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right] + \eta' \frac{\partial f}{\partial y''} = 0$ at $x_1$ so, we individually set each of these quantities to 0 to get 2 natural boundary condition. So, then the solution, I am going to right away use the solution to the previous test case.

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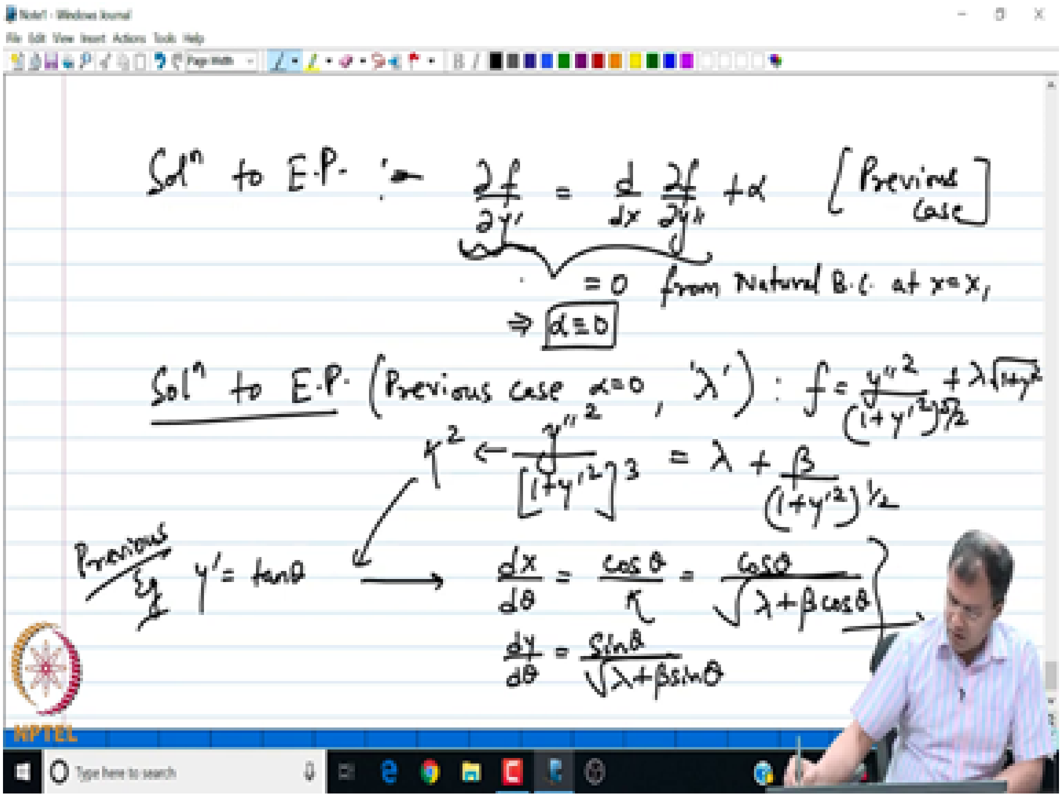


The solution to my Euler-Poisson equation from our previous example is directly found out to be $\frac{\partial f}{\partial y'} = \frac{d}{dx} \frac{\partial f}{\partial y''} + \alpha$ that is from our previous case, and finally we note that this is nothing but equal to 0 at x = $x_1$. So, this is nothing but from the natural boundary condition this is 0.

So, that brings in the conclusion that $\alpha = 0$ identically . So, which means that from my previous case I can directly write the solution to the Euler-Poisson from previous case with$\alpha = 0$ and we have the introduction of a new constraint $\lambda$. We see that our equation reduces to the following form.

$$f = \frac{{y''}^2}{\left(1+{y'}^2\right)^{\frac{5}{2}}} + \lambda \sqrt{1+{y'}^2}$$

For this f my equation reduces to the following $\kappa^2 = \frac{{y''}^2}{\left(1+{y'}^2\right)^3} = \lambda + \frac{\beta}{\sqrt{1+{y'}^2}}$

again we use the same set of substitution to solve this equation that is $y^{'} = \tan\theta$ and we are going to get a set of 2 equations. So, this is again from the previous example.

$$\frac{dx}{d\theta} = \frac{\cos\theta}{\kappa} = \frac{\cos\theta}{\sqrt{\lambda + \beta\cos\theta}}$$
$$\frac{dy}{d\theta} = \frac{\sin\theta}{\sqrt{\lambda + \beta\sin\theta}}$$

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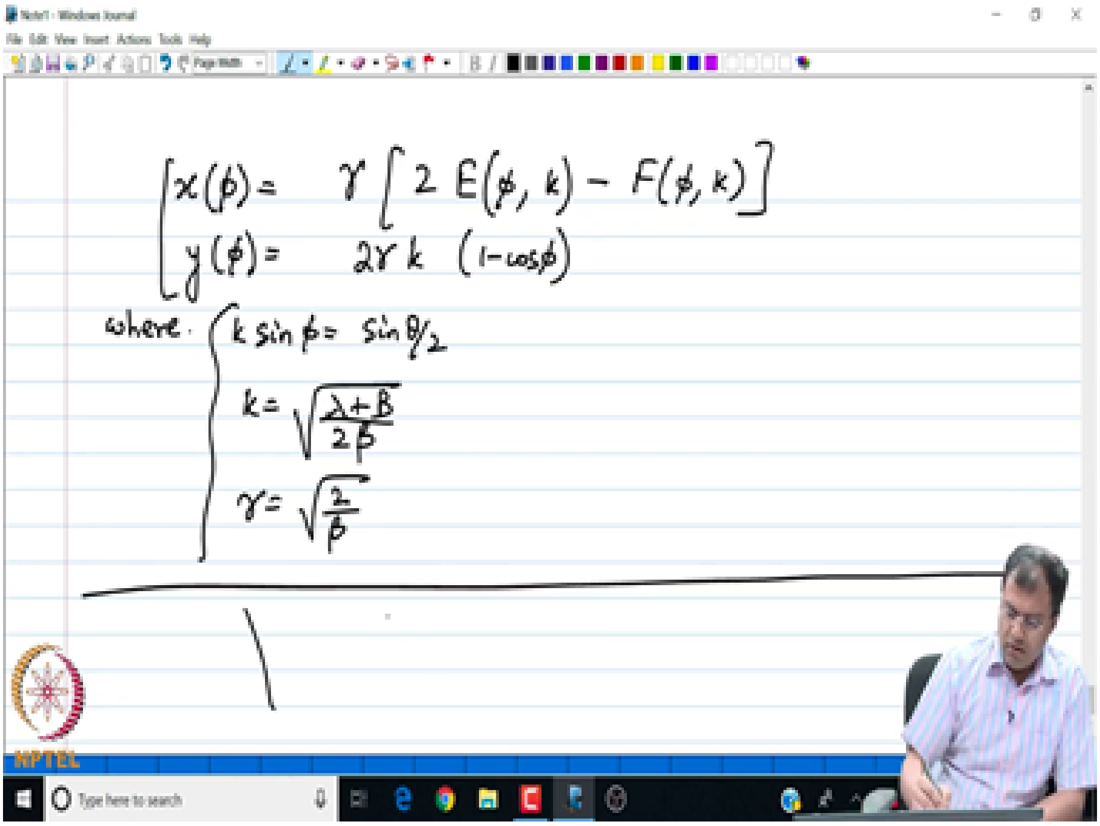


The final solution for this case comes out to be the following

$$x(\phi) = \gamma\left[2E(\phi,k) - F(\phi,k)\right], \;\; y(\phi) = 2\gamma k(1-\cos\phi)$$

Where $k\sin\phi = \sin\frac{\theta}{2}$
$k = \sqrt{\frac{\lambda+\beta}{2\beta}}$
$\gamma = \sqrt{\frac{2}{\beta}}$
that completes the solution to this constraint elastica problem