Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 24
Isoperimetric Problems 6

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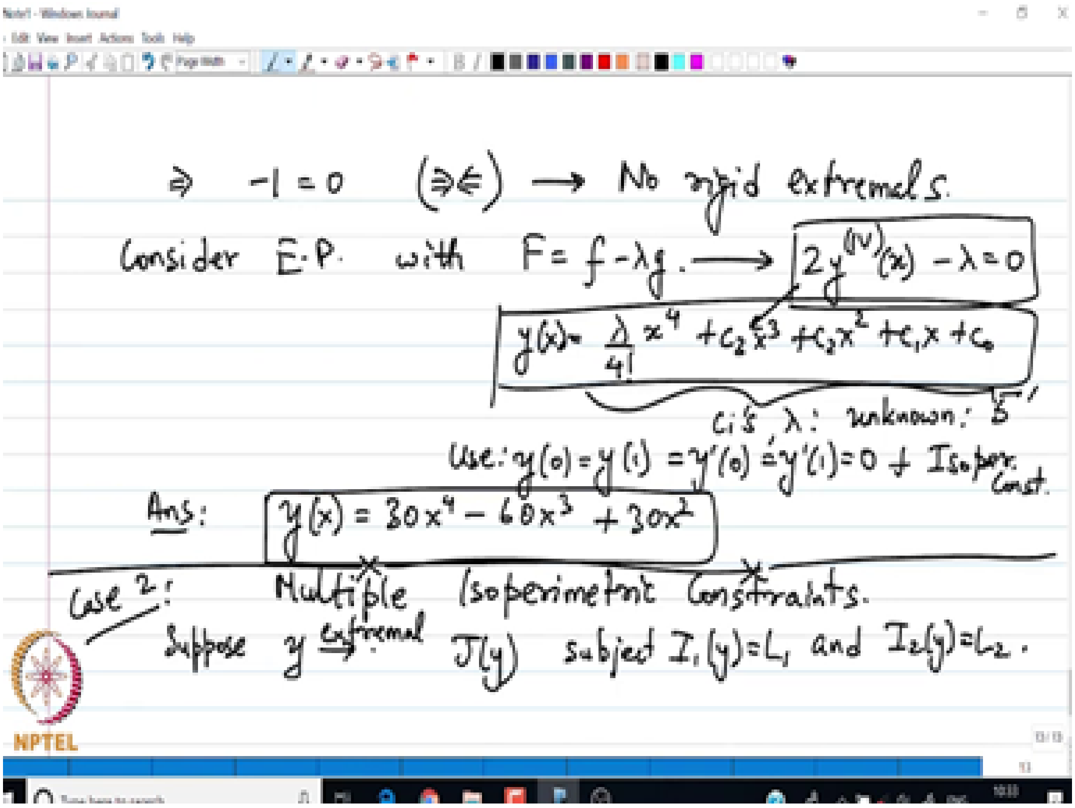

So then we can look at another generalization of the problem and the second case of the generalized isoperimetric problem is when we have multiple isoperimetric constraints, then what to do in this case? Let us set up the problem, Suppose I have 'y' being the extremal J(y) subject to $I_1(y) = L_1$ and $I_2(y) = L_2$, so we have two isoperimetric constraints

let us go back to our discussion on the standard derivation of Euler LaGrange. We were perturbing in a standard derivation of Euler Lagrange. We were trying to find the first variation $\delta J(y)$ and then we looked at subject to one constraint that is I(y)= L.

Now, given the fact that we have two constraints we cannot perturb freely. We have to perturb in such a way. We have to perturb our extremal in such a way so that the extremals always satisfy both the constraints simultaneously, right? So what I just said is the following.

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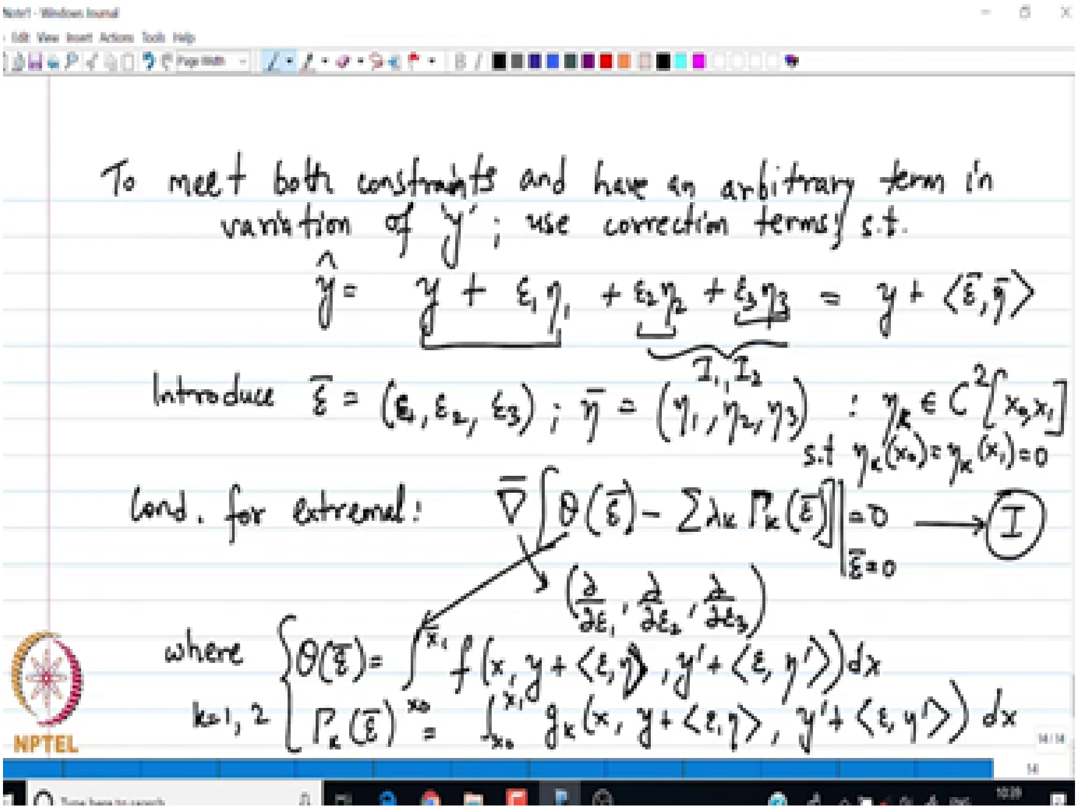

To meet both constraints and have an arbitrary term in variation of y, we use the correction terms such that $\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2 + \epsilon_3\eta_3 = y + \langle \epsilon, \bar{\eta} \rangle$

let me introduce everything in the vector notation $\bar{\epsilon}(\epsilon_1, \epsilon_2, \epsilon_3)$ and $\bar{\epsilon}(\epsilon_1, \epsilon_2, \epsilon_3)$; $\bar{\eta}(\eta_1, \eta_2, \eta_3)$ where $\eta_k \in C^2[x_o, x_1]$ such that $\eta_k(x_o) = \eta_k(x_1) = 0$.

So the perturbations are such that they vanish on the boundary that is needed for all sets of perturbation, again approaching in a similar way that we approached earlier for the Lagrange for isoperimetric constraints with one constraint, we set up the condition for extremal

$$\bar{\bigtriangledown} [\theta(\bar{\epsilon}) - \sum \lambda_k \Gamma_k(\bar{\epsilon})]_{\bar{\epsilon}=0} = 0 \qquad \mathbf{I}$$

These are the respective derivatives with respect to $\epsilon_1, \epsilon_2, \epsilon_3$

Where $\theta(\bar{\epsilon}) = \int_{x_o}^{x_1} f(x, y + \langle \epsilon, \eta \rangle, y' + < \epsilon, \eta' >)dx$ and
$\Gamma_k(\bar{\epsilon}) = \int_{x_o}^{x_1} g_k(x, y + \langle \epsilon, \eta \rangle, y' + < \epsilon, \eta' >)dx, \quad k = 1, 2$

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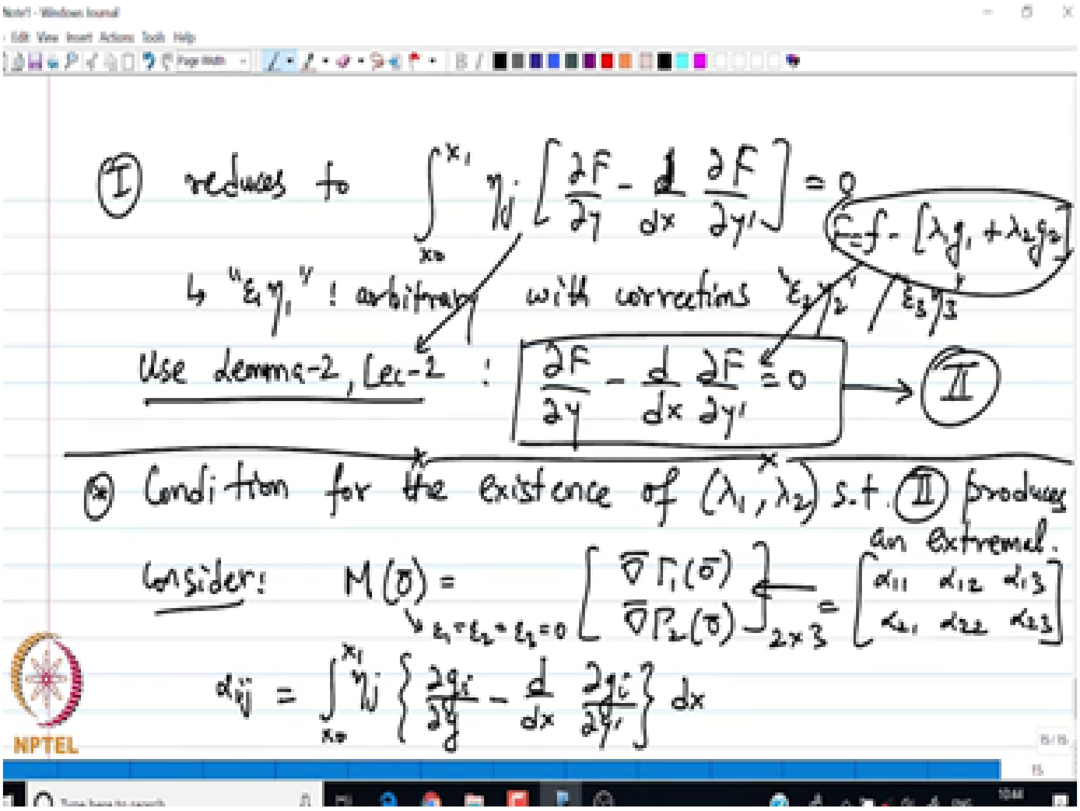

We see that **I** eventually reduces to the condition $\int_{x_o}^{x_1} \eta_j \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right] = 0$ with F = F - $[\lambda_1 g_1 + \lambda_2 g_2]$. Notice that the term "$\epsilon_1 \eta_1$" is arbitrary with corrections $\epsilon_2 \eta_2$ and $\epsilon_3 \eta_3$.

The corrections are due to the respective isoperimetric constant, which means the condition can now be reduced if we use if you use Lemma 2 in a lecture 2 we can reduce this integral constraint into the differential constraint with the condition $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$ **II**
which is the necessary condition for the extremal.

that concludes the necessary derivation of the necessary condition, except we need to remark a little bit about the existence of the Lagrange multiplier (λ_1, λ_2) whether under what conditions that these constants (λ_1, λ_2) exists.

We are discussing is the condition for the Existence, does this (λ_1, λ_2) all really exists or not? Such that the condition to produces an extremal, so what is the condition let us recall the case for single isoperimetric constraint problem.

We checked the rank of the Jacobian matrix and the rank of the augmented Matrix and it turned out that the rank in the isoperimetric problem with single constraint and the rank of the augmented Matrix has an upper bound, which is the rank of the Jacobian matrix and similarly it is the same situation in this case as well.

Consider Jacobian matrix at 0 which is $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ i.e M(0) = $\begin{bmatrix} \bar{\bigtriangledown}\Gamma_1(\bar{\theta}) \\ \bar{\bigtriangledown}\Gamma_2(\bar{\theta}) \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{bmatrix}$

Where $\alpha_{ij} = \int_{x_o}^{x_1} \eta_j \left\{\frac{\partial g_i}{\partial y} - \frac{d}{dx}\frac{\partial g_i}{\partial y'}\right\} dx$ and my augmented Matrix will be an additional row replace the gradient of θ.

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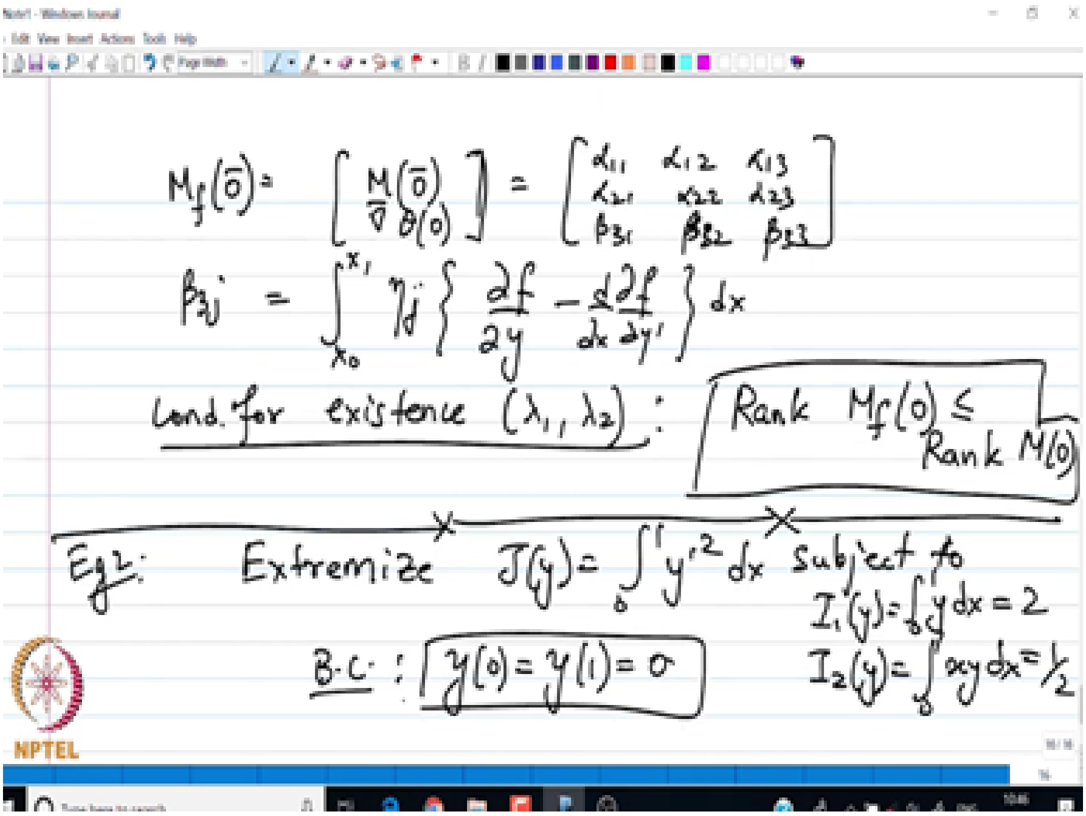

$$M_f(\bar{0}) = \begin{bmatrix} M(\bar{0}) \\ \bar{\bigtriangledown}\theta_1(0) \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}$$

Where $\beta_{ij} = \int_{x_o}^{x_1} \eta_j \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right\} dx$
So the condition for the existence finally we are ready now to state the condition for existence of(λ_1, λ_2) is that the rank of the augmented Matrix must be less than or equal to the rank of the Jacobian Matrix, when that happens we are guaranteed the existence of LaGrange multiplier leading to the extremal y.

We are going to end the discussion on this topic by looking at few Examples: eExtremize $J(y) = \int_0^1 {y'}^2 dx$ subject to the constraint $I_1(y) = \int_0^1 y dx = 2$ and $I_2(y) = \int_0^1 xy dx = \frac{1}{2}$, we have two boundary conditions y(0) = y(1) = 0 , So we need to extremize J subject to the two constraints I_1 and I_2.

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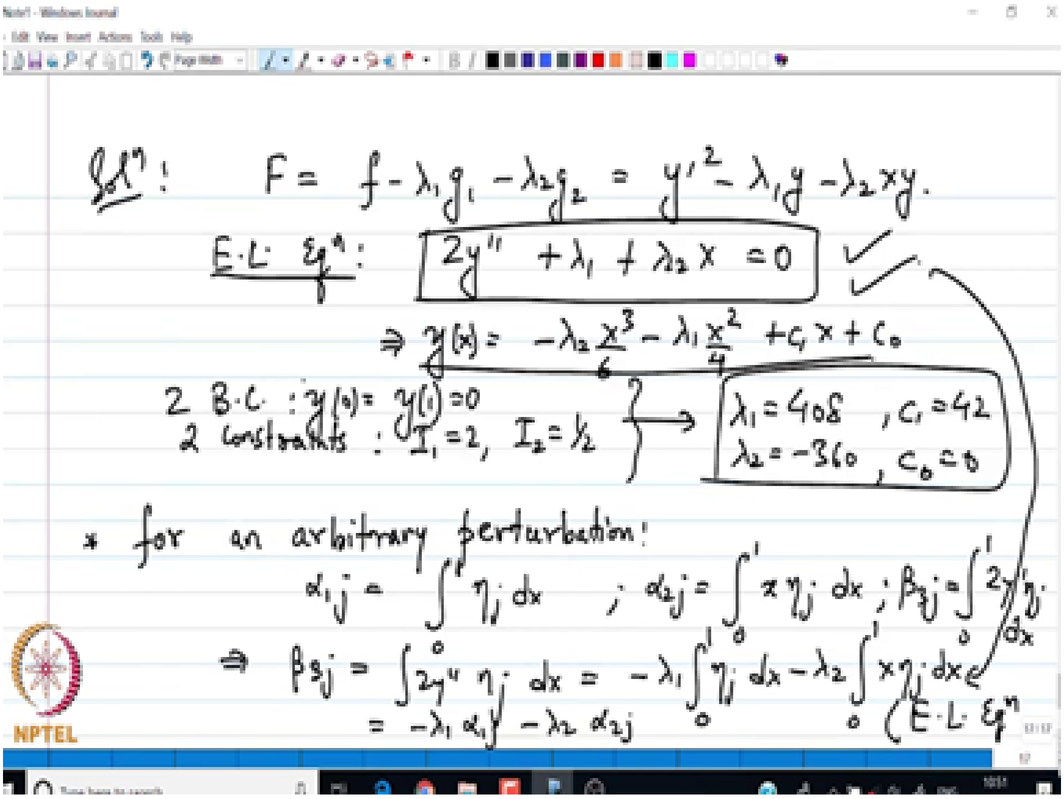

Solution: We directly look at the Euler LaGrange equation here, we have our extended function $F = f - \lambda_1 g_1 - \lambda_2 g_2 = {y'}^2 - \lambda_1 y - \lambda_2 xy$

If I use my Euler Lagrange equation, I am going to directly write down the ODE that we get $2y'' + \lambda_1 + \lambda_2 x = 0$

$\Rightarrow y(x) = -\lambda_2 \frac{x^3}{6} - \lambda_1 \frac{x^2}{4} + C_1 x + C_o$, so we have four constraints $\lambda_1, \lambda_2, C_1$ and C_o. Note that we have two boundary conditions y(0) = y(1) = 0 and we have two constraints $I_1 = 2$ and $I_2 = \frac{1}{2}$.

From these four conditions I can readily find these four constants of integration and as well as the Lagrange constants, students can directly check that we get the answer $\lambda_1 = 408, \lambda_2 = -360, C_1 = 42$ and $C_o = 0$, that completes the solution to this problem with where extremal is given by this underlined equation and before we end the discussion on this Example, let us also briefly look at the existence when we already shown that the values λ_1, λ_2 but whether these values are unique or not or we could possibly get another extremal for a different value of λ_1, λ_2

We just do not know unless and until we check the rank of the Jacobian and the argument Matrix, so let us look at those quantities.

For an arbitrary perturbation, we see that $\alpha_{1j} = \int_0^1 \eta_j dx$; $\alpha_{2j} = \int_0^1 x\eta_j dx$; $\beta_{3j} = \int_0^1 2y' \eta_j dx = -\lambda_1 \int_0^1 \eta_j dx - \lambda_2 \int_0^1 x\eta_j dx = -\lambda_1 \alpha_{1j} - \lambda_2 \alpha_{2j}$ (directly from our Euler Lagrange equation), which means that my third row is linearly dependent or the rank of the conclusion is Rank $M_f(\bar{0}) \leq$ Rank $M(\bar{0})$, that concludes the discussion on this example

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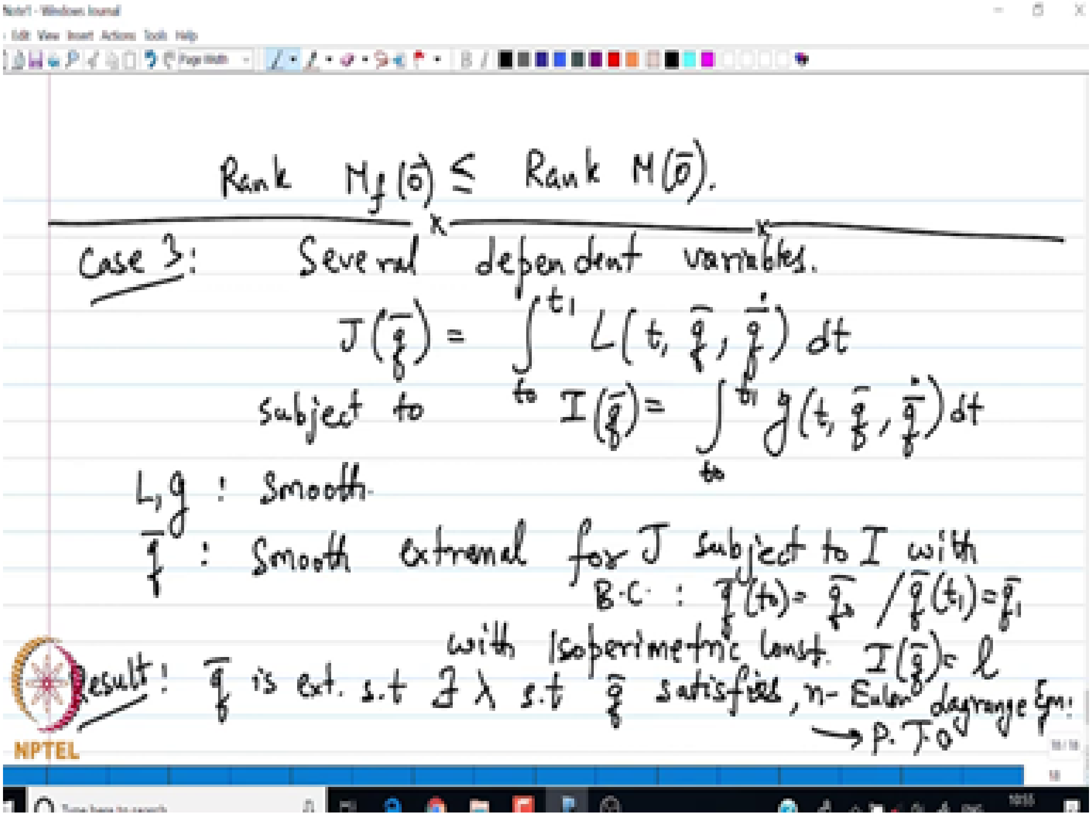

let us now finally look at Case 3 of the generalization of the isoperimetric problem, that is the case of several dependent variables, we are talking about functional of the form $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ subject to the isoperimetric constraint of the form $I(\bar{q}) = \int_{t_o}^{t_1} g(t, \bar{q}, \dot{\bar{q}})dt$

we have that L and g smooth, they have derivatives up to continuous derivatives up to second order and let us say that $\bar{q}$ is a smooth extremal for J subject to I with my boundary condition $\bar{q}(t_o) = \bar{q_o}$ and $\bar{q}(t_1) = \bar{q_1}$, so we have sets of vector set of two boundary conditions each of them are vector conditions and with my isoperimetric constraint of the form I($\bar{q}$) = l

Result I am going to directly state the result because the proof follows the same similar case for the problem with one variable that $\bar{q}$ is an extremal such that there exists a constant λ so that $\bar{q}$ satisfies the equation the n-Euler Lagrange equation of the form $\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}\right] F = 0, \quad k = 1., ...n$, i.e we have n constraints and function $F = L - \lambda g$ let us quickly look at an example to this problem.
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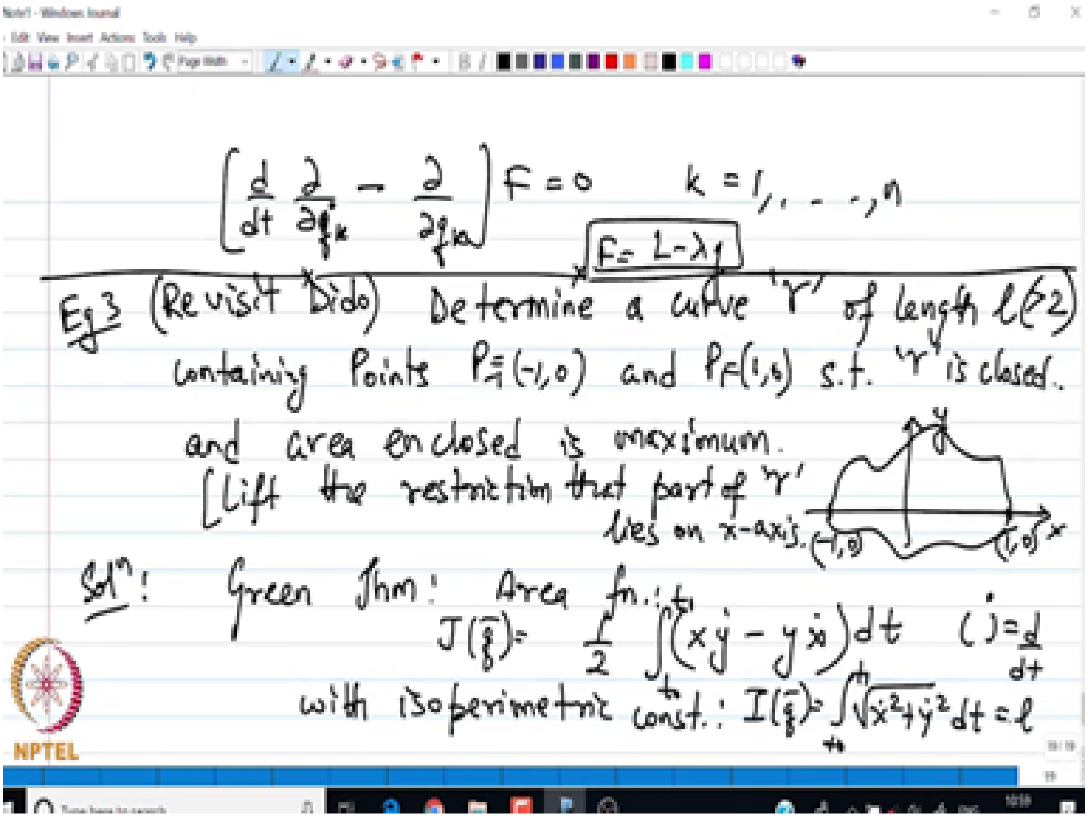

Example 3: (Revisit Dido problem) Determine the curve γ length $L(>2)$ so that I remove the case of rigid extremals containing the points $P_{-1} = (-1, 0)$ and the points $P_1 = (1, 0)$ such that γ is closed and the line segment from P_1 and P_2 is closed.

We do not say anything beyond that and the area enclosed is maximum. So essentially what we have done is, in the earlier version of Dido's problem we were trying to stipulate the condition that a part of the curve lies on the x axis. So now we have lifted we have lifted that condition so we have lift the restriction.

Restriction that part of γ lies on the x-axis. Well, certainly it passes through (-1, 0) and (1, 0) but now the other half can very well be below the Y-axis, which means that we can have a much more generalized problem in this case.

So, in this case now the area functional is going to be described by Green's theorem because this is a much more general scenario.
Green's theorem says that area functional is given by $J(\bar{q}) = \frac{1}{2}\int_{t_o}^{t_1}(x\dot{y} - y\dot{x})dt$ and $\dot{()} = \frac{d}{dt}$, withe isoperimetric constraints $I(\bar{q}) = \int_{t_o}^{t_1}\sqrt{\dot{x}^2 + \dot{y}^2}dt = l$

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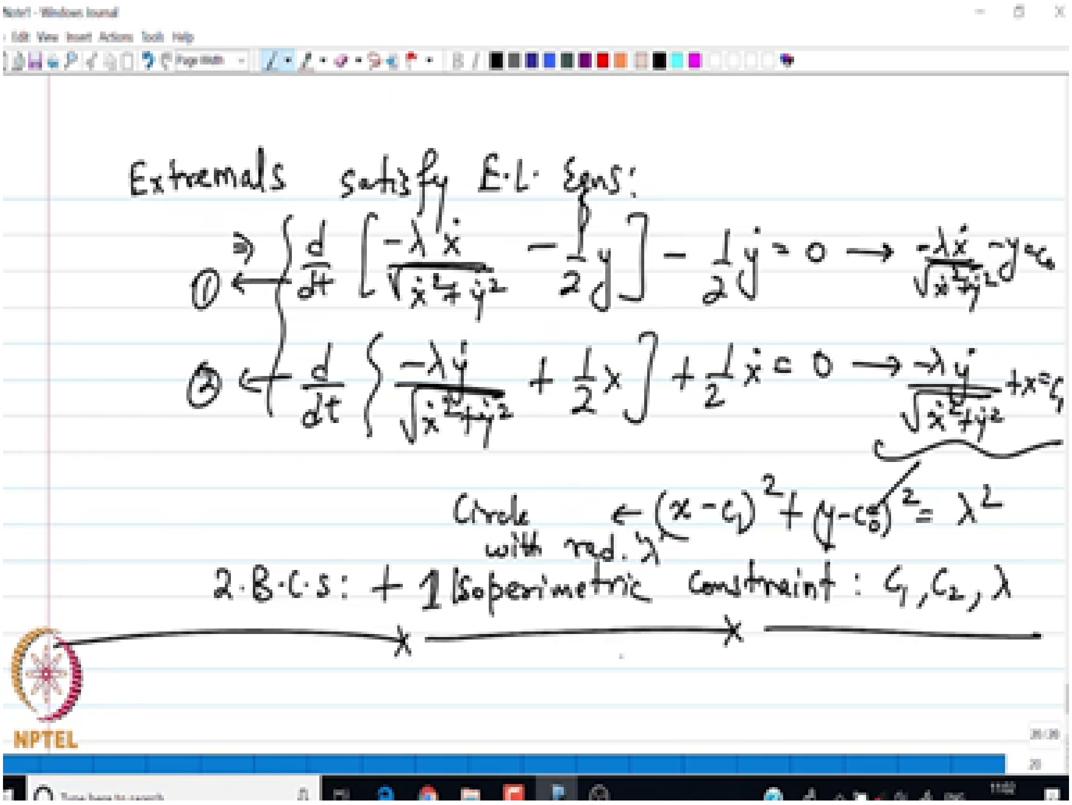

Now I have a case of constraint optimization with two dependent variables and we directly set the system of two Euler LaGrange equations;

$$\frac{d}{dt}\left[\frac{-\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}-\frac{1}{2}y\right]-\frac{1}{2}\dot{y}=0 \Rightarrow \quad \frac{-\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}-y=C_O \qquad \mathbf{1}$$

$$\frac{d}{dt}\left[\frac{-\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}+\frac{1}{2}x\right]+\frac{1}{2}\dot{x}=0 \Rightarrow \quad \frac{-\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}+x=C_1 \qquad \mathbf{2}$$

I can directly square and add from these two equations and see that extremal will follow this particular curve $(x-C_1)^2+(y-C_o)^2=\lambda^2$

So x and y are such that they lie on a circle with radius λ and we have three this equation has three unknown C_0, C_1 and λ but we also have two boundary conditions plus we have one isoperimetric constraint and that will fully determine the system.

So I end my lecture I end by discussion at this point and in the next lecture, I am going to talk about the situation where we deal with constraints of the form which are algebraic or holonomic constraints as well as non-algebraic a differential or non-holonomic constraints.
Thank you very much.