Variational Calculus and its Applications in Control Theory and Nano mechanics

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Problems with Holononomic and non-Holononomic Constraints Part 4

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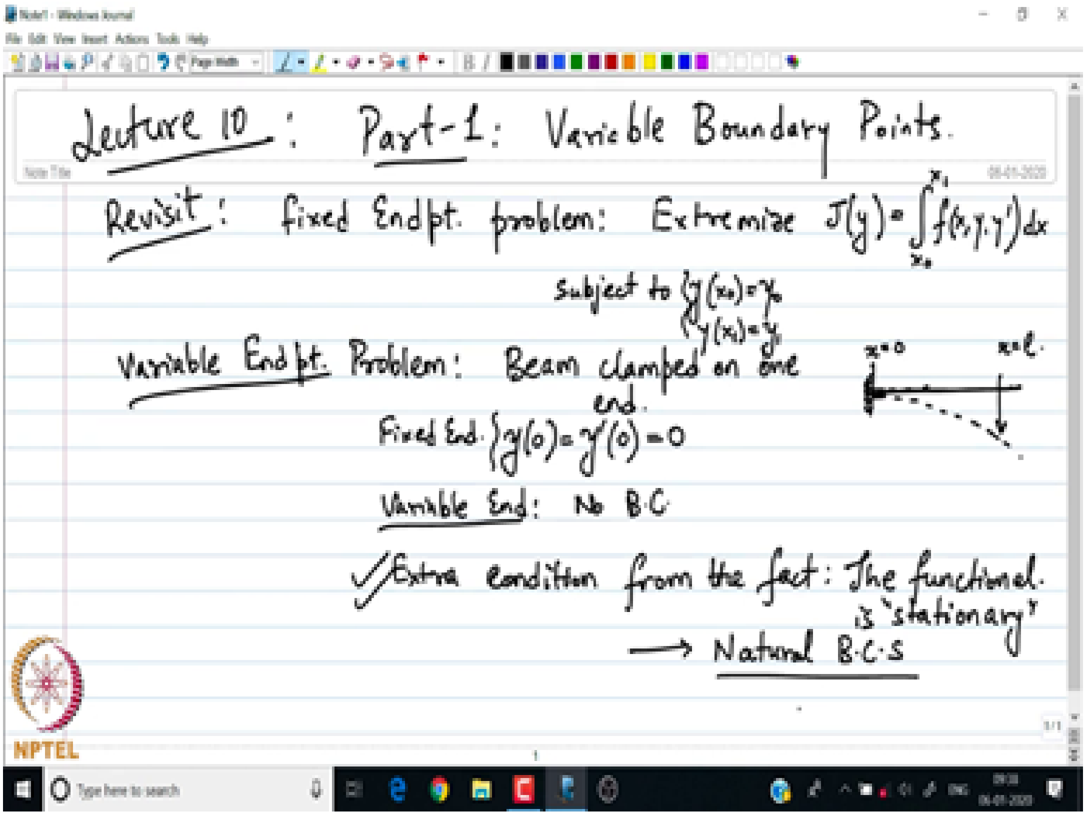

Topic of our today's lecture is on the boundary conditions such that the boundary conditions are not fixed but variable, In this lecture and in the next one I am going to talk about specific cases when we do not have fixed boundary condition on the functional end points.

In this lecture I am going to look at a very special case of variable boundary points followed by the more general case in the next lecture.

Let us revisit our original functional optimization problem. Note that for the fixed end point problem we had the following functional that we need to extremize the functional J(y) such that $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'})dx$ and subject to $y(x_o) = y_o$ and $y(x_1) = y_1$ these two boundary condition.

We are going to look at the class of variable end point problems where we only vary the y component, the x component is still fixed. So, the problem that I have the variable end point problem. A very common example in this category which we are going to discuss in detail is the problem of bending beams.

For example if we have a beam or a steel rod, let us say clamped onto the wall at one end and it is possible to bend the beam under a action of a load. So, in that case one of the end being clamped, the boundary condition still applicable at this clamped end is the fixed point boundary condition. Since, the other end is varying, we will have to figure out a new set of boundary conditions for this variable end points.

A beam clamped on one end and in that case as the diagram shows we can very well assume that. So, this is x = 0 and this is x = L . So, y at 0 is equal to the slop at 0 is equal to 0. So, we still use the fixed point boundary condition at the fixed end. However, at the variable end there is no such boundary condition because there is no boundary at all.

Well, the question is then, we do need extra conditions to completely describe our extremal solution, so, where are we going to get those extra condition, the answer lies in the setup of the extremal itself, so the extra conditions that are needed to fully solve and find the extremal of the solution of the functional is coming from the fact that the functional is stationery. So, we want to find the conditions at the variable end points such that the variation in the functional is at most of $O(\epsilon^2)$ or in other words the variation in the functional is negligibly small.

So, the extra condition comes from the fact that the functional that we are trying to optimize is stationery or has extremal solution and the imposition of this particular statement will lead us to the development of new conditions knowns as the so called natural boundary condition. We will see that the natural boundary conditions, they reduce to the fixed point boundary condition if we fix the free end. So, the natural boundary condition turns out to be a more general class of boundary conditions of which the fixed point boundary is a subset. So, let us now build the background on finding these set of natural boundary condition.

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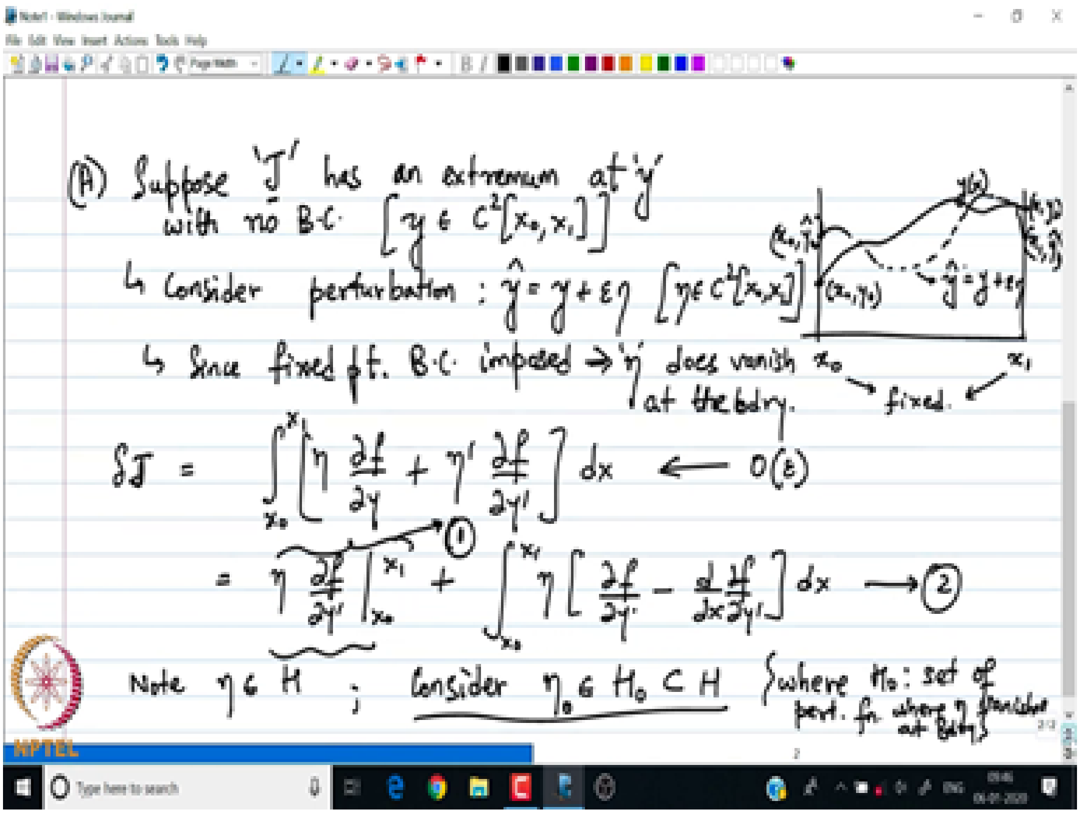

Let us say we are given a domain with x coordinate from x_0 to x_1, I allow these x coordinates to be fixed, but I allow the y coordinates to vary. So, let us say we have a function y(x) and now if we want to perturb the function $\hat{y}$.

Let us say this is my $\hat{y}$ and now the end points are at (x_o, y_o) and the new perturbed quantity or the perturbed function has the end points $(x_o, \hat{y}_o)$ and from (x_1, y_1) to $(x_1, \hat{y}_1)$. So, if we were to describe this perturbation as the original function plus $\epsilon\eta$, then no longer does the perturbation η going to satisfy the

0 end point condition because the end points are not going to match for the perturbed and the original function.

let me call this case study as **A** because we have more case studies along the similar lines. Suppose I have the functional J has an extremum at 'y' with no boundary condition and $y \in C^2[x_o, x_1]$.

Let us consider the perturbation $\hat{y} = y + \epsilon\eta$ Now, again we still require η to be continuously differentiable upto second order. However, this time η will not satisfy the 0 boundary condition because the end points do not match, since no fixed point boundary conditions imposed, it implies that η does not vanish at the boundary condition problems η use to vanish the perturbation function.

Let us now write down the variation of this functional J.

$$\delta J = \int_{x_o}^{x_1} \left[\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right] dx$$

$$= \eta \frac{\partial f}{\partial y'}\Big|_{x_o}^{x_1} + \int_{x_o}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right] dx \qquad \mathbf{2}$$

Note that eta belongs to the class of perturbation functions i.e $\eta \in H$ but the only new thing now is that eta does not vanish on the boundary.

However, If we consider a new perturbation $H_o \subset H$, where my subset H_o is the set of perturbation functions where η vanishes at the boundary, Consider $\eta_o \in H_o \subset H$

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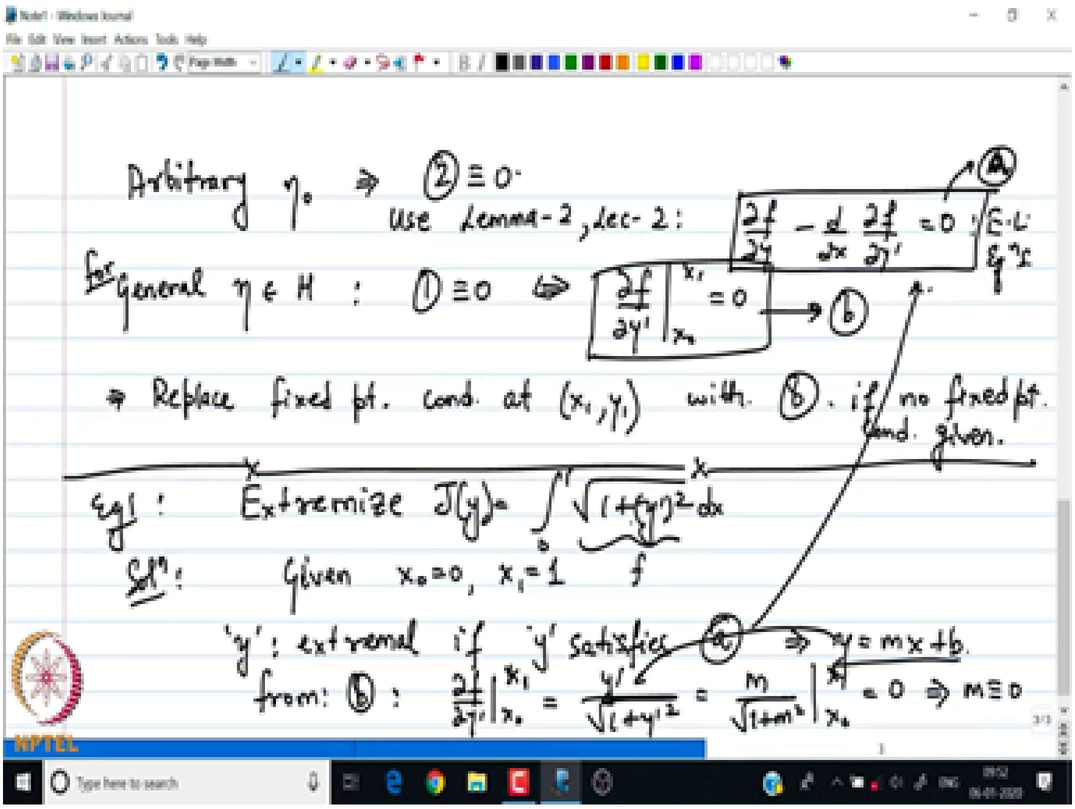

So, for η_o certainly equation **2** is satisfied, and η_o is quite arbitrary, which means for an arbitrary η_o this integral will be 0.

For an arbitrary η_o , it turns out that η_o vanishes on the boundary. It turns out that 2 must be identically equal to 0 which means that again using Lemma 2 in our lecture two we can come again to the same set of conclusion that $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$ (**a**)
where this is nothing but our original Euler Lagrange equations which are recovered.

Now let us consider the most general η for the most general perturbation $\eta_o \in H$. Note that, equation **1** is going to vanish only when the coefficient of η vanishes. So, for a very general η we must have that 1 is identically 0 if and only if $\frac{\partial f}{\partial y'}|_{x_o}^{x_1} = 0$ (**b**)
Equations **a** , **b** are new class of natural boundary conditions.

We will replace fixed point conditions at the point (x_1, y_1) with condition **b**, if no fixed point conditions are given. So, let us quickly look at an example to see what this new condition, how this new condition works. So, a simple example is finding the extremal of the arc length. Extremize the functional
$J(y) = \int_0^1 \sqrt{1 + (y')^2} dx$

Solution: we are not given any boundary condition, we are only given that $x_o = 0, x_1 = 1$. We have not given what is the value of y at x_o and y at x_1. So, no fixed point boundary conditions given, now it is assumed that the boundary conditions in general class of such problems is for the variable case or we are going to use natural boundary condition.

Before we go ahead we know that for the fixed point problem the solution to this arc length problem was a straight line, which means y is extremal if y satisfies our condition **(a)** which is the Euler Lagrange equation and from here we can directly get that the solution is a straight line, y = mx + b, and then since we are not given any extra boundary condition we can impose condition **(b)**, from **(b)** I get that $\frac{\partial f}{\partial y'}|_{x_o}^{x_1} = \frac{y'}{\sqrt{1+(y')^2}} dx = \frac{m}{\sqrt{1+m^2}}|_{x_o}^{x_1} = 0 \Rightarrow m \equiv 0$

So, the conclusion here is that the extremal that we get are constant y = b, where b is constant or the extremal that we get are lines which are parallel to the x axis.

So, that should be quite intuitive because the functional J will attain the minimum only when $y' = 0$ and that is only possible when y is a constant.

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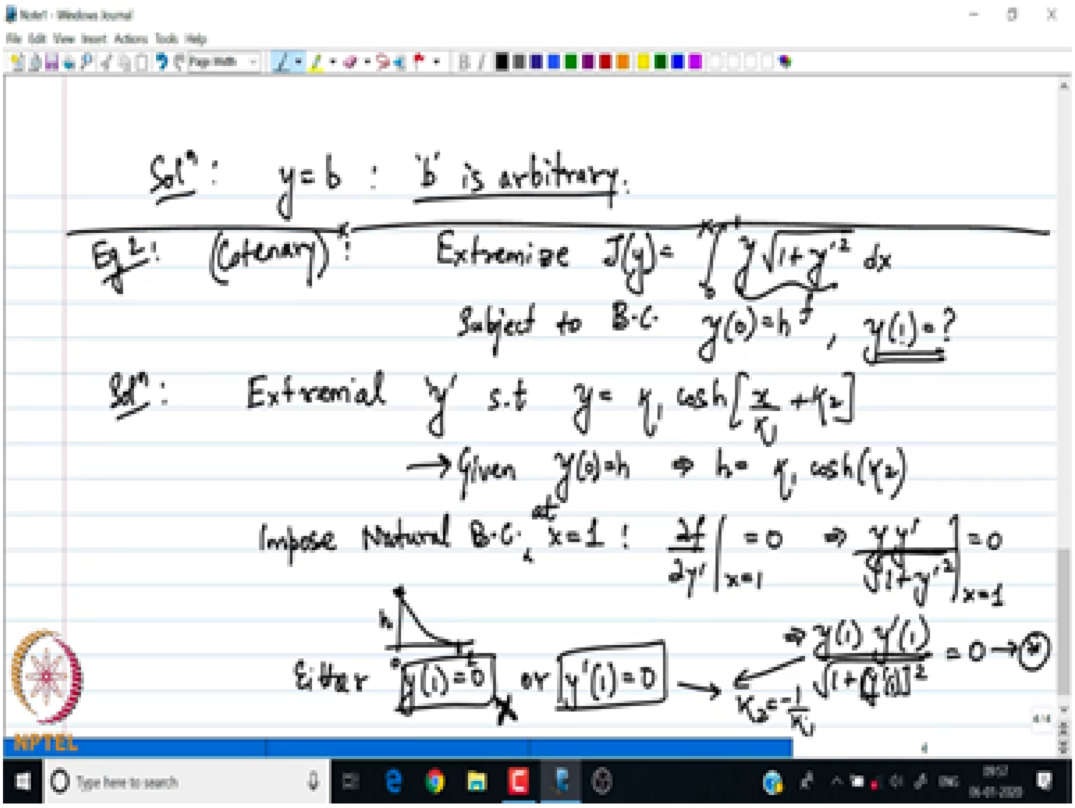

Let us look at another example of the catenary that we have done so many times, Extremize $J(y) = \int_0^1 y\sqrt{1+{y'}^2}dx$J subject to the boundary condition y(0) = h , however we do not know what is y(1)

When we extremize 'y', we already know we are going to the extremal $y = \kappa_1 \cosh\left[\frac{x}{\kappa_1} + \kappa_2\right]$ and also we are given one boundary condition y(0) = h which implies $h = \kappa_1 \cosh(\kappa_2)$.

And then to find the other constant out of κ_1 and κ_2, we impose the natural boundary condition at x = 1, I get $\frac{\partial f}{\partial y'}|_{x=1} = 0 \Rightarrow \frac{yy'}{\sqrt{1+{y'}^2}}|_{x=1} = 0 \Rightarrow \frac{y(1)y'(1)}{\sqrt{1+y'(1)^2}} = 0$ $*$

This cos hyperbolic function will only have one minima. Certainly that will not be equal to one, so from here what we have is either y(1) = 0 or $y'(1) = 0$. Now, if y(1) = 0, what is going to happen? We have a catenary where at 0 the curve is having a height h and at 1 the curve is lying on the ground.

So, we have a very abnormal catenary where part of the catenary is touching the ground, this is not a very plausible situation. So, the only other plausible situation is where the slope vanishes. So, if the slope vanishes, we evaluate the slope at 1 and from here I get a relation between κ_1 and κ_2, I get that $\kappa_2 = -\frac{1}{\kappa_1}$.

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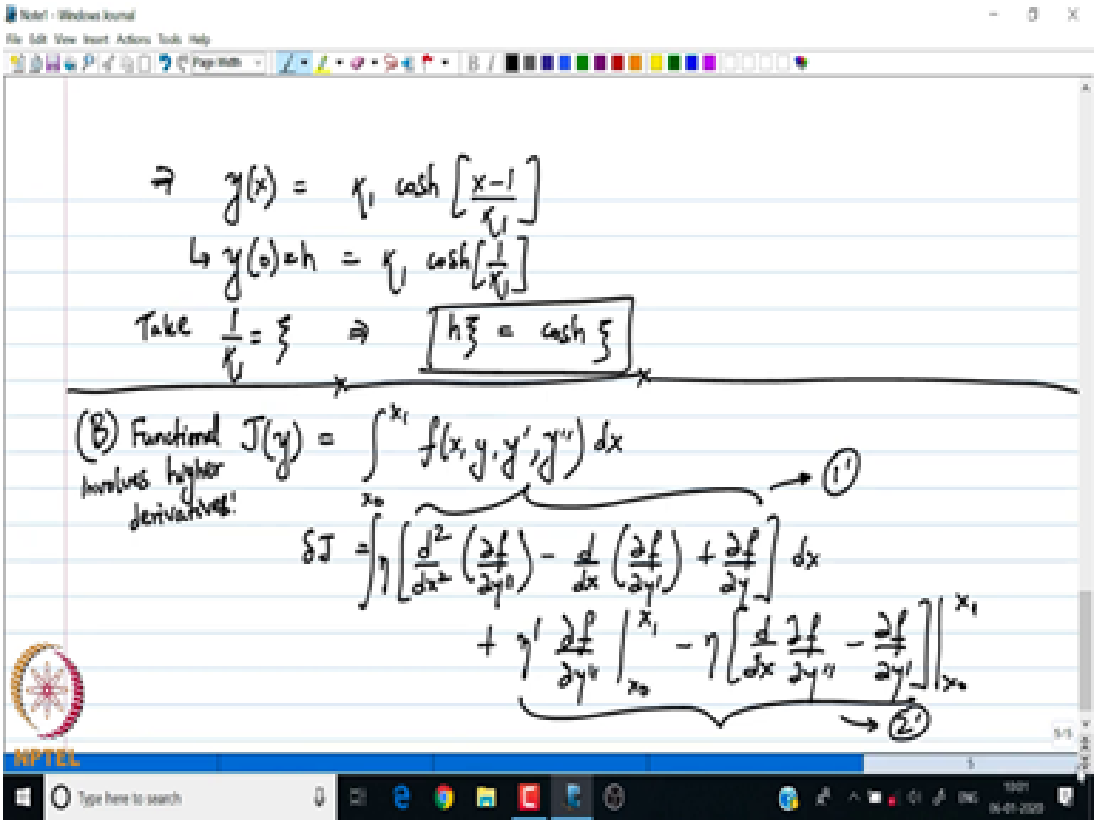

$\Rightarrow y(x) = \kappa_1 \cosh\left[\frac{x-1}{\kappa_1}\right]$ and we have y$(0) =$ h $= \kappa_1 \cosh\left[\frac{1}{\kappa_1}\right]$, which means if I take $\frac{1}{\kappa_1} = \xi \Rightarrow h\xi = \cosh\xi$ So, this is the same standard equation that we get in order to solve for the roots of the catenary. It is a transcendental equation and in general for h large enough it is going to give us two solutions.

So, I am going to stop the discussion on this example because the most general solution has already been discussed in our previous lecture but this example highlighted the way how we can use the natural boundary conditions. So, then we are going to look at another case where we see how does the absence of fixed point boundary condition changes that case.

The case that we are interested where the functional involves higher derivatives. Let $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'}, y^{''})dx$, then again we setup the variation set it equal to 0 and do integration by parts and we are going to come to a stage where we have the following expressions.

$$\delta J = \int \eta\left[\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y}\right]dx + \eta^{'}\frac{\partial f}{\partial y''}|_{x_o}^{x_1} - \eta\left[\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right]|_{x_o}^{x_1}$$

So, we see that again using our specific class of perturbations and from lemma 2 of lecture 2, we can get that from the first bracket I get my standard Euler Poisson equation. So, the conditions for extremal are going to be along the similar lines we can say.
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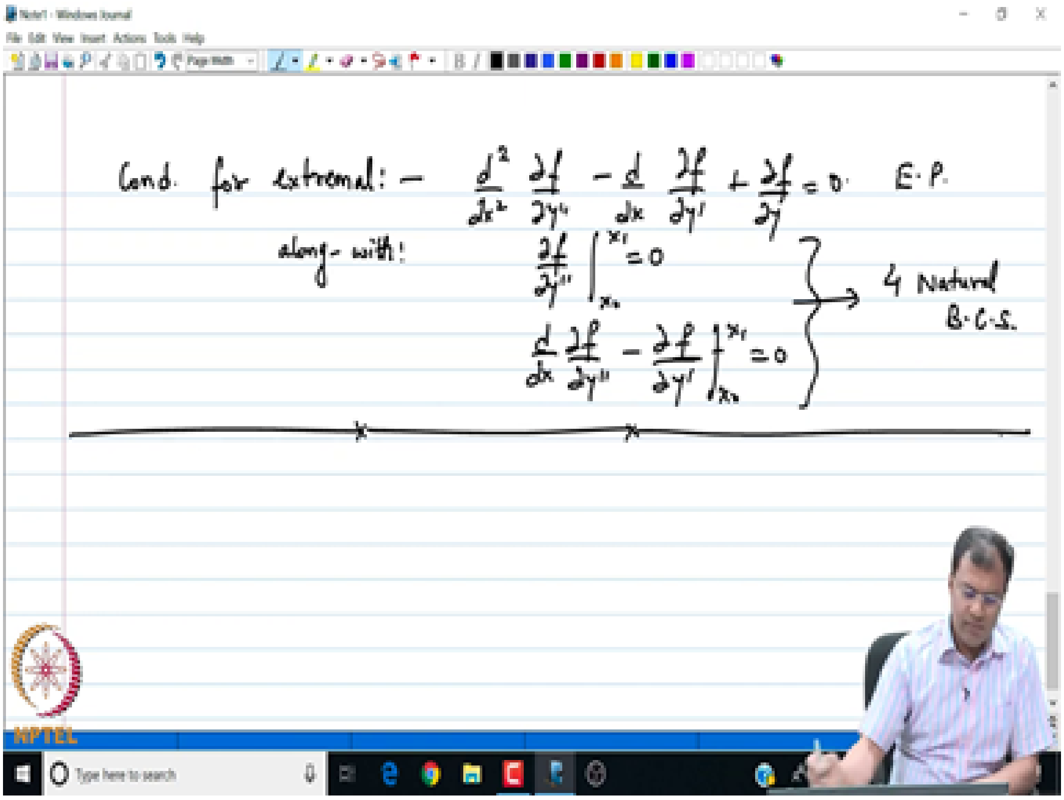

We can conclude that the conditions will be such that the extremal will satisfies $\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y} = 0$ along with $\frac{\partial f}{\partial y''}|_{x_o}^{x_1} = 0$ and $\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}|_{x_o}^{x_1}$

So, these are four natural boundary conditions replacing the fixed point conditions. Now, certainly if there are fixed points in the problem then for those fixed points condition are going to replace the natural boundary condition without any confusion.