Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 23

Isoperimetric Problems 5

Now move on to another important discussion as to what exactly is the role of this Lagrange Multiplier.

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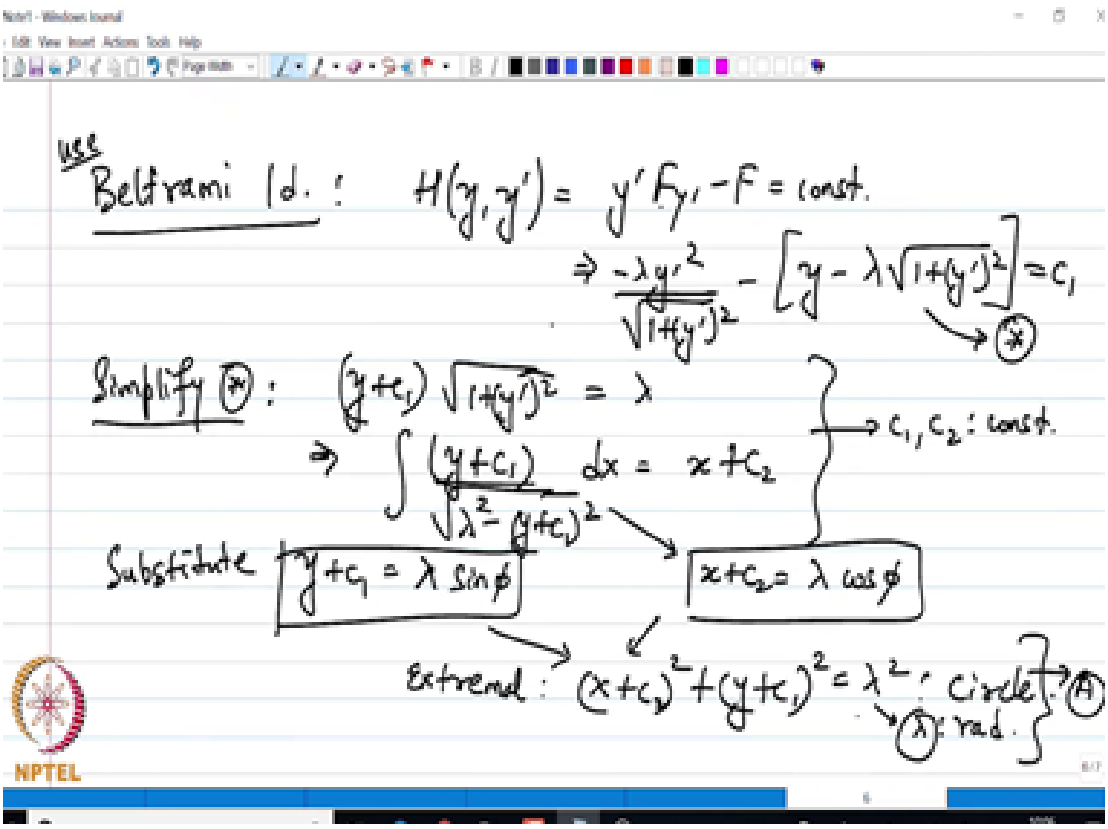

So far we have not told anything about this constant λ, not in this example but in general. So, what is the role of λ? let me briefly discuss the role of λ.

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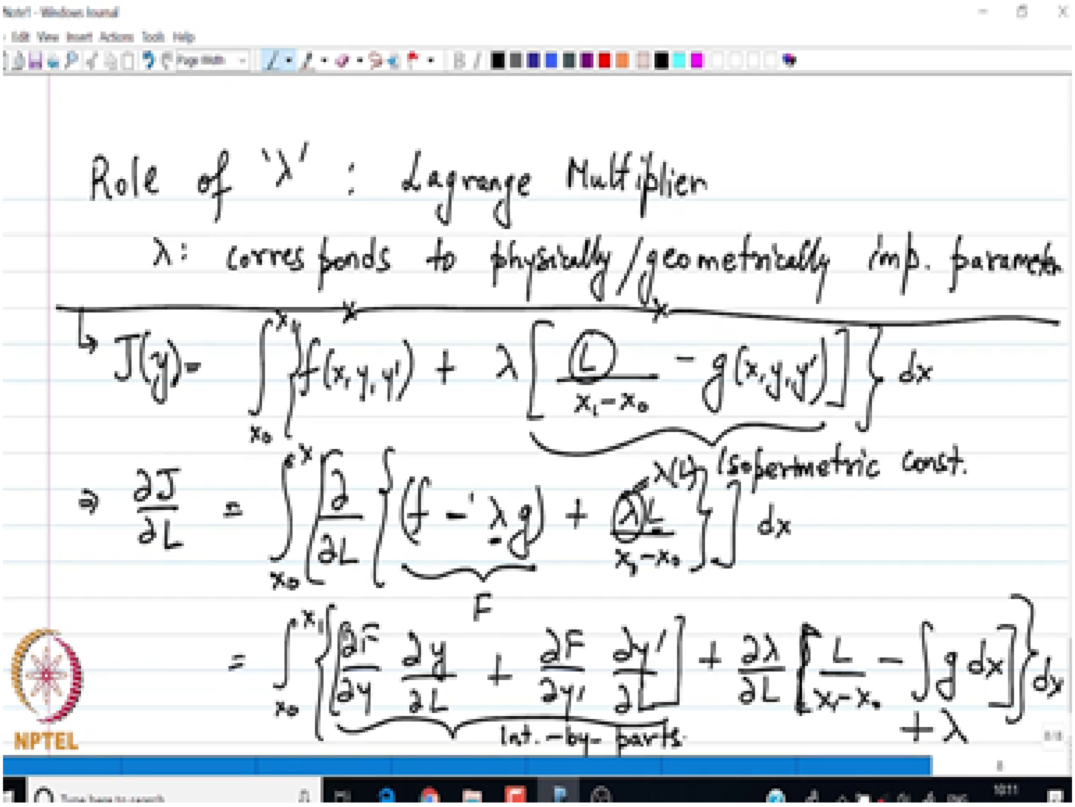

The role of λ which is our Lagrange multiplier in isoperimetric constraints, notice that in the previous example λ was the radius. So, λ definitely corresponds to some physically or geometrically important parameter in the problem, we see that λ corresponds to a physically or a geometrically important parameter.

Now, we are going to exactly highlight what exactly is the importance of this Lagrange Multiplier λ because noting down the significance of λ will help us to look at the problem from another point of view. let us look at the functional J(y) which is now I am including the isoperimetric constraints within this functional.

$$\text{Consider the functional } \ J(y) = \int_{x_o}^{x_1} \left\{ f(x, y, y^{'}) + \lambda \left[\frac{L}{x_1 - x_o} - g(x, y, y^{'}) \right] \right\} dx$$

Note that what we have done is this is my Isoperimetric constraint that is now included in my functional.

$$\Rightarrow \frac{\partial J}{\partial L} = \int_{x_o}^{x_1} \left[\frac{\partial}{\partial L} \left\{ (f - \lambda g) + \frac{\lambda L}{x_1 - x_o} \right\} \right] dx$$

$$= \int_{x_o}^{x_1} \left[\left\{ \frac{\partial F}{\partial y} \frac{\partial y}{\partial L} + \frac{\partial F}{\partial y'} \frac{\partial y^{'}}{\partial L} \right\} + \frac{\partial \lambda}{\partial L} \left\{ \frac{L}{x_1 - x_o} - \int g dx \right\} \right] dx + \lambda$$

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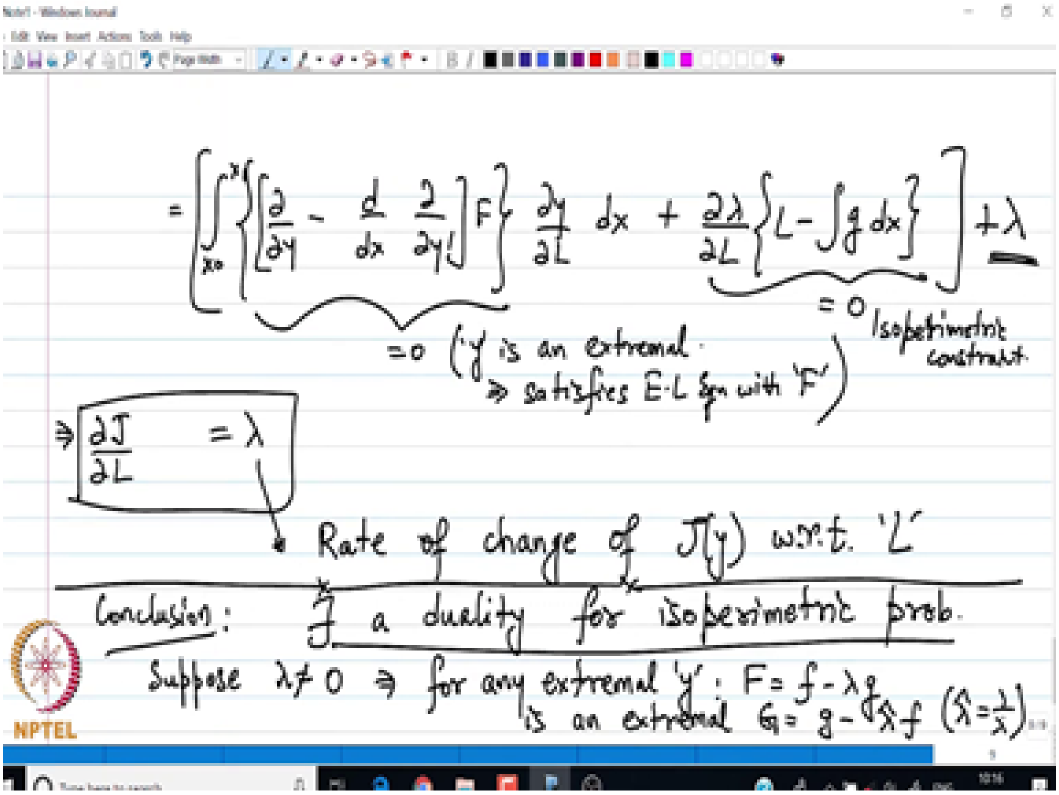

$$= \left[\int_{x_o}^{x_1} \left\{ \left[\frac{\partial}{\partial y} - \frac{d}{dx}\frac{\partial}{\partial y'}\right] F \right\} \frac{\partial y}{\partial L} dx + \frac{\partial \lambda}{\partial L}\left\{L - \int g dx\right\}\right] + \lambda = 0$$

It is equals to zero because y is an extremal and it satisfies Euler Lagrange equation with my integrand F and this is also equal to 0 because of the Isoperimetric constraint, finally what we get is that everything inside the integral is 0 because of the Euler Lagrange equation and the Isoperimetric constraint and the only quantity that survives is the quantity outside the integral which is equal to λ.

$\Rightarrow \frac{\partial J}{\partial L} = \lambda$, λ is the rate of change of the functional J(y) with respect to the parameter L
there is another way to look at this result, it shows a duality of the isoperimetric problem so which means what we are trying to show is that there exists a duality for isoperimetric problem and note that what this means is that we are trying to minimize J subject to I as the constraint is also equivalent to saying that we are trying to maximize I subject to J as the constraint, We can flip the role of I and J for isoperimetric problem

Suppose that λ Lagrange Multiplier is non zero, then for any extremal y, I have $F = f - \lambda g$ is also an extremal for g which is $G = g - \hat{\lambda} f$ where $\hat{\lambda} = \frac{1}{\lambda}$, which means that the same Euler Lagrange equation holds for the extremal y with the function F or with the function g, with the roles of λ reversed instead of λ now we have $\frac{1}{\lambda}$ for the other situation. Now, how is this possible?

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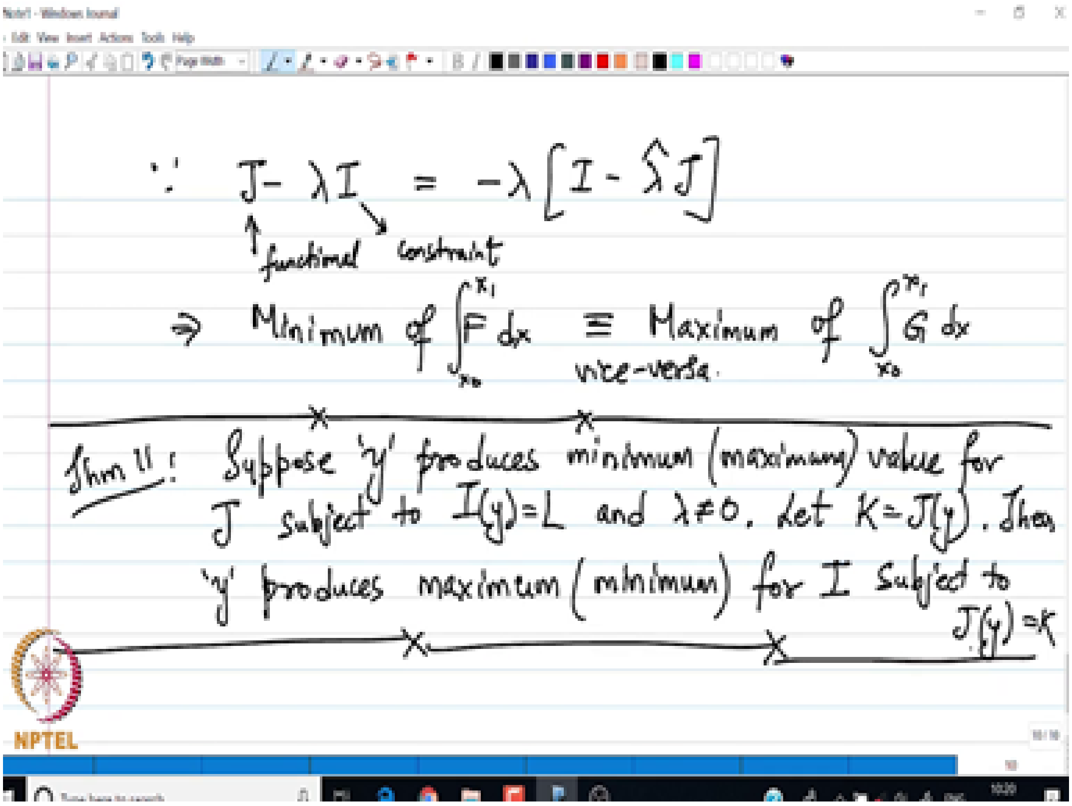

It is quite simple to see that because note that $J - \lambda I = -\lambda[I - \hat{\lambda} J]$, as if you were to extremize the constraint on the left hand side, it is saying that we are extremizing the constraint on the right hand side with the roles of I and J reversed which means that the minimum of $\int_{x_o}^{x_1} F dx$ is equal to the maximum of $\int_{x_o}^{x_1} G dx$ and this can be vice versa. So, what have we got here? let me just state this duality in the form of a theorem

Theorem 11: Suppose y produces a minimum (respectively a maximum value) for the functional J subject to I(y) = L and $\lambda \neq 0$
let K = J(y), then y produces for maximum(minimum) for I subject to the constraint J(y) = k subject to the constraint J and the functional I.

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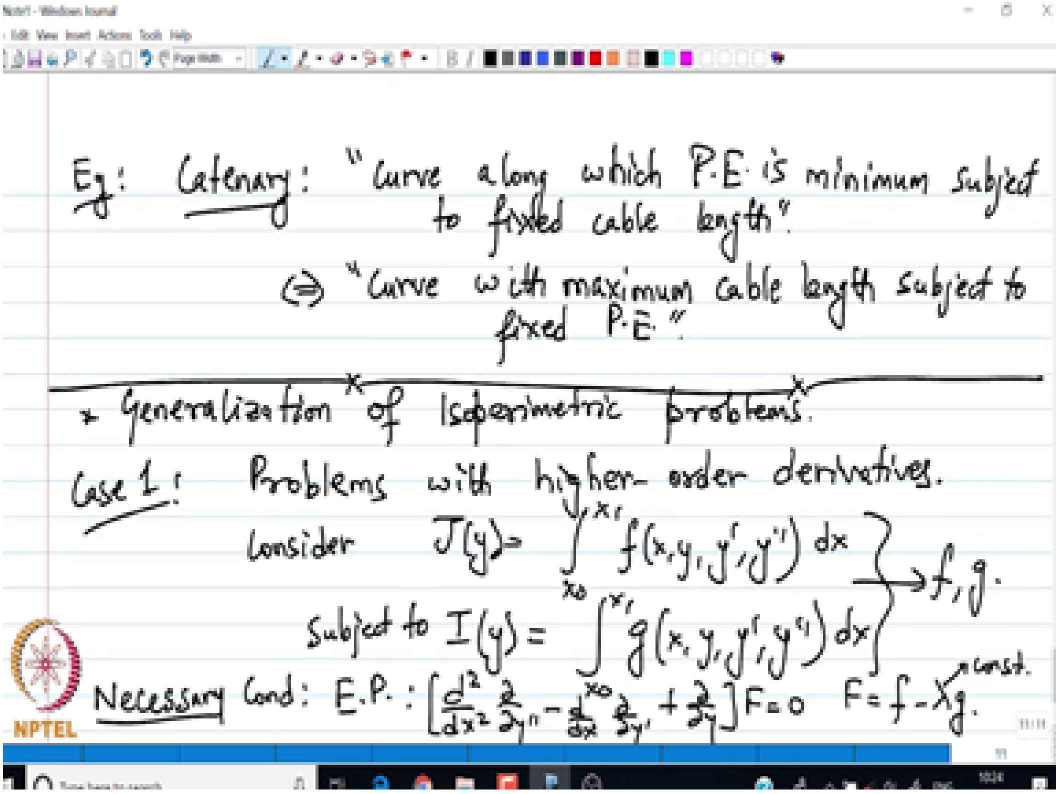

let us again revisit the problem of catenary very quickly, in the problem of catenary we were trying to minimize the potential energy subject to the length of the cable being fixed, there is another way to state the catenary problem, we are trying to maximize the length of the rope subject to the fixed potential energy and that is how the roles are reversed.

So, in the catenary problem we were saying the catenary is a curve along which the potential energy is minimum subject to fixed cable length and this is also equivalent to saying this is the curve with maximum cable length subject to fixed potential energy, so that concludes the discussion on the role of the Lagrange Multiplier.

We can now proceed with the generalization of the isoperimetric problems. let us look at the different cases, we could generalize in several different ways. The first case could be for functional with integrant having higher order derivatives.

Consider $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'}, y^{''})dx$ and we have the isoperimetric constraint subject to the constraint integral $I(y) = \int_{x_o}^{x_1} g(x, y, y^{'}, y^{''})dx$, here we assume that f and g are smooth having all the necessary derivatives required for the necessary condition or the Euler Lagrange or Euler Poisson equation.

The necessary condition in this case, I am not going to go into the derivation but the necessary condition in this case is the Euler Poisson equation which tells us that the extremal need to satisfy the following equation $\left[\frac{d^2}{dx^2}\frac{\partial}{\partial y''} - \frac{d}{dx}\frac{\partial}{\partial y'}\right] F = 0$ where $F = f - \lambda g$, where $\lambda$is a constant.

So, we have given the generalized version of the isoperimetric problem for this case of generalization. So, the generalized version of the Euler Lagrange equation for this case.

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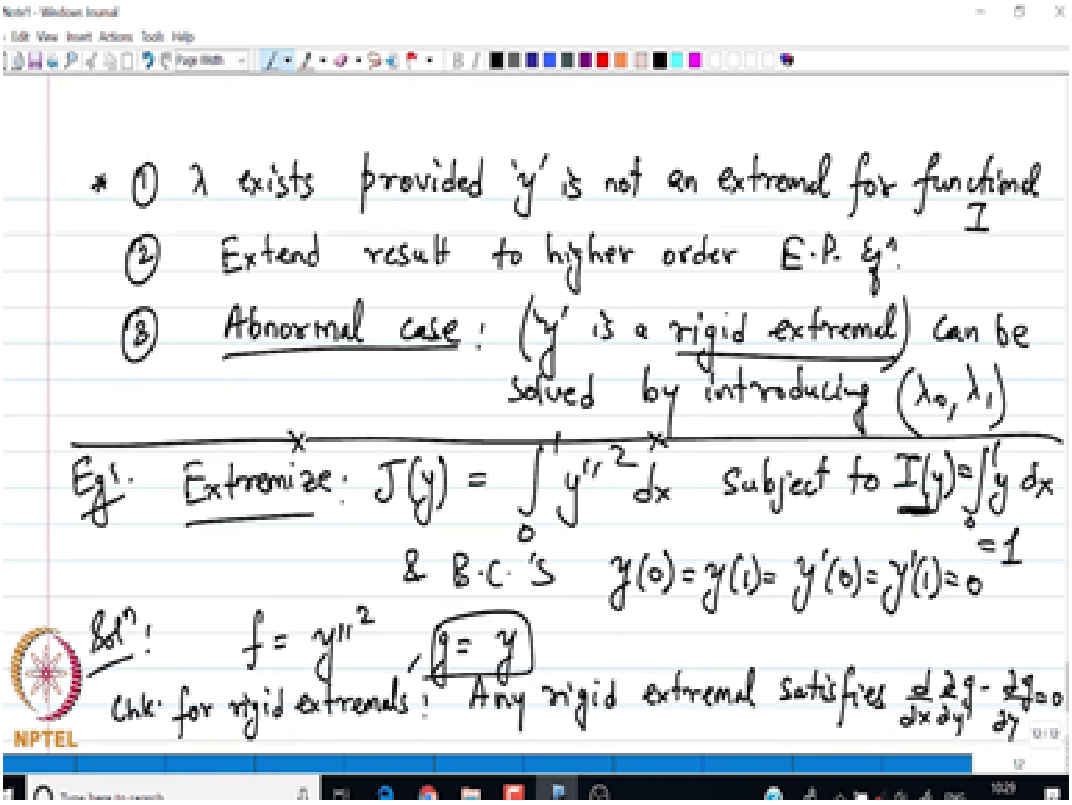

There are a few more points that I want to highlight before we look at an example,
1) λ or Lagrange Multiplier exists provided 'y 'is not a rigid extremal for the functional I.
2) We can extend the result to higher-order Euler Poisson equation that is third derivative, fourth derivative and so on so forth.
3) For the abnormal case scenario we need to introduce two Lagrange Multipliers λ_o and λ_1, where that includes the case where y is a rigid extremal that is not included in discussion 1, so in this case y is a rigid extremal, when I say rigid extremal it means that y is an extremal for the isoperimetric constraint I(y).

Example: Extremize $J(y) = \int_0^1 {y''}^2 dx$ subject to $I(y) = \int_0^1 y dx = 1$
we have the boundary conditions y(0) = y(1) = $y^{'}$(0) = $y^{'}$(1) = 0 Note that this is the extremal to this set up is going to be the solution to the Euler Poisson equation subject to these four sets of boundary conditions .

Solution: $f = {y''}^2$ and g = y , we solve for the extremal, we just do not know whether we have an abnormal problem or not or whether y is a rigid extremal or not. so we want to eliminate the case of rigid extremal. let us check for rigid extremal, suppose, so if I have any rigid extremal then y will satisfy the Euler Lagrange equation or y will be the extremal to the isoperimetric constraint I(y).

Any rigid extremal will satisfy the Euler Lagrange equation with the function g, this is set equal to 0, let us see what happens, note that g is purely a function of y, so the first quantity is 0 and the derivative of g with respect to y is 1.

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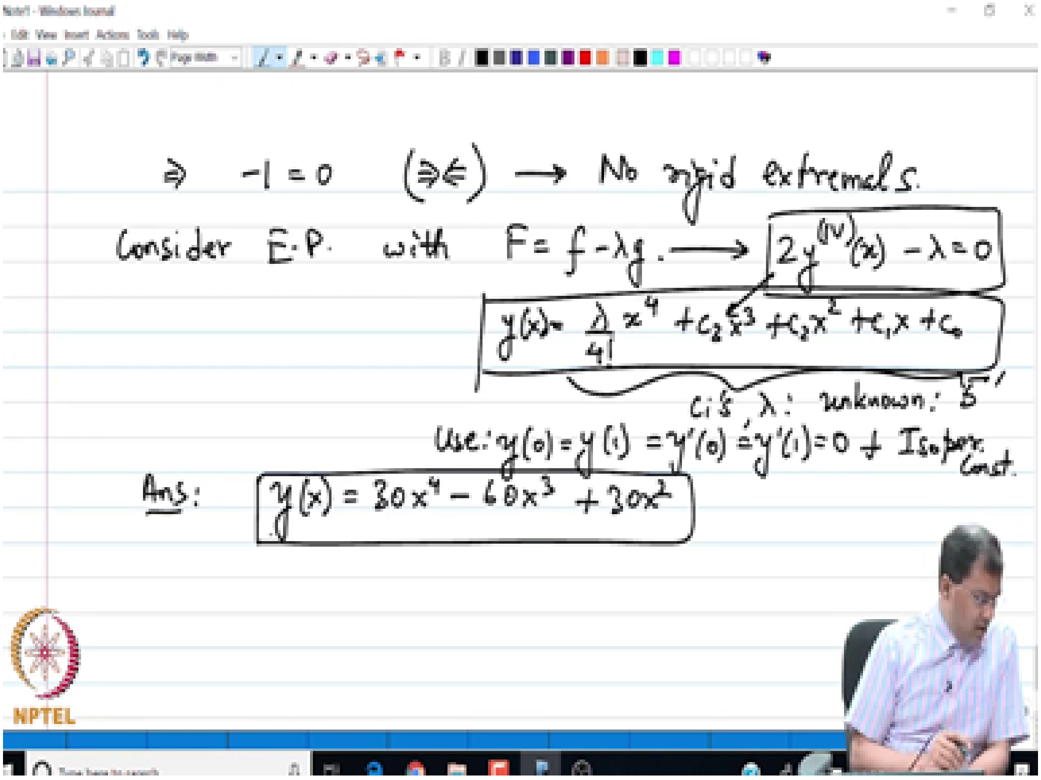

which means that I get - 1 = 0 or I get a contradiction here, which means that there are no rigid extremals to the problem, so we are in good shape, we are in the case of normal problem category, we can peacefully solve for the extremals. let us look at the Euler Poisson equation with $F = f - \lambda g \rightarrow 2y^{(IV)}(x) - \lambda = 0$.

Solution of ODE is $y(x) = \frac{\lambda}{4!}x^4 + C_3x^3 + C_2x^2 + C_1x + C_o$
So, there are C_i's and λ's are unknown, there are five of unknowns in total but we have also the boundary condition on y, so we use these boundary conditions y(0) = y(1) = $y^{'}(0) = y^{'}(1) = 0$ So, we have four boundary condition plus we have the isoperimetric constraint that makes it five conditions for five unknowns and hence the system is fully determinable and students are asked to check that the system can be fully solved and they should find the values of C_i's and λ's as an exercise.

I am just going to write the final answer the answer $y(x) = 30x^4 - 60x^3 + 30x^2$, this is the extremal that we get.