Variational Calculus and its Applications in Control Theory and Nano mechanics
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Problems with Holononomic and non-Holononomic Constraints Part 3

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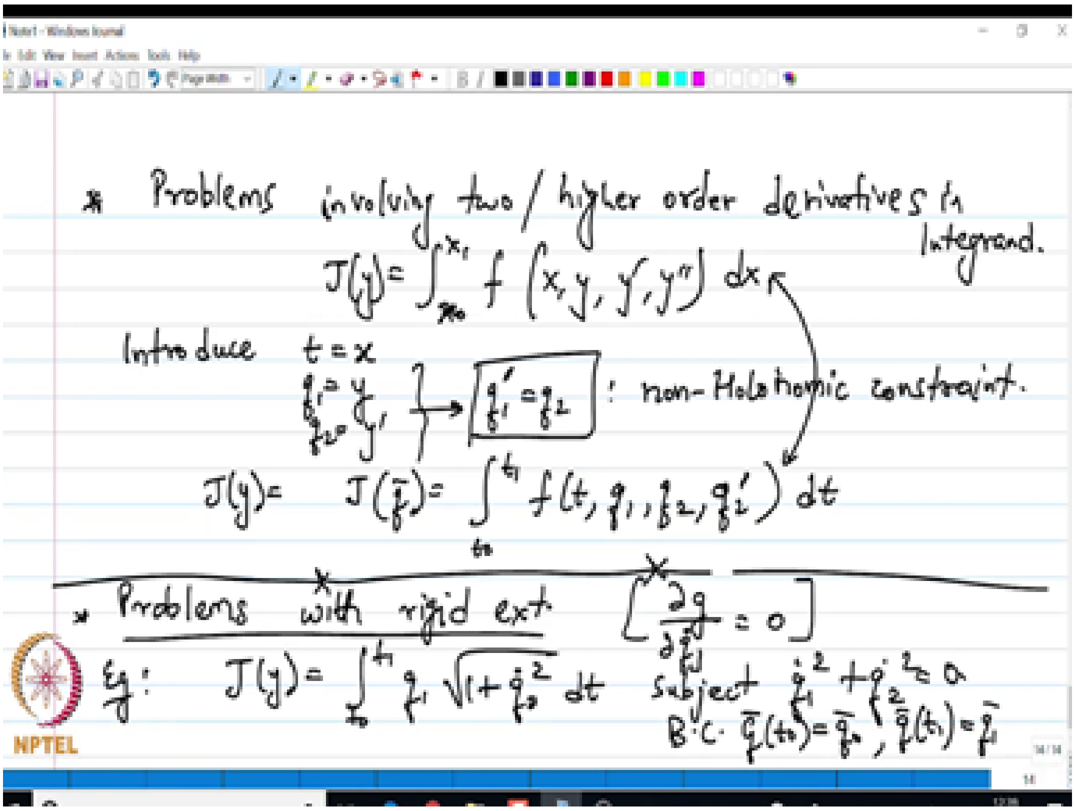

Further before we discuss the problems in this category. Let me just eliminate the pathetic cases or the problems which are abnormal or the problems where we have rigid extremals. Let us look at problems with rigid extremals and we are not going to discuss problems of this category from after this. so, what happens with problems with rigid extremals? I am talking about problems where the derivative of the constraint $\frac{\partial g}{\partial \dot{q}_j} = 0$.

Let us see what happens to this class of problems with an Example: We have the functional of the form $J(y) = \int_{t_o}^{t_1} q_1\sqrt{1+\dot{q}_2^2}dt$ subject to $\dot{q}_1^2 + \dot{q}_2^2 = 0$ and the boundary conditions are $\bar{q}(t_o) = \bar{q}_o, \bar{q}(t_1) = \bar{q}_1$, we need real valued solution, the non-holonomic constraints imply that the only solution that we may have is that $\dot{q}_1 = \dot{q}_2 = 0$ and that gives us the solution that q_1 and q_2 are constants

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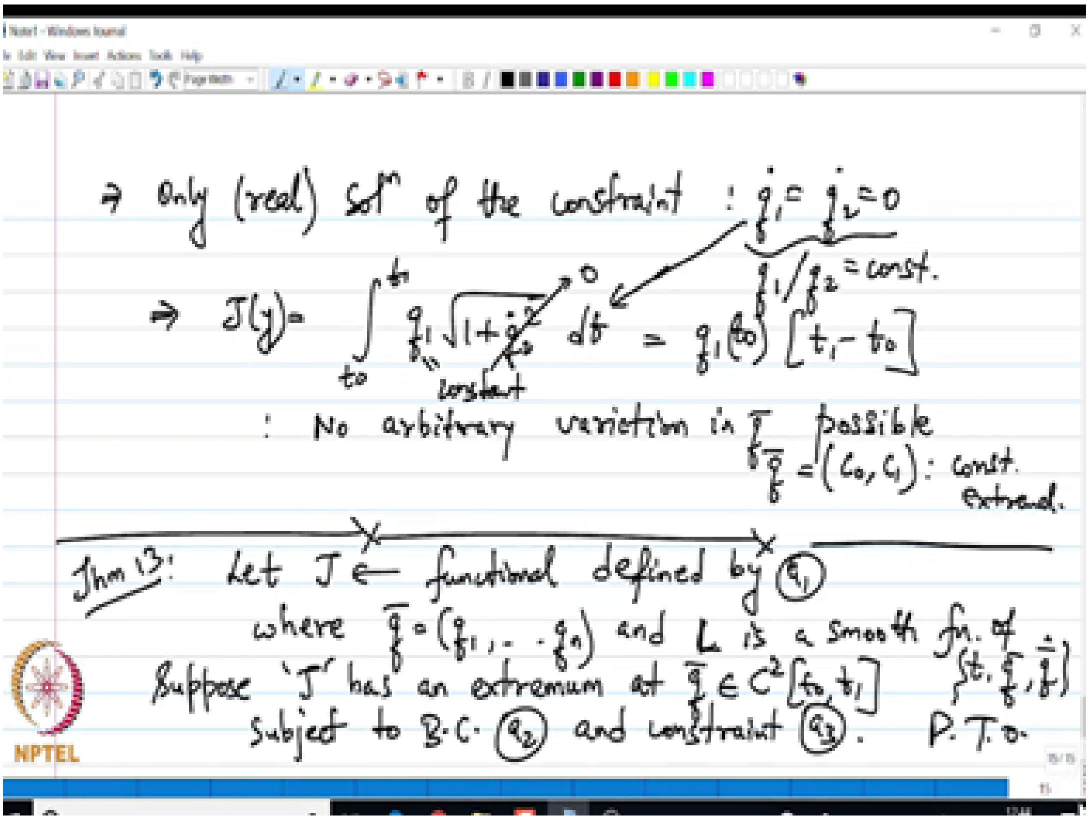

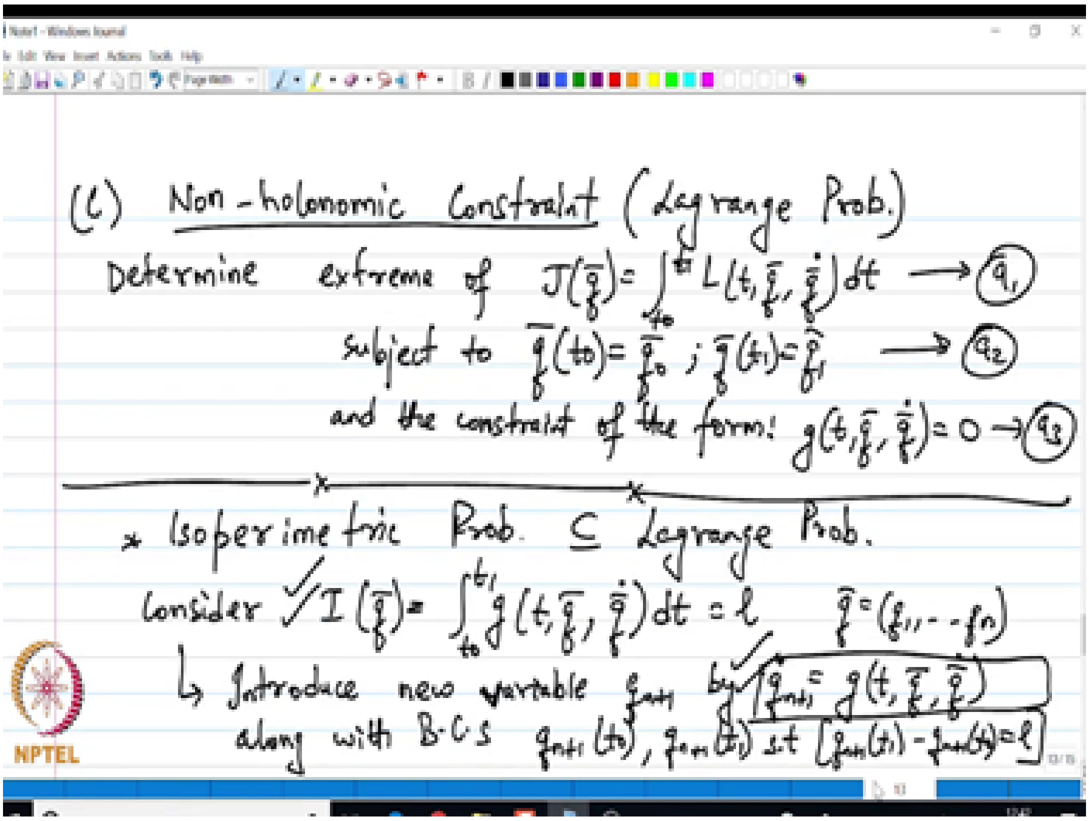

So what this means is that if we look at our J(y) = $\int_{t_o}^{t_1} q_1\sqrt{1+\dot{q}_2^2}dt = q_1(t_o)[t_1 - t_o]$.

So the functional gives us a straight line without any possibility of varying the function q, so in the case of rigid extremal there is no arbitrary variation in $\bar{q}$ possible and the only extremal of $\bar{q} = (C_o, C_1)$ is a constant extremal.

We get an extremal which is also the only solution to the problem. So, the rigid extremal case is going to be handled in the manner that has been shown in this example. Later on, when we state a more general result, we are going to separate out the case when the problem will have rigid extremals versus the case when the problems will not have rigid extremals. So, let us state a summarizing result in the form of a theorem.

Theorem 13: Let J be a functional defined by $\mathbf{a_1}$, For $\mathbf{a_1}$ we need to go back few slides, we have defined $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ in the description of the non-holonomic constraint problem, where $\bar{q} = (q_1, ..., q_n)$ and L is a smooth function of the variables t, $\bar{q}$, $\dot{\bar{q}}$ and suppose J has an extremum at $\bar{q} \in [t_o, t_1]$ subject to the boundary condition $\mathbf{a_2}$ and the constraint $\mathbf{a_3}$

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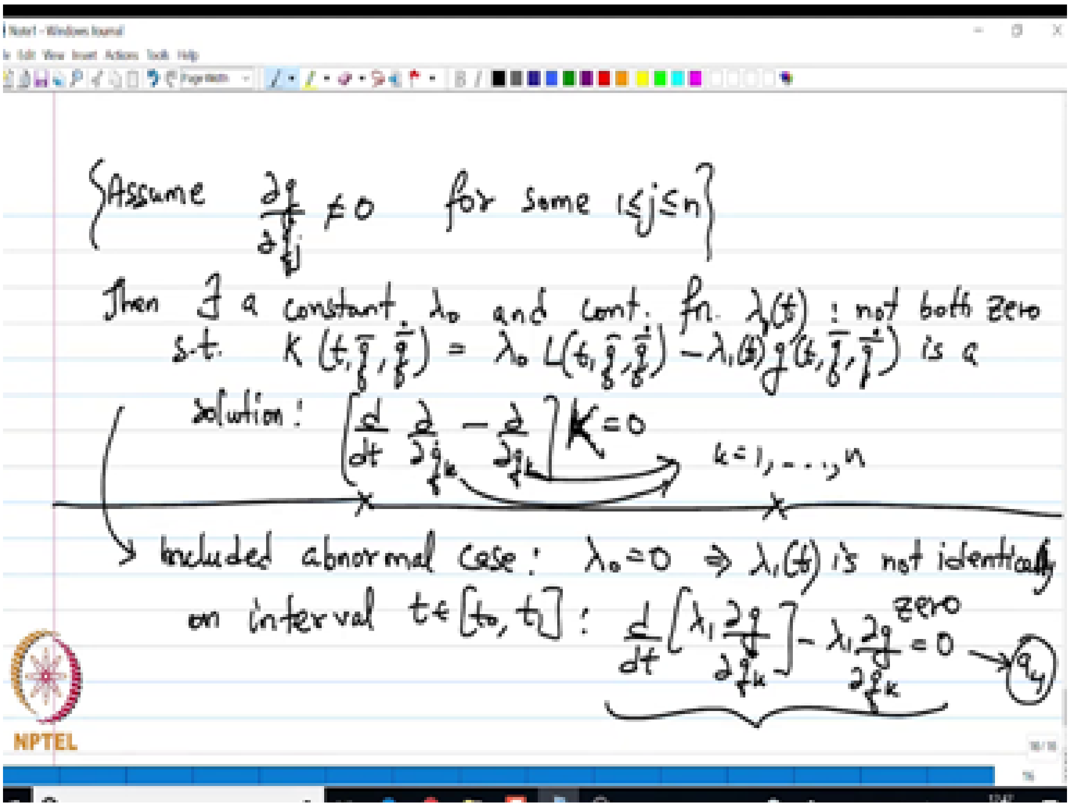

Further we assume that $\frac{\partial q}{\partial \dot{q}_j} \neq 0$, for some $1 \leq j \leq n$. Then there exist a constant λ_o and a continuous function $\lambda(t)$ not both 0 such that $K(t, \bar{q}, \dot{\bar{q}}) = \lambda_o L(t, \bar{q}, \dot{\bar{q}}) - \lambda_1(t) g(t, \bar{q}, \dot{\bar{q}})$ is a solution to the Euler Lagrange equation given by $\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}\right] K = 0 \;\; k = 1, ..., n$

Now, we have already included the abnormal case, this is when $\lambda_o = 0 \Rightarrow \lambda_1(t)$ is not identically 0 on the interval $t \in [t_o, t_1]$ and we have $\frac{d}{dt}\left[\lambda_1 \frac{\partial g}{\partial \dot{q}_k} - \lambda_1 \frac{\partial g}{\partial q_k}\right] = 0$ $\quad \mathbf{a_4}$
So the moment we are dealing with an abnormal problem we must check this criteria.

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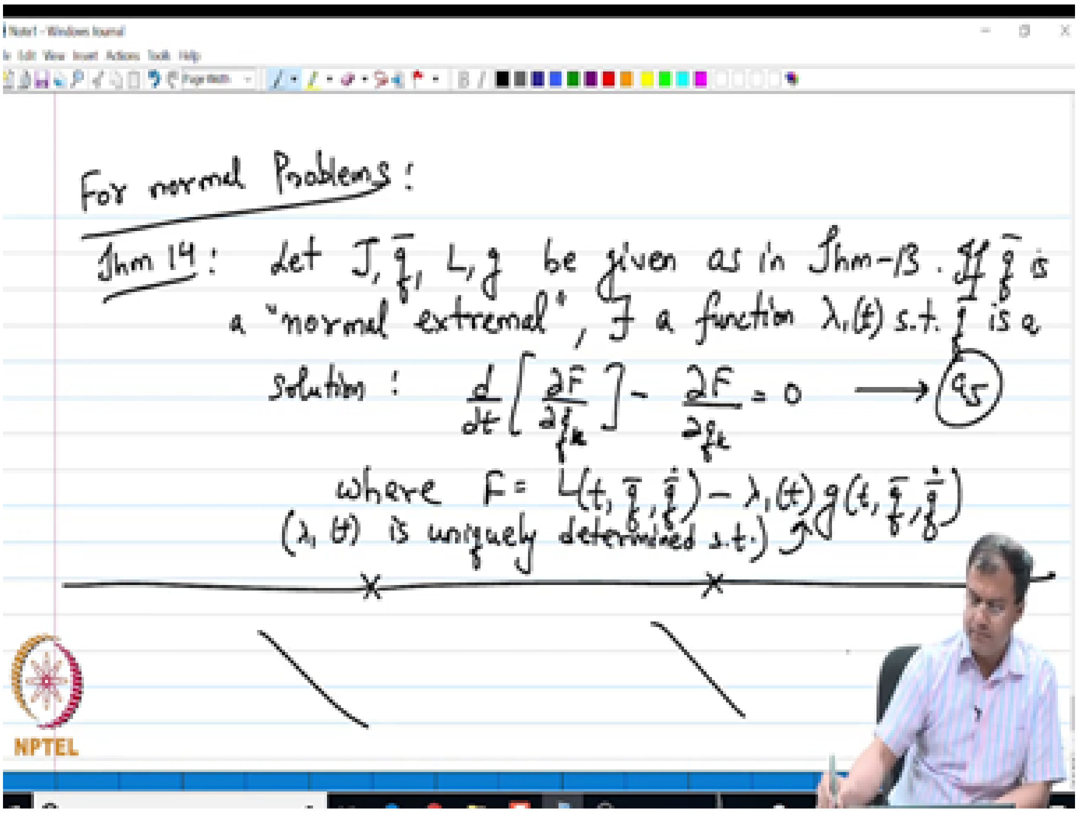

We are generally going to solve in non-holonomic constraint optimization. For normal problems

Theorem 14: Let J, $\bar{q}$, L and g be given as in the Theorem 13, If $\bar{q}$ is a normal extremal, there exists a function $\lambda_1(t)$ such that $\bar{q}$ is a solution to $\frac{d}{dt}\left[\frac{\partial F}{\partial \dot{q}_k}\right] - \frac{\partial F}{\partial q_k} = 0$ **a_5**

Where function $F = L(t, \bar{q}, \dot{\bar{q}}) - \lambda_1(t) g(t, \bar{q}, \dot{\bar{q}})$, $\lambda_1(t)$ is uniquely determined such that the above criteria **a_5** holds.
We end the discussion on the theory of non-holonomic problems and follow it up with some examples. A quick example that we have is as follows.

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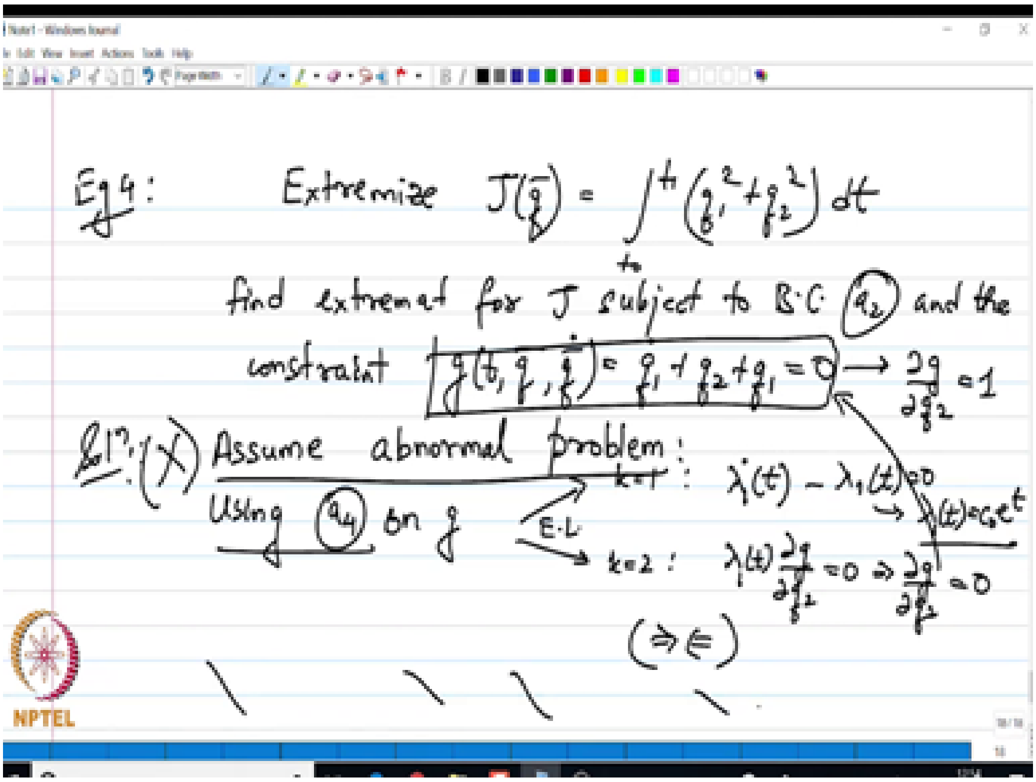

Example 4: Extremize $J(\bar{q}) = \int_{t_o}^{t_1} (q_1^2 + q_2^2)dt$, we need to find the extremal for J subject to the boundary condition $\mathbf{a_2}$, it is the same set of boundary conditions we described in the definition of non-holonomic problems and the constraint $g(t, \bar{q}, \dot{\bar{q}}) = \dot{q}_1 + q_2 + q_1 = 0$

So, this is the constraint that we imposed which is a non-holonomic constraint and we do not know whether the problem that we posed is a normal or an abnormal problem. So, to find what sort of problem is that, first we assume that the problem is abnormal. If that be the case, we will be able to find out the solution to this system, if we use our condition $\mathbf{a_4}$ that we have mentioned for abnormal problem on g.

So, we see that we are going to get two sets of equations that is we solve the Euler Lagrange equation for g with λ_1 also as the unknown, with respect to the first component for k = 1, I get $\dot{\lambda}(t) - \lambda_1(t) = 0$ and from here I get a solution that $\lambda_1(t) = C_o e^t$.

For the second case I have K = 2 and from here I get the solution $\lambda_1(t)\frac{\partial g}{\partial q_2} = 0$. since $\lambda_1 \neq 0$ which means that $\frac{\partial g}{\partial q_2} = 0$.

However notice this constraint tells us that $\frac{\partial g}{\partial q_2} = 1$. So, we arrive at a contradiction. It cannot be 0 and 1 at the same time which means that this is a wrong assumption. Assuming that the problem is abnormal is a wrong assumption, which means that the problem belongs to the normal category or we need to solve $\mathbf{a_5}$ version of the Euler Lagrange.

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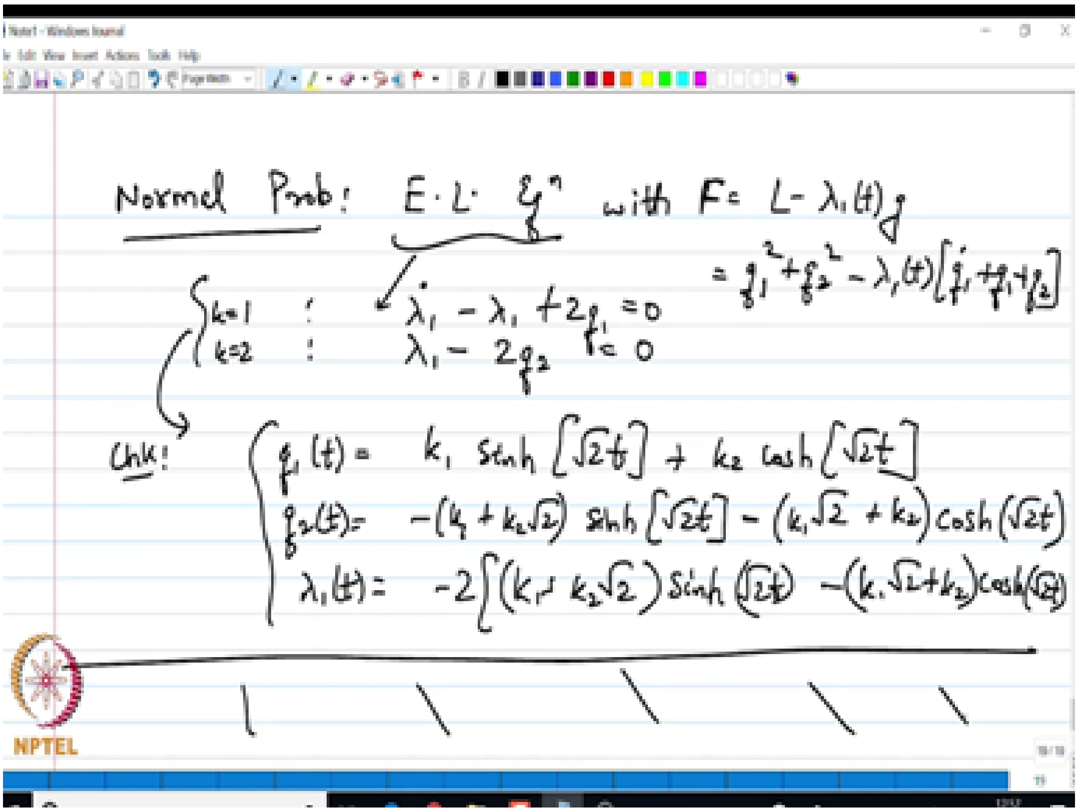

We expect that my Euler Lagrange equation will be satisfied with F $= L - \lambda_1(t)g$ $=q_1^2+q_2^2-\lambda_1(t)[\dot{q}_1+q_1+q_2]$ and we see that we get a set of two equations for K = 1 we have $\dot{\lambda}_1-\lambda_1+2q_1 = 0$ and for k = 2 we have $\lambda_1 - 2q_2 = 0$

Note that now we have two equations with three unknowns λ_1, q_1 and q_2 So, the third equation is given by the non-holonomic constraint and that is going to completely solve our system. So, I am going to directly give the solution and the students are asked to check that this is indeed the solution satisfying these two equations along with the holonomic constraints.

Solution is as follows

$$q_1(t) = k_1 \sinh[\sqrt{2}t] + k_2 \cosh[\sqrt{2}t]$$

$$q_2(t) = -(k_1 + k_2\sqrt{2}) \sinh[\sqrt{2}t] - (k_1\sqrt{2} + k_2) \cosh[\sqrt{2}t]$$

$$\lambda_1(t) = -2\left[(k_1 + k_2\sqrt{2}) \sinh[\sqrt{2}t] - (k_1\sqrt{2} + k_2) \cosh[\sqrt{2}t]\right]$$

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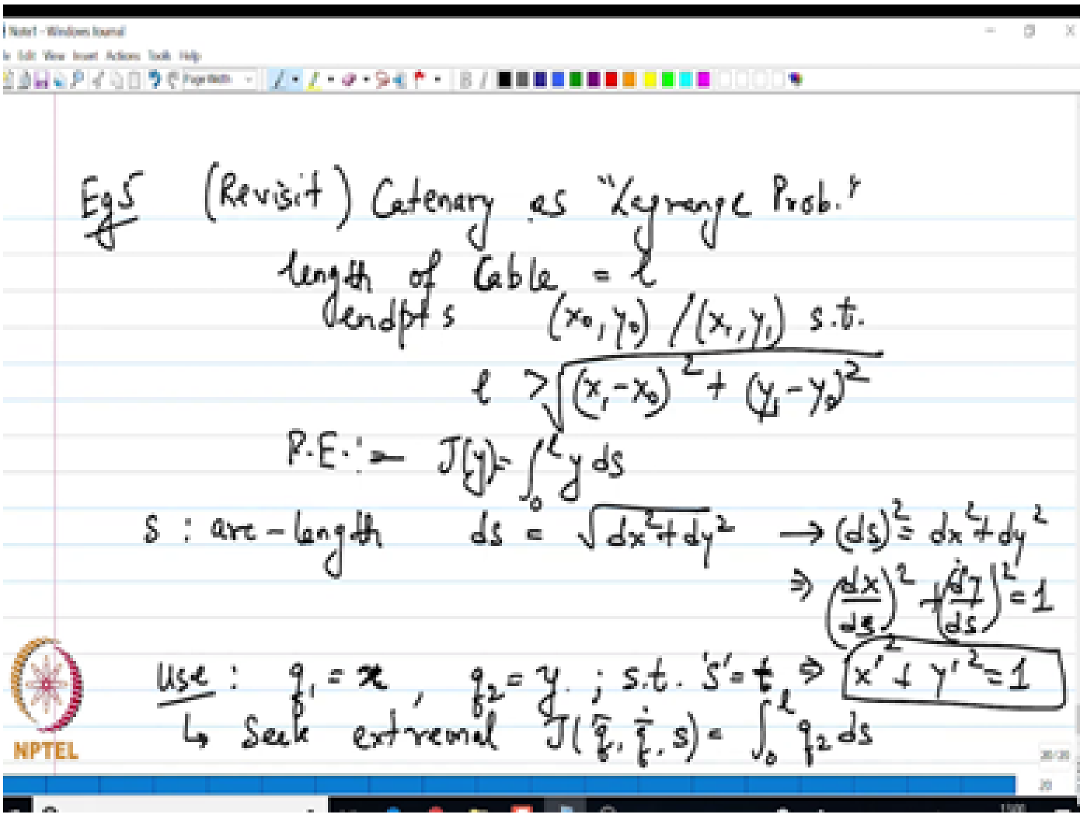

Let us finally look at one quick example before we wrap up. The example that I have on discussion is the revisitation of the catenary problem. So, we revisit catenary, and pose it as a Lagrange Problem, Suppose the length of the cable is L, I am not writing the entire statement of the problem because this has been done several times but we are posing the problem as a form a non-holonomic problem.

So, suppose the length of the cable is l and given that the end points are (x_o, y_o) and (x_1, y_1) such that given the condition which removes the rigid extremal criteria. So, $l > \sqrt{(x_1 - x_o)^2 + (y_1 - y_o)^2}$ and I have the potential energy functional given by J(y) = $\int_0^l y ds$ where s is arc length of the problem such that $ds^2 = \sqrt{dx^2 + dy^2} \Rightarrow (ds)^2 = dx^2 + dy^2 \Rightarrow \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \Rightarrow x'^2 + y'^2 = 1$
Where the primes denote the derivative with respect to s.

let us introduce new sets of variables $q_1 = x, q_2 = y$ such that arc length parameter 's' is the independent variable and we denote it by t, so we are now going to seek the extremal of this functional $J(\bar{q}, \dot{\bar{q}}, s) = \int_0^l q_2 ds$ subject to the constraint $x'^2 + y'^2 = 1$

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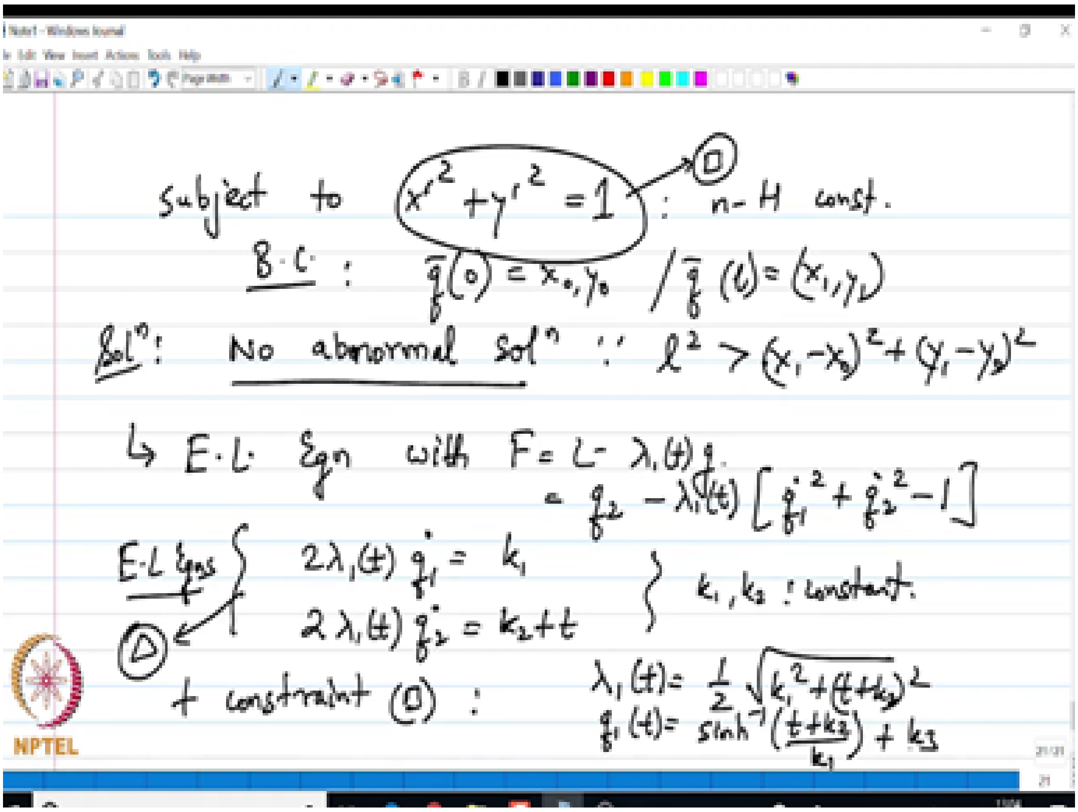

Notice that this is our non-holonomic constraint. So, we have posed the problem in the form of a non-holonomic constraint optimization problem and further we close this problem by stating the boundary conditions $\bar{q}(0) = (x_o, y_o)$ and $\bar{q}(l) = (x_1, y_1)$.

We can quickly solve for the extremal by showing that the abnormal solutions because we have already mentioned that our length is greater than the distance between the two points. So, we have already avoided the case of rigid extremals, we do not expect any abnormal solutions which means that there will be a solution to our system $\mathbf{a_5}$.

Euler Lagrange equations are to be satisfied with our function $F = L - \lambda_1(t)g = q_2 - \lambda_1(t)[\dot{q}_1^2 + \dot{q}_2^2 - 1]$ This is set equal to 0 and so we need to satisfy the Euler Lagrange equation for this function.

Let me just quickly write down the set of two Euler Lagrange equation. We get $2\lambda_1(t)\dot{q}_1 = k_1$ and $2\lambda_1(t)\dot{q}_2 = k_2 + t$ where my k_1 and k_2 are constants. I have already integrated my Euler Lagrange once to get a first order, ordinary differential equation and so, we use, we use this let me call this system as triangle.

We use this system and the constraint to get the solutions. Let me just write down the entire solution q_1, q_2 and λ, students are asked to check that these are indeed the solution, so $\lambda_1(t) = \frac{1}{2}\sqrt{k_1^2 + (t + k_2)^2}$, $q_1(t) = \sinh^{-1}\left(\frac{t+k_2}{k_1}\right) + k_3$ and $q_2(t) = \sqrt{k_1^2 + (t + k_2)^2} + k_4$.

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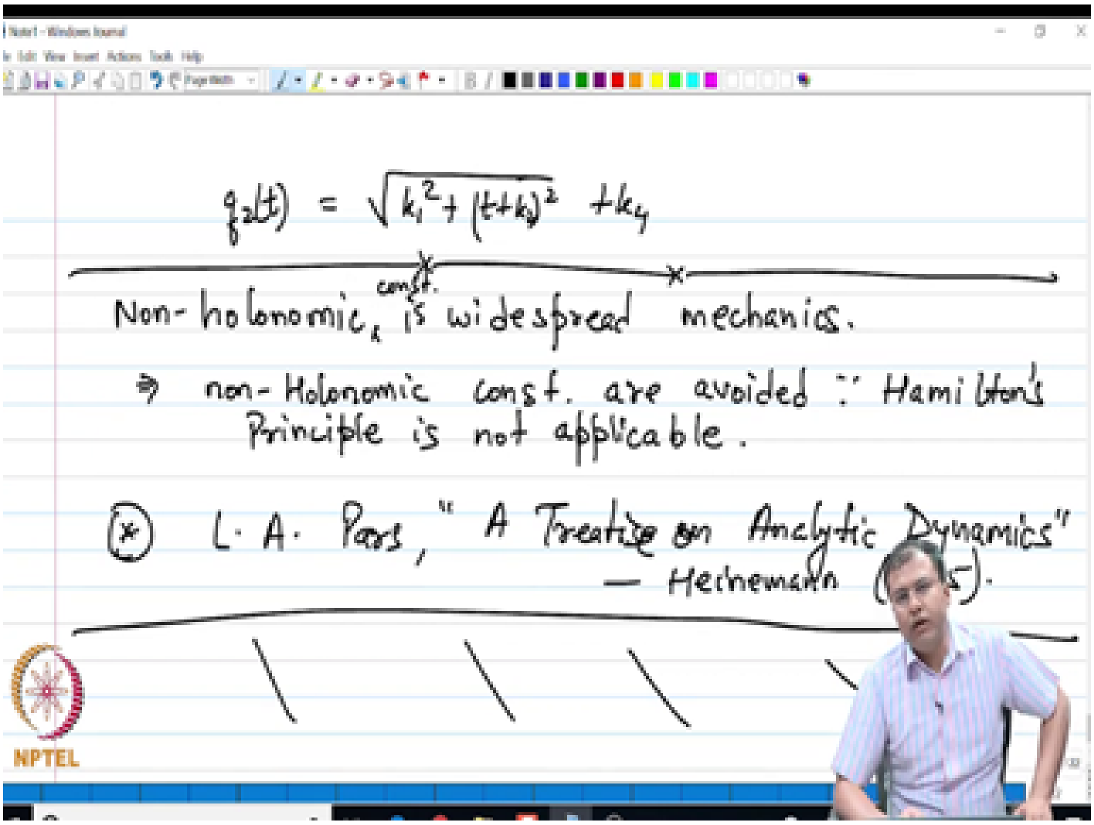

Now, notice that this solution is also of the form of cosh form and whatever problems we have done for catenary, we have shown that the solution boils down to the hyperbolic cosine function form as well. So, we end our discussion with just one particular statement by saying that the class of non-holonomic problems are widespread in mechanics.

However, the non-holonomic constraint problems are more or less avoided because in this class of problems the Hamilton's principle are not applicable, so non-holonomic constraints are avoided because in this class of problems the Hamilton's principle or the principle of least action which is widespread in this Newtonian mechanics is not applicable.

So, this I state without going into depth, these are not applicable and for students who want to understand more in depth about this statement. They are referred to this book by L.A. Pars, "A Treaties on Analytical Dynamics" and this is by Heinemann publishers. A very classical book published in 1965.