Variational Calculus and its Applications in Control Theory and Nano mechanics Professor Sarthok Sircar

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Lecture – 03

Introduction –Euler Lagrange Equations Part-3

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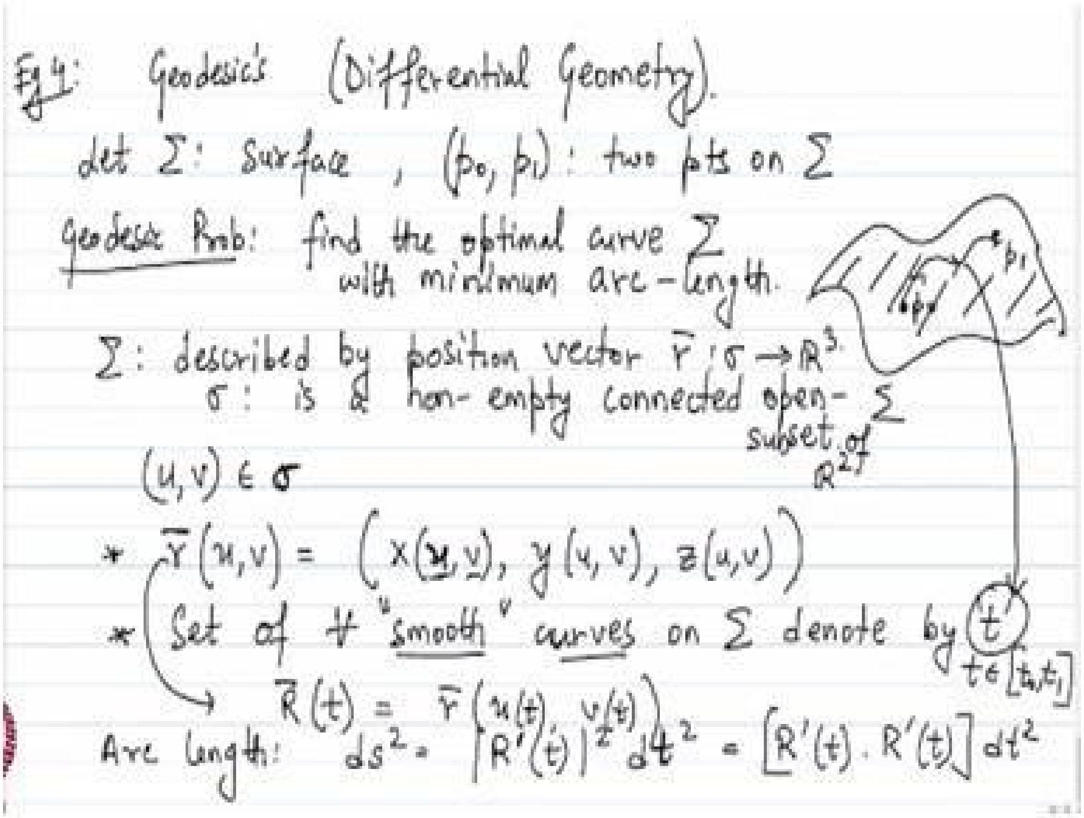

So another problem that I want to describe in this lecture is the problem of geodesic. So this is a problem which is widespread in the area of differential geometry, the problem is as follows, let us assume that σ is some surface, it could be a surface of a sphere, it could be any particular surface and now for easier reference let us say this is my surface denoted by σ and let p_o and p_1 be two points on this surface sigma.

The geodesic problem says, we need to figure out the optimal curve σ with minimum arc length. So, we have to figure out the curve passing between the two points p_o and p_1 so that the total length of the curve is minimum. Now, intuitively we can see that the length will be minimum when this curve let us say on a two dimensional surface this curve is a straight line, but we will show using calculus of variations that indeed the extremal solution that we get from the geodesic problem is indeed a straight line.

So a bit more in depth detail of this problem, let us say we denote σ by the position vector r from small σ to R^3 where in terms of notation my small σ is a non empty connected open set or I would say a subset of R^2. So sigma is a subset over the class of all these surfaces which is the subset of R^2 and then I describe any of my surface by a two parameter family let us say u and v.

Now let us say u and v belongs to σ which is a class of this surface and in that case family of curves on this surface is described by

So I can further simplify this description of these optimal curve using a single parameter, let us say t.

$$\Rightarrow \qquad r(u,v) = (x(u,v), y(u,v), z(u,v))$$

let us say the set of all smooth curves When I say smooth I denote continuous/piecewise differentiable curves, so set of smooth curves on σ we denote it using the parameter t, such that t is taken from some interval along the real axis t_o to t_1. So, in this case the set of all smooth curves which is going to lie on this surface we denote by R(t) which is eventually going to be r(u(t), v(t)). So we see that the curves will be found by substituting particular values of u and v in this position vector.

And as we move along these curves, the value of this parameter t changes from t_o to t_1 in case of arc length, let us say arc length element ds^2 is going to be by chain rule, it is going to be $R^0(t)$ this is the absolute value because it is length, as length can never be negative. So arc length square is absolute value square well absolute value square times dt^2.

$$ds^2 = [R^{'}(t)]^2\, dt^2 = [R^{'}(t).R^{'}(t)]dt^2$$

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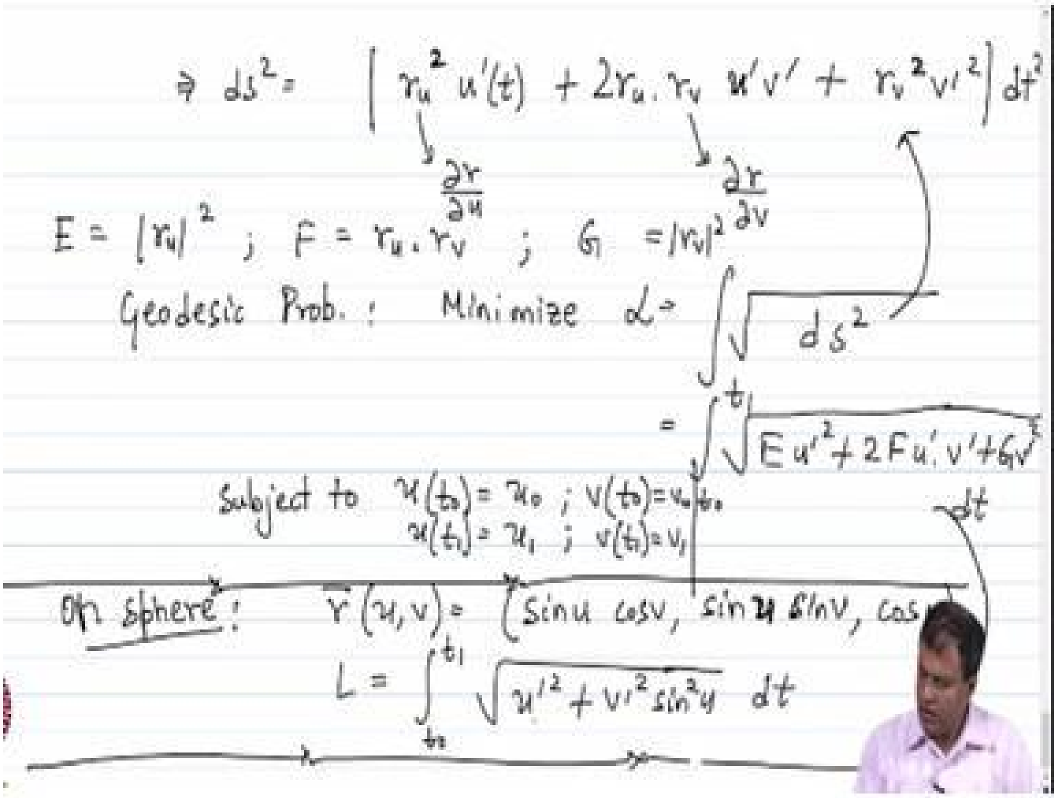

We can rewrite this arc length element as follows,

$$\Rightarrow \qquad ds^2 = \left[r_u^2 u^{'}(t) + 2r_u r_v u^{'} v^{'} + r_v^2 v^{'}(t)\right] dt^2$$

where my r_u is $\frac{\partial r}{\partial u}$ and my r_v is $\frac{\partial r}{\partial v}$

Let us simplicity we denote

$E = |r_u|^2$; $F = r_u.r_v$; $G = |r_v|^2$

i.e E to be the absolute value of r_u^2 and F to be r_u dot product with r_v and G to be absolute value of r_v^2

So, the geodesic problem says that we want to minimize L which is

$$L = \int \sqrt{ds^2}$$

So, the integral of the square root of the arc length square which we substitute from this following expression above.

$$\Rightarrow \qquad L = \int_{t_o}^{t_1} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt$$

Subject to $u(t_o) = u_o$, $u(t_1) = u_1$, $v(t_o) = v_o$, $v(t_1) = v_1$, these are the boundary conditions in terms of the parameters of the system. So just to give a brief overview let us say we are considering the surface as the sphere.

So on a sphere, we denote

$$r(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

Then if we have to use this formula above we use see that the arc length L turns out to be by substituting all this values of x, y and z

$$L = \int_{t_o}^{t_1} \sqrt{u'^2 + v'^2 \sin u^2}dt$$

Where my ' denote the derivatives of this parameters u and v with respect to the parameter t.

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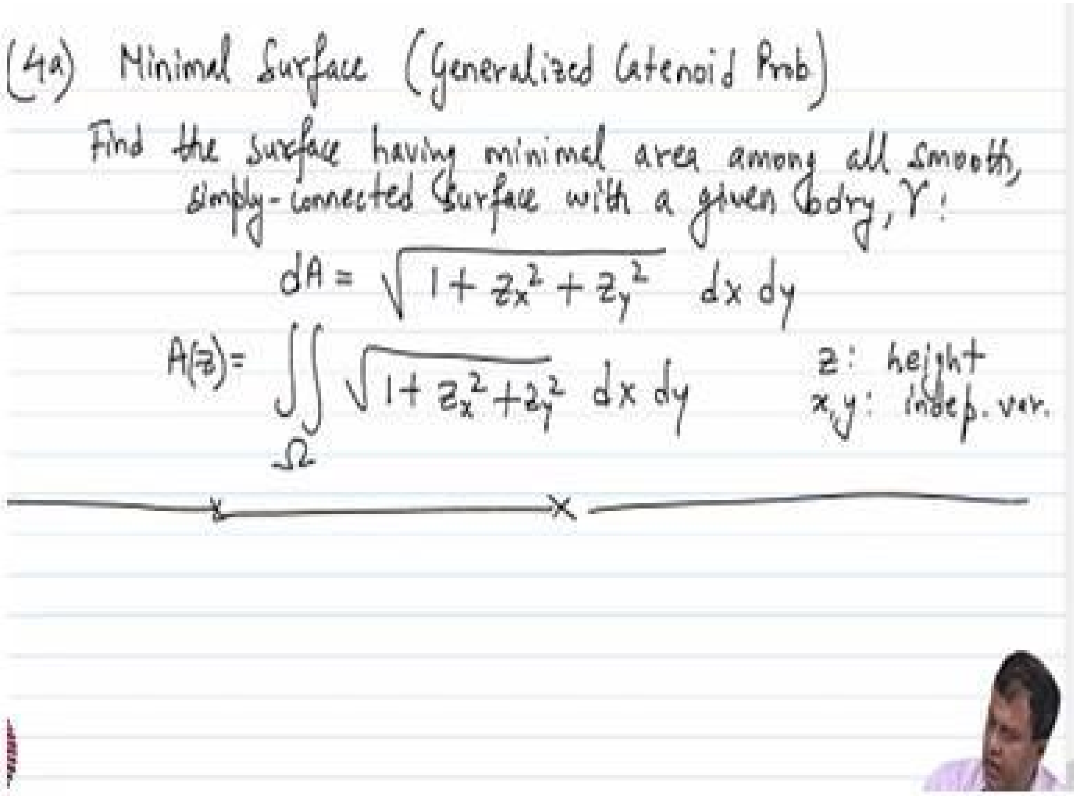

Now, let us look at a related problem. that is to find the minimal surface similar to catenoid problem. I can refer this problem as the generalized catenoid problem that showed earlier. So in this case, again this is the geodesic problem, but in two independent variables. The problem says we have to find the surface having minimal area among all smooth simply connected surface with a given boundary γ.

Such that, my area element in this case is going to be similar to the arc length

$$dA = \sqrt{1 + z_x^2 + z_y^2}\ dx\, dy$$

where my z is the height of the surface and x and y are the independent variables. So, in this case area functional A(z) is

$$A(z) = \iint_{\Omega} \sqrt{1 + {z_x}^2 + {z_y}^2}\, dx dy$$

So we see that in this case the optimization involves a functional of two independent variables x and y and we will deal with similar class of problem later on. So then, let us now move on to another problem in economics.

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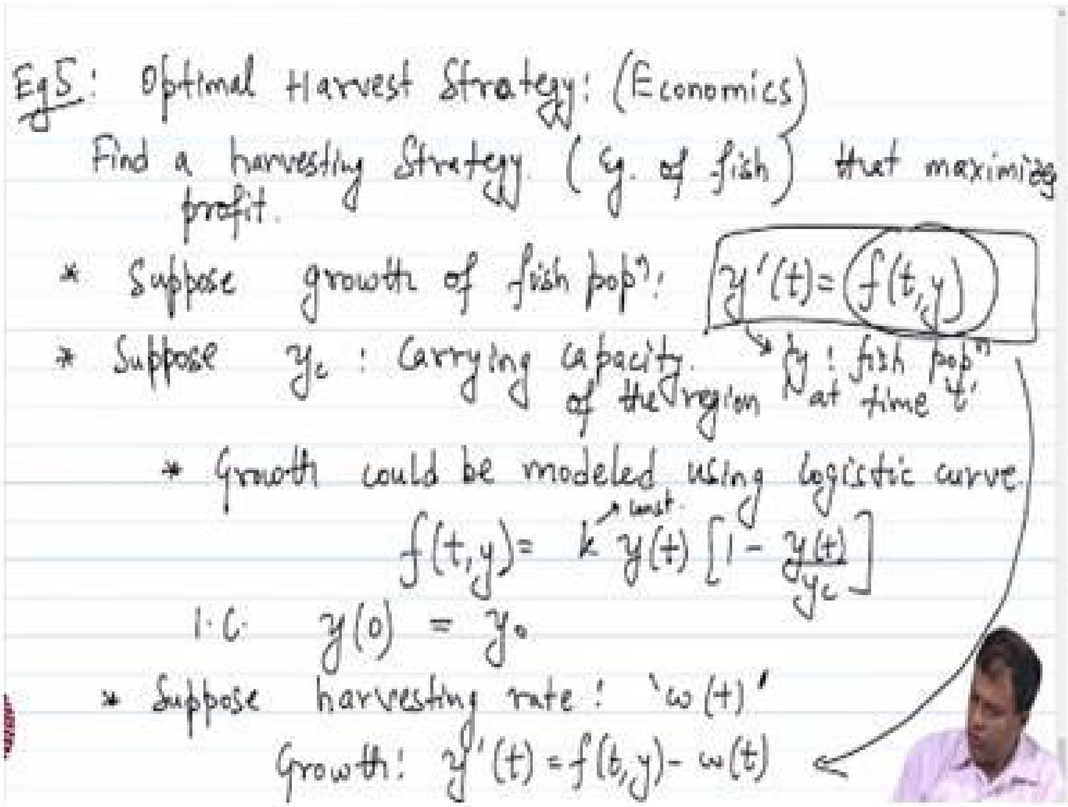

So, this fifth one or the last problem of this lecture series is the problem of finding the optimal harvesting strategy. So, this problem could be modified into problem in several areas, but the classical problem arises in economics. So the problem is to find a harvesting strategy, let us say of fishes in a pond. So we have to find the harvesting strategy that maximizes profit. So the fisherman has to harvest so much quantity of fish.

So that he or she can get the maximum profit out of it. So let us say that the growth of the fish population follows an ODE of the form

$$y'(t) = f(t, y)$$

where I denote y to be fish population at any given time point t. So then, let us further assume that the fish cannot grow indefinitely which means that there is a carrying capacity of the fish let us denote the carrying capacity by y_c.

So y_c is the carrying capacity of the fish, which means that I am going to model the growth of the fish which is I am going to denote this right hand side of this ODE given here by the logistic curve. So, the growth could be modeled using the logistic curve. So, which means that my right hand side of my ODE will be of the following form

$$f(t, y) = ky(t)\left[1 - \frac{y(t)}{y_c}\right]$$

where k is constant.

We see that when the population exceeds the carrying capacity y_c, the right hand of this OD becomes negative. Well we assume that the constant k is positive, so which means that after the y_c the fish population is going to start to fall. Further, we assume that there is an initial condition that the initial population of the fish at t = 0 is y_o and let us further assume that the harvesting rate the rate at which the fish population is taken out of the pond is denoted by this function w (t).

So, which means that this particular ODE can now be remodeled as follows. We are going to remodel the growth of the fish population by this modified growth equation

$$y'(t) = f(t, y) - w(t)$$

which is the quantity per unit time of the fish which is taken out of the system. So then, the problem is as follows.

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Problem: Determine '$w(t)$' so that profit in a period (T) is maximum

Let : $c(t,y,w)$: cost to harvest / unit biomass

$p(t)$: price / unit sold in market.

Profit $P(y,w) = \int_0^T [p(t) - c(t,y,w)]\, w(t)\, dt \longrightarrow (5)$

* Find $w(t)$ s.t (5) is maximum

Subject to the constraint $\boxed{y' = f(t,y) - w(t)}$

+ B.C / I.C $y(0) = y_0$ (Holonomic Const.)

* B.C $y(T) = 0$: not optimal case ($\because$ harvesting is most costly with fewer fish).

The problem is we need to determine the harvesting rate i.e. we need to determine w (t), so that the profit is maximum in a given period. So, typically the fisherman they want to take out fish in a particular season of the year, so let us say this period is a particular season of the year where the fisherman wants to maximize the profit. Let us denote this period by T.

So then, let us now quantify this problem let us say that c be the cost of harvesting so c is the cost to harvest per unit biomass or biomass of the fish. So of course the cost will depend on that particular season it will depend on the fish population in the pond as well as the harvesting rate and then let us say that p of t is the price of the fish that is sold in the market the price per unit mass of the fish which is sold in the market.

So, which means that profit functional P(y, w) is

$$p(y,w) = \int_0^T [p(t) - c(t,y,w)]\, w(t) dt \qquad \mathbf{5}$$

so we need to find the goal of this problem i.e we need to find w(t), such that **5** is maximum and subject to the constraint

$$y'(t) = f(t,y) - w(t)$$

So, we have that the fish population that must satisfy the ODE constraint, you see that this particular constraint now is not an integral constraint it is a differential constraint and hence the so called Holonomic constraint. So, we are going to deal with the class of problems of extremizing functions with

Holonomic constraints later on and of course, we have boundary conditions or I would say initial conditions in this case which is given by the fact that $y(t = 0) = y_0$.

Now, one can always say that an intuition tells us that perhaps the profit of this functional **5** can be maximized when all the fish is taken out of the pond. So, one intuition tells us that the boundary condition when over the entire length of the catching season the total fish is taken out of the pond or y(T) = 0 is perhaps the optimal solution it turns out that this is not the case.

So, by the boundary conditions this is not the optimal case because it turns out that as we harvest more and more fish from the pond it becomes more and more expensive to harvest the fish. So, because harvesting is more costly with fewer fish in the pond. So, let me wrap up this lecture session by giving you some more problems that we are going to disclose and talk about in more depth.

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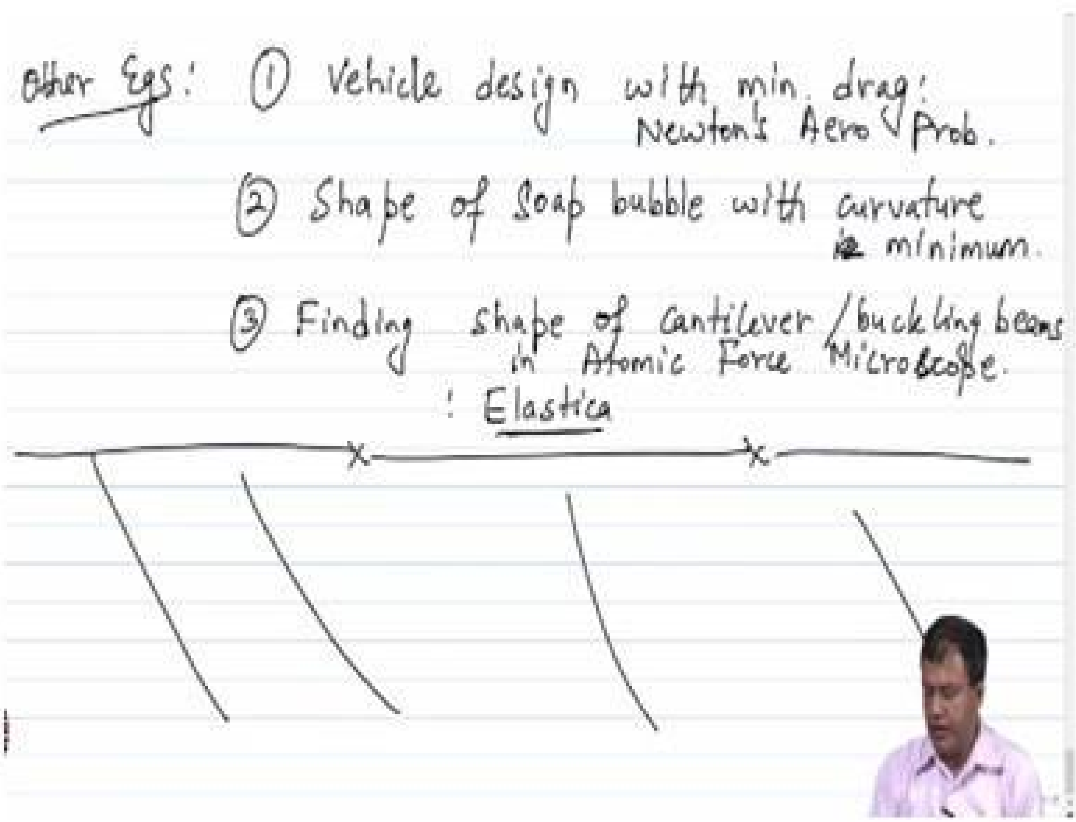

So, the problems that we are going to talk about over the later lectures are the problems of reduction of vehicle drag also known as the class of problems related to the Newton's Aerodynamic problem, so the vehicle design with minimum drag the class of this problem belongs to the Newton's Aerodynamic problem. We will go through this in detail over another lecture. Another problem that I am interested to go in depth is to find the optimal shape of a soap bubble with such that the curvature is minimum.

It turns out that the lower the curvature of the soap bubble, the lower is the total energy or the surface energy of the soap bubble. So, nature provides the shape of the soap bubble such that the total energy of this system is minimum, then another problem that we are going to talk in detail is finding the shape of cantilever or bending beam problems cantilever or buckling beams in atomic force let us say atomic force microscopy.

This is just one case microscope so this is just one case of the class of problems that we are going to describe and this class of problems belong to the class known as Elastica which we will also talk in more depth. So, thank you very much so that is the end of this session. Thank you for listening.