Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 25
Problems with Holonomic and non-Holonomic Constraints (Part 01)

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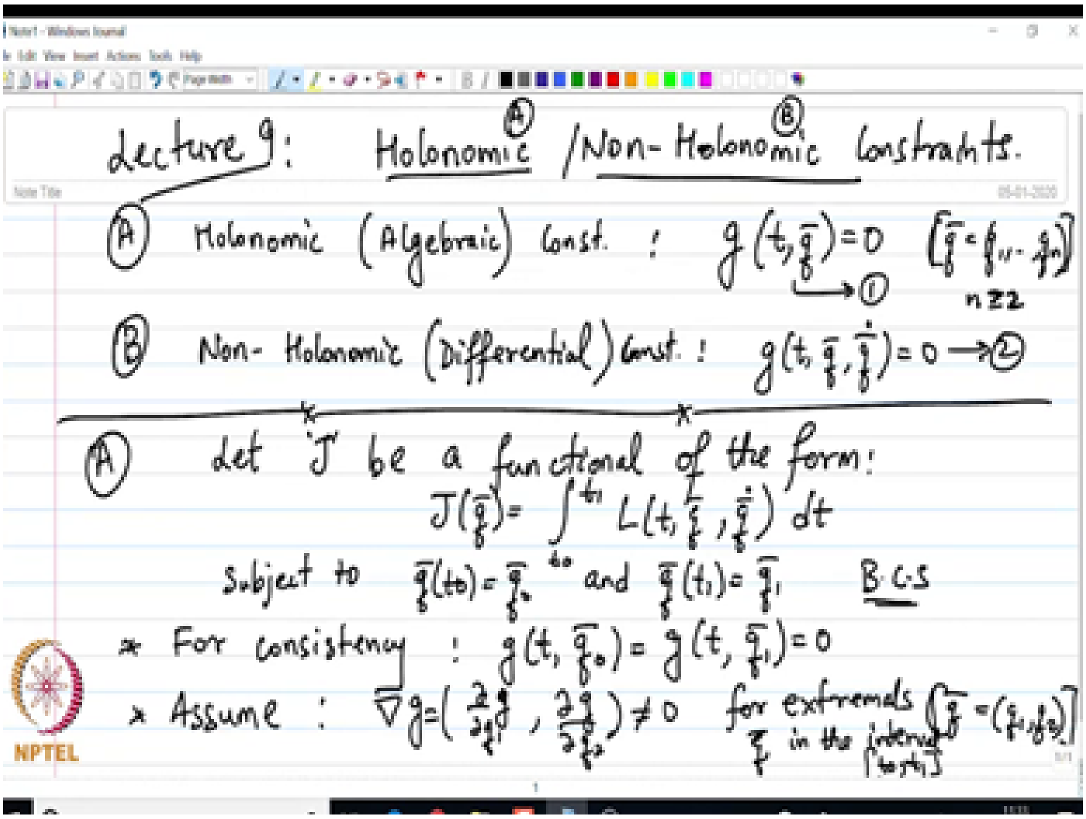

In this lecture, I am going to talk about the other set of constraints namely the holonomic constraints problems as well as a non-holonomic constraints problems. So far we have looked at functional optimization subject to the isoperimetric constraints. So in this lecture, I am going to introduce the two other different types of constraint

A Holonomic or the algebraic constraints will be of the form
$g(t, \bar{q}) = 0$, $[\bar{q} = (q_1, ..., q_n)](n > 2)$ **1**
B Non-holonomic constraints or the differential constraints will be of the form $g(t, \bar{q}, \dot{\bar{q}}) = 0$ **2**

Let us set up the problem in for case **A**: let J be a functional of the form $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ subject to the boundary conditions $\bar{q}(t_o) = \bar{q}_o$ and $\bar{q}(t_1) = \bar{q}_1$

Now, further for consistency purposes we must have these boundary conditions, the constraint also should satisfy the boundary condition, otherwise we will run into some trouble. So, we must have a prior assumption for the consistent purpose that $g(t, \bar{q}_o) = g(t, \bar{q}_1) = 0$ for the consistency purpose and also we assume we are dealing with normal problems.

We assume that $\bar{\nabla} g = \left(\frac{\partial g}{\partial q_1}, \frac{\partial g}{\partial q_2}\right) \neq 0$. Let us say in $\mathbb{R}^2$ for higher orders that there will be more components in this expression, so these conditions are for extremals $\bar{q}$in the interval t_o to t_1 .

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Intuitively it seems that this class of problems is relatively simple to solve. In fact, this is not even a new set of problems we are discussing, it seems intuitive. Why? Because notice that the constraint possibly we could use the constraint to solve one variable q. Let us say q_1 with respect to the other variable q_2 and then replace this constraint problem with the corresponding unconstrained problem of one variable.

So rather than solving a constrained optimization problem with two variables q_1 and q_2, We could solve the unconstrained problem with respect to q_1. If we could solve q_1 as a function of q_2 using the constraint, like we mentioned briefly in the case of finite dimensional calculus. But again, we will see that we run into some problems.

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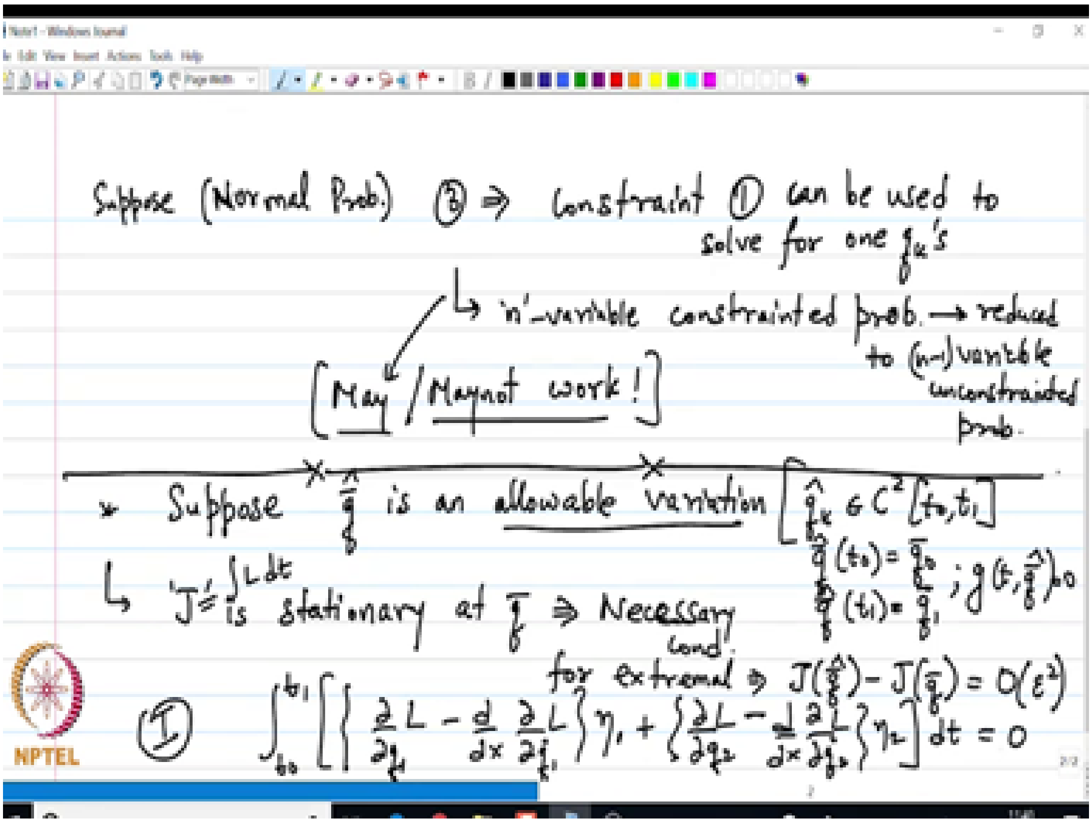

Suppose for normal problems, **3** implies that the holonomic constraint **1** can be used to solve for one q_k's, then we could possibly change a constrained problem into an unconstraint optimization problem. The standard Lagrange multiplier, which means it seems intuitively that we have n variable constraint problem to be reduced to reduce to (n-1) variable unconstrained problem.

It seems that many times it's possible but many times it not so it may it may or may not work. As we have seen a several similar scenarios in finite dimensional calculus and and in the case of finite dimensional calculus, we saw that it may or may not work.

So then what is the alternative out of it? Let us consider the following. Suppose I look at a perturbation in the extremal, suppose $\hat{\bar{q}}$ is an allowable variation, where each of the components $\hat{\bar{q}}_k \in C^2[t_o, t_1]$ and $\hat{\bar{q}}(t_o) = \bar{q}_o$, $\hat{\bar{q}}(t_1) = \bar{q}_1$

Further I have that the constraint is also satisfied, so when I say allowable variations, these are the set of conditions that $\hat{\bar{q}}$ must satisfy. So when I say that J is stationary or q is an extremal of J stationary at $\bar{q}$, then it implies that the necessary condition for extremal is $J(\hat{\bar{q}}) - J(\bar{q})$ will be 0 or at most of $O(\epsilon^2)$.

Only the higher order terms survive, the lower Order terms with respect to ϵ terms they vanish, we are guaranteed that we get necessarily extremal and we saw that , If this following integral constraints is satisfied we get

$$\int_{t_o}^{t_1} \left[\left\{ \frac{\partial L}{\partial q_1} - \frac{d}{dx}\frac{\partial L}{\partial \dot{q}_1} \right\} \eta_1 + \left\{ \frac{\partial L}{\partial q_2} - \frac{d}{dx}\frac{\partial L}{\partial \dot{q}_2} \right\} \eta_2 \right] dt = 0 \qquad \mathbf{I}$$

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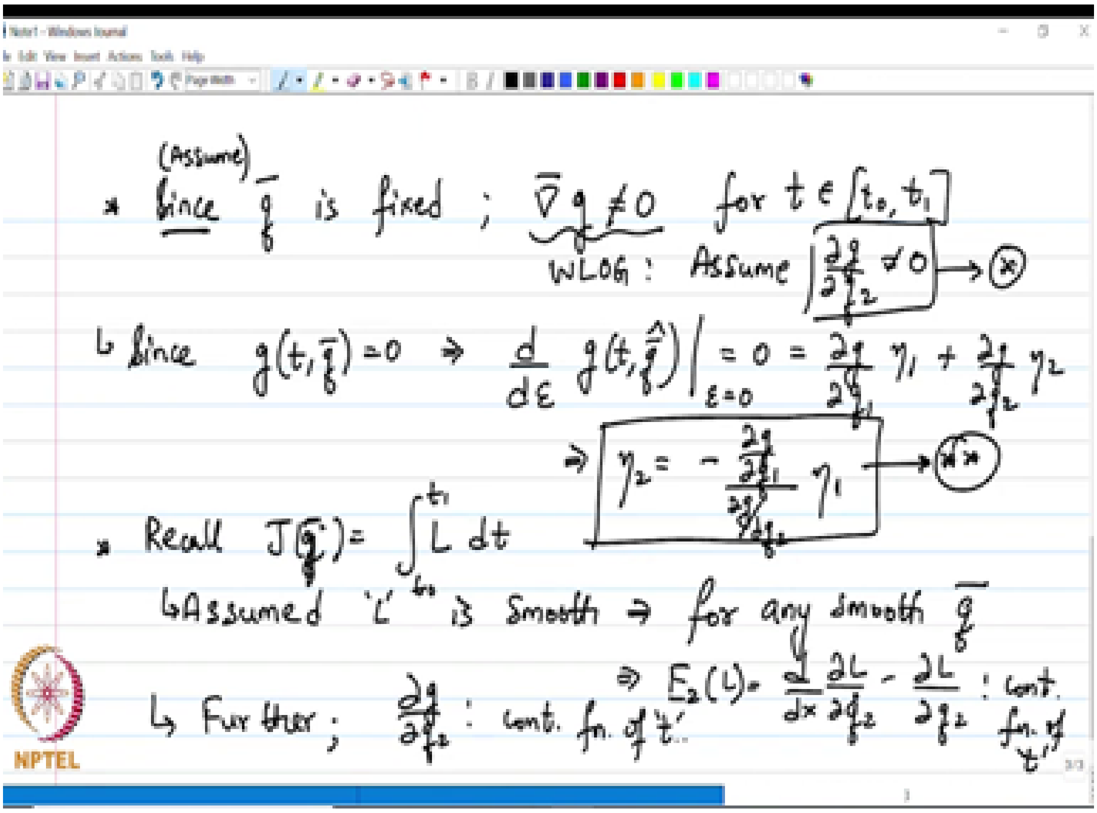

We need some few other assumptions before we move ahead since I have that $\bar{q}$ is an extremal and it is fixed and I am also assuming that $\bar{\bigtriangledown} g \neq 0$ fot $t \in [t_o, t_1]$

This means without loss of generality we can assume $\frac{\partial g}{\partial q_2} \neq 0$ $\qquad *$
We could assume the otherwise or we could also assume that both components are non-zero.

Since $g(t, \bar{q}) = 0$ the constraint is satisfied that implies $\frac{d}{dx} g(t, \hat{\bar{q}})|_{\epsilon=0} = 0 = \frac{\partial g}{\partial q_1}\eta_1 + \frac{\partial g}{\partial q_2}\eta_2$

$$\Rightarrow \eta_2 = \frac{-\frac{\partial g}{\partial q_1}}{\frac{\partial g}{\partial q_2}}\eta_1 \qquad **$$

Let me state another fact, Recall $J(\bar{q}) = \int_{t_o}^{t_1} Ldt$, here we have assumed that L is smooth function, which implies that for any smooth extremal $\bar{q}$ implies that $E_2(L) = \frac{d}{dx}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2}$, this is also continuous pf parameter t

Because if L is smooth and it has derivatives up to second order I must have that this operator acting on L must also produce a smooth function of t, we know that $\frac{\partial g}{\partial \partial q_2}$ is a continuous function of 't', which means that we can express one continuous function in the form of the other continuous function. As both are continuous, so the ratio of the two quantities function will also be a continuous function.

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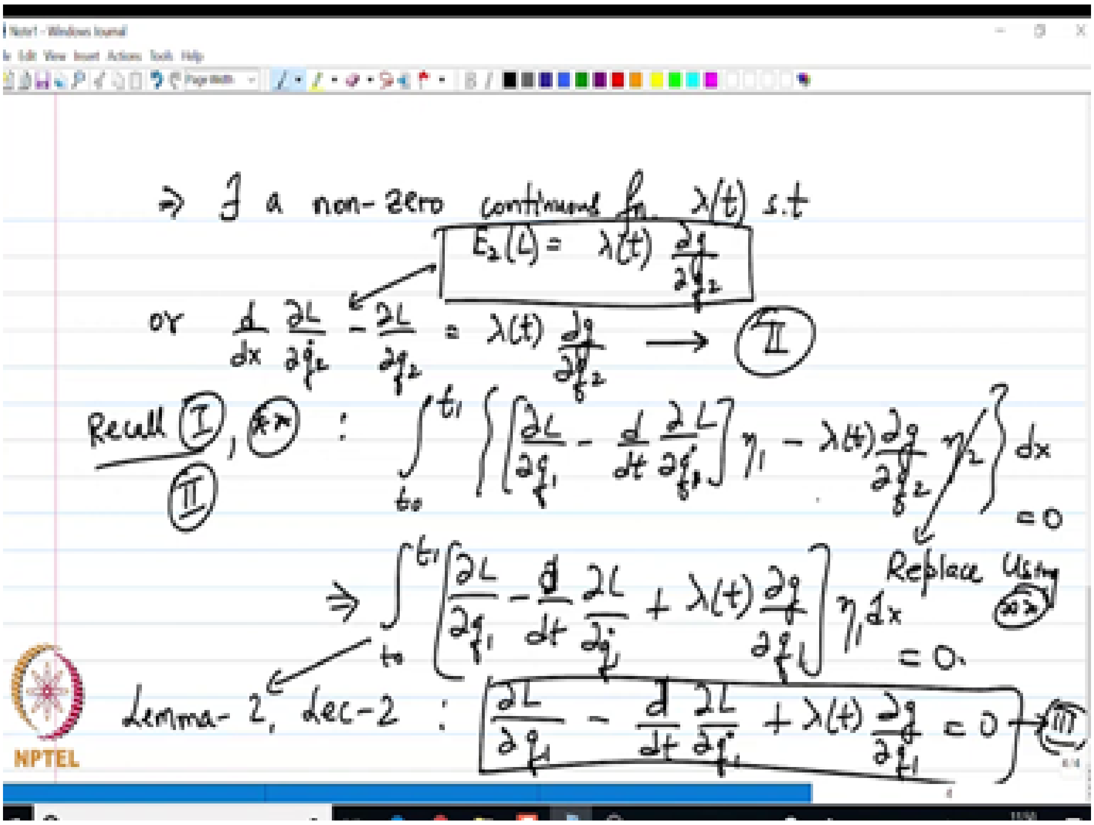

Since both of them are continuous functions that means there exists a non-zero continuous function $\lambda(t)$ such that $E_2(L) = \lambda(t) \frac{\partial g}{\partial q_2}$ 0r $\frac{d}{dx} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} = \lambda(t) \frac{\partial g}{\partial q_2}$ **II**

If we recall Result **I**, **II**, and ** we see that

$$\int_{t_o}^{t_1} \left\{ \left[\frac{\partial L}{\partial q_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \right] \eta_1 - \lambda(t) \frac{\partial g}{\partial q_2} \eta_2 \right\} dx = 0$$

So this is η_2 is replaced by the relation between η_2 and η_1 to come at this particular integral constraint and again using a version of Lemma 2 discussed in lecture 2 I come to the point where I have the Euler Lagrange equation for L with the holonomic constraints g for the component q_1.

$$\Rightarrow \int_{t_o}^{t_1} \left[\frac{\partial L}{\partial q_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} + \lambda(t) \frac{\partial g}{\partial q_1} \right] \eta_1 dx = 0$$

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} + \lambda(t) \frac{\partial g}{\partial q_1} = 0 \qquad \textbf{III}$$

Equations **II** and **III** are identical but with component q_2 replaced with q_1.

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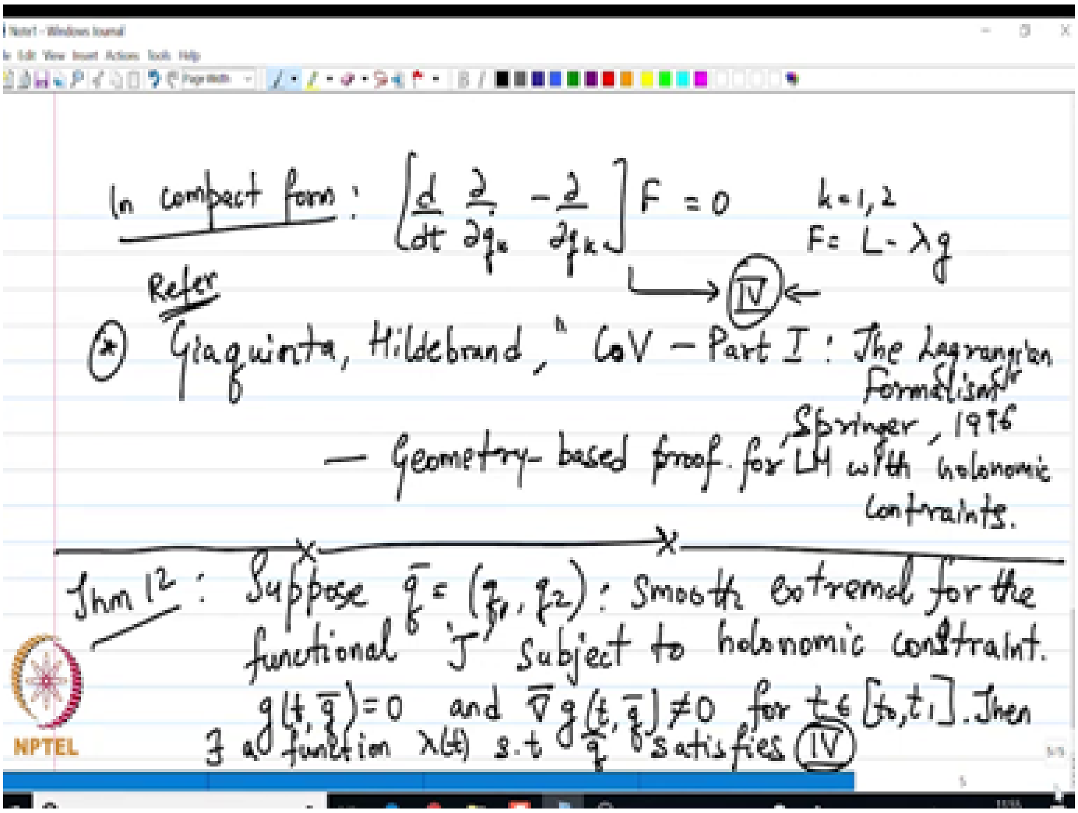

We are ready to describe Euler Lagrange necessary condition in this situation. In compact form our necessary condition is $\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}\right] F = 0, \quad k = 1, 2$ **IV**
and $F = L - \lambda g$, where L is the Integrand of the functional and G is the holonomic constraints.

Another reference to all the students where a proper proof based on geometry is provided. So the students are asked to also refer this particular reference to look at a more detailed geometry based proof for this situation of functional optimization with holonomic constraints, the author is Giaquinta and Hildebrand.

Students are requested to refer to the books named as calculus of variations part 1 the Lagrangian formulism and published by Springer in 1996, this is a useful reference for geometry based proof of these class of problems that I have just shown for the Lagrange multiplier method for with holonomic constraints.

So I am going to end by discussion on holonomic constraints by summarizing my entire discussion in the form of a theorem and also providing some example to highlight how this Euler Lagrange necessary condition is utilized.

Theorem 12: Suppose $\bar{q} = (q_1, q_2)$ which is a smooth extremal for the functional J subject to the holonomic constraints $g(t, \bar{q}) = 0$ and $\bar{\nabla} g(t, \bar{q}) \neq 0$ for $t \in [t_o, t_1]$, then $\exists$ a real valued function $\lambda(t)$ such that $\bar{q}$ satisfies our relation **IV** which is the Euler Lagrange necessary condition.

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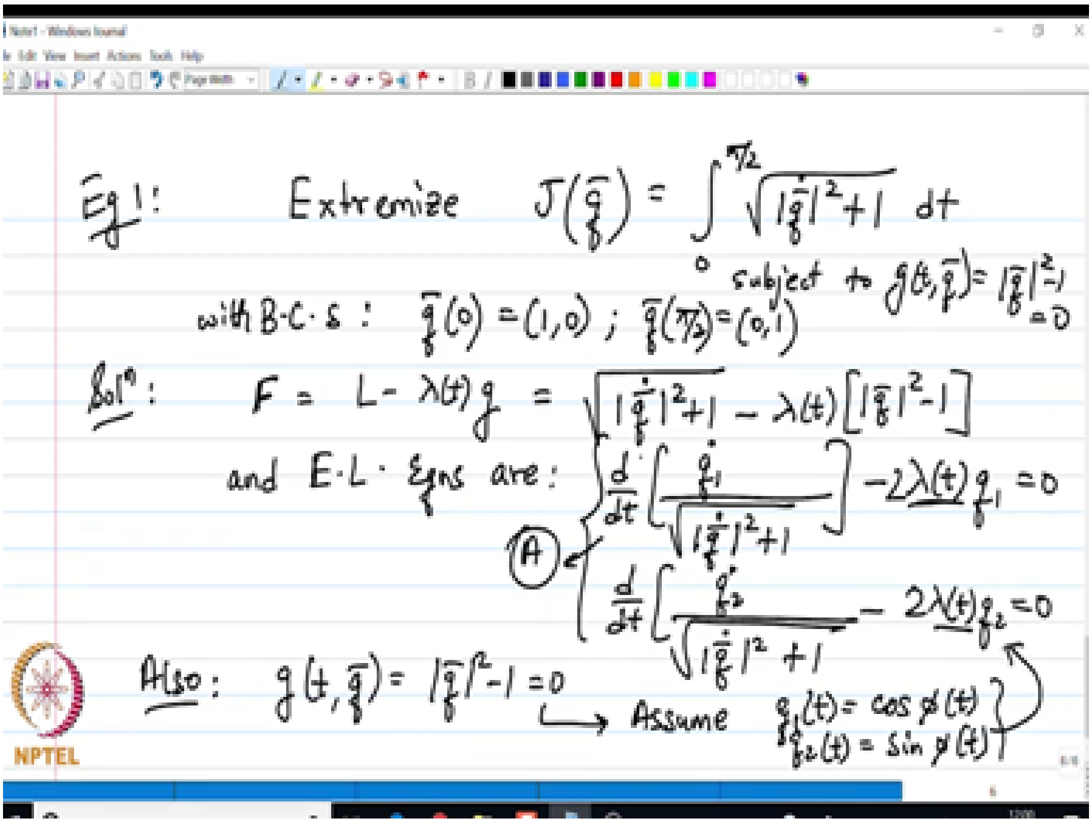

Example 1: Extremize $J(\bar{q}) = \int_0^{\frac{\pi}{2}} \sqrt{|\dot{\bar{q}}|^2 + 1} dt$ subject to $g(t, \bar{q}) = |\bar{q}|^2 - 1 = 0$, this is a case of the constrained optimization subject to holonomic constraint with boundary conditions $\bar{q}(0) = (1, 0)$: $\bar{q}(\frac{\pi}{2}) = (0, 1)$

Solution: $F = L - \lambda(t)g = \sqrt{|\dot{\bar{q}}|^2 + 1} - \lambda(t)[|\bar{q}|^2 - 1]$ and Euler Lagrange equations are

$$\left\{ \frac{d}{dt}\left[\frac{\dot{q}_1}{\sqrt{|\dot{\bar{q}}|^2+1}}\right] - 2\lambda(t)q_1 = 0, \frac{d}{dt}\left[\frac{\dot{q}_2}{\sqrt{|\dot{\bar{q}}|^2+1}}\right] - 2\lambda(t)q_2 = 0 \right. \qquad \mathbf{A}$$

Now we have to solve this system of equation, Also we have the constraint $g(t, \bar{q}) = |\bar{q}|^2 - 1 = 0$, Assume that $q_1(t) = \cos\phi(t)$ and $q_2(t) = \sin\phi(t)$ are lying on a unit circle.

So instead of two variables q_1 and q_2 now we have just one variable ϕ to solve and we have the second functional variable $\lambda(t)$ So we have two unknowns ϕ and λ and we have two equations which are given by **A**.

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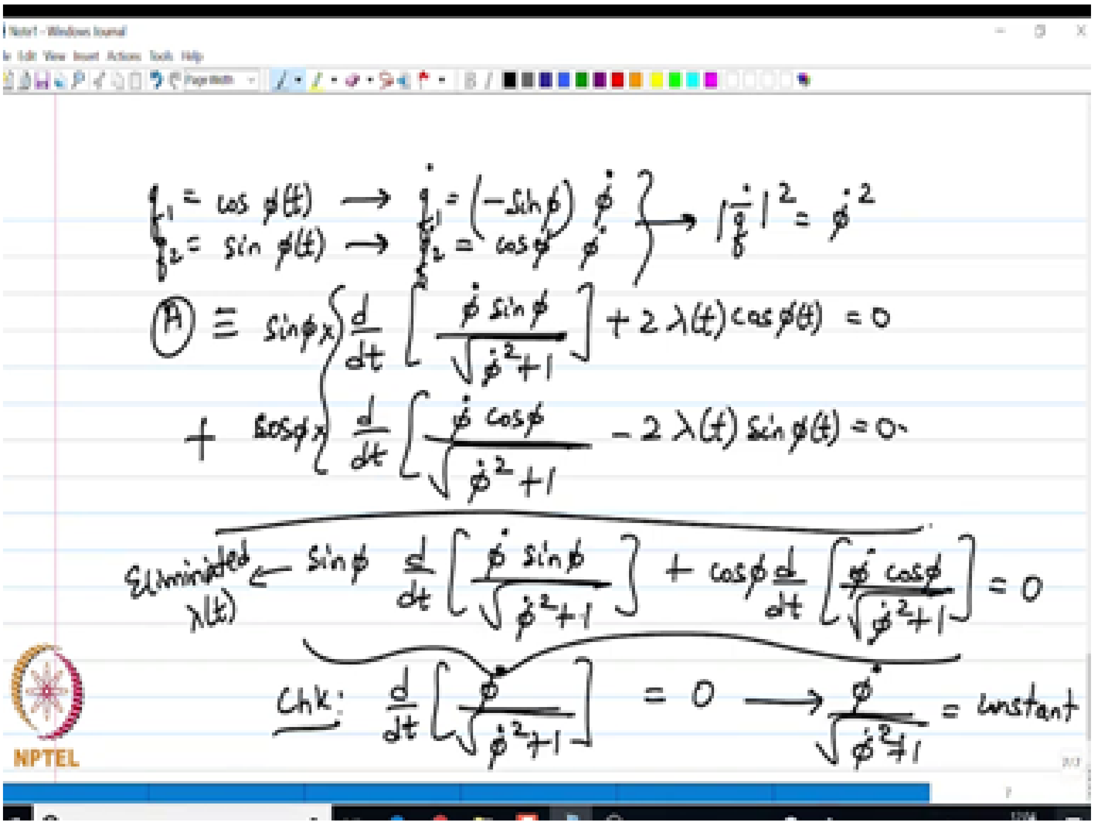

$q_1(t) = \cos\phi(t) \quad \Rightarrow \dot{q}_1 = (-\sin\phi)\dot{\phi}$
and $q_2(t) = \sin\phi(t) \quad \Rightarrow \dot{q}_2 = (\cos\phi)\dot{\phi}$

After squaring and adding the above two equation we get $|\dot{\vec{q}}|^2 = \dot{\phi}^2$ and then from **A**, we have

$$\left\{\frac{d}{dt}\left[\frac{\dot{\phi}\sin\phi}{\sqrt{\dot{\phi}^2+1}}\right] + 2\lambda(t)\cos\phi(t) = 0,\ \frac{d}{dt}\left[\frac{\dot{\phi}\cos\phi}{\sqrt{\dot{\phi}^2+1}}\right] - 2\lambda(t)\sin\phi(t) = 0\right.$$

$$\Rightarrow \quad \sin\phi\left[\frac{d}{dt}\left[\frac{\dot{\phi}\sin\phi}{\sqrt{\dot{\phi}^2+1}}\right] + 2\lambda(t)\cos\phi(t)\right] + \cos\phi\left[\frac{d}{dt}\left[\frac{\dot{\phi}\cos\phi}{\sqrt{\dot{\phi}^2+1}}\right] - 2\lambda(t)\sin\phi(t)\right] = 0$$

$$\Rightarrow \sin\phi\frac{d}{dt}\left[\frac{\dot{\phi}\sin\phi}{\sqrt{\dot{\phi}^2+1}}\right] + \cos\phi\left[\frac{\dot{\phi}\cos\phi}{\sqrt{\dot{\phi}^2+1}}\right] = 0$$

The students can check $\frac{d}{dt}\left[\frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+1}}\right] = 0 \Rightarrow \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2+1}} = \text{Constant}$

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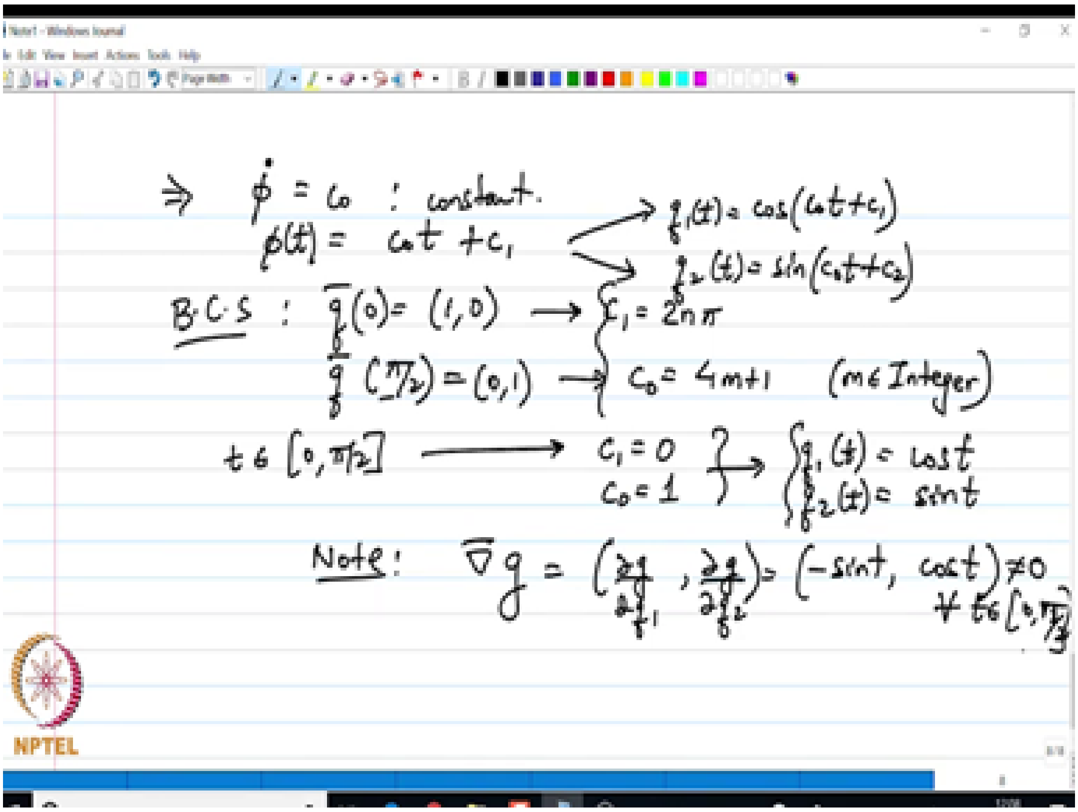

$\Rightarrow \dot{\phi} = C_o$: Constant $\Rightarrow \phi(t) = C_o t + C_1 \Rightarrow q_1(t) = \cos{(C_o t + C_1)}$ and $q_2(t) = \sin{(C_o t + C_2)}$, now all we have to eliminate C_o and C_1 but we have sets of we have two boundary conditions $\bar{q}(0) = (1, 0) \Rightarrow C_1 = 2n\pi$ and $\bar{q}(\frac{\pi}{2}) = (0, 1) \Rightarrow C_o = 4m + 1 \;\; (m \in Integer)$.

Also we know that $t \in [0, \frac{\pi}{2}]$, which tells us that it is only possible when you $C_1 = 0$ and $C_o = 1$ and in that case $q_1(t) = \cos t$ and $q_2(t) = \sin t \;\forall\; t \in [0, \frac{\pi}{2}]$

So those are my parameter representation of the function and that completes the discussion that this extremal lies on the rim of a unit circle. Note, when we did not even check whether the problem was normal or abnormal, but note that if we were to calculate the $\bar{\bigtriangledown} g = \left(\frac{\partial g}{\partial q_1}, \frac{\partial g}{\partial q_2}\right) = (-\sin t, \cos t) \neq 0 \;\forall\; t \in [0, \frac{\pi}{2}]$, which means that we are working with a normal problem in this case.