Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 22
Isoperimetric Problems 4

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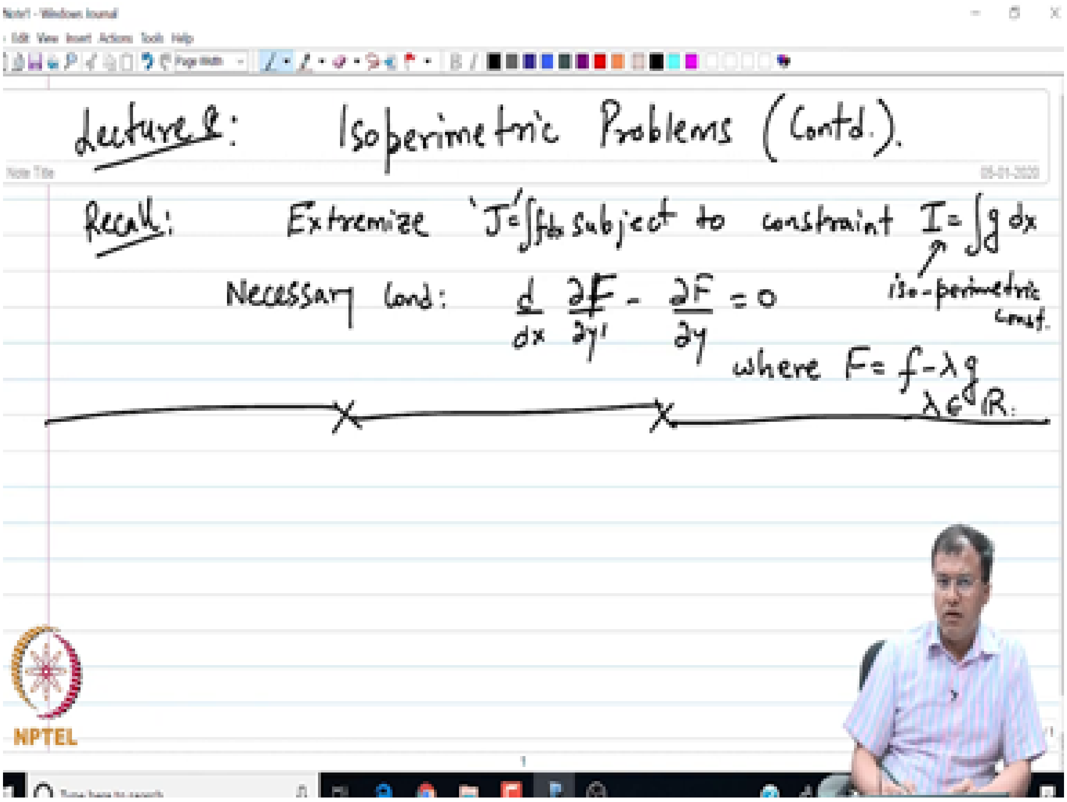

In today's lecture I am going to continue my discussion on Isoperimetric Problems. let us just recall the major result that we derived in the last lecture, what we derived was if we have to extremize a functional J subject to the constraint g which is an isoperimetric constraint, me denote it by I which was an Isoperimetric constraint.

The necessary condition for the extremal was the satisfaction of the following ODE $J = \int f dx$ so what we have is the following

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \text{ where } F = f - \lambda g \quad \lambda \in \mathbb{R}$$

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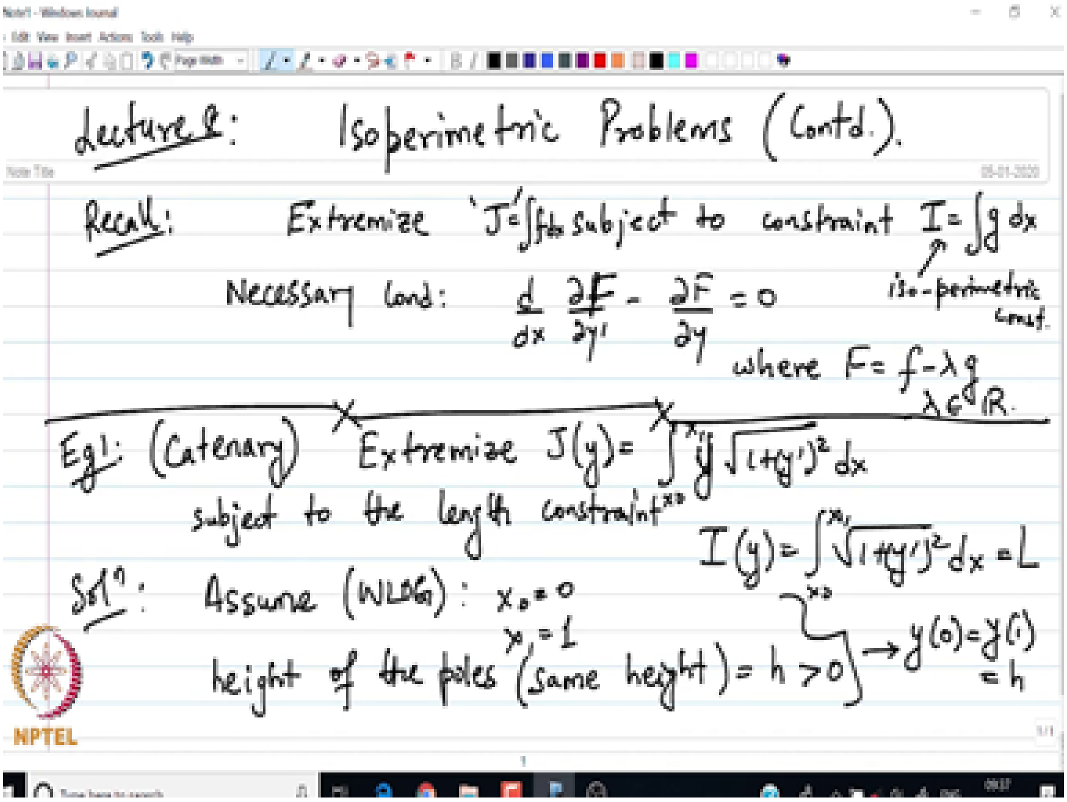

let us continue our discussion for the class of these set of problems and let us look at some examples where we impose Isoperimetric constraints.

Example 1: The problem of catenary are finding the optimal shape of the cable subject to some constraint on its weight or on its potential energy.

We have to extremize the functional $J(y) = \int_{x_o}^{x_1} y\sqrt{1+(y')^2}dx$ subject to the length constraint given by I(y) $= \int_{x_o}^{x_1} \sqrt{1+(y')^2}dx = L$ which is given to be a fixed number, this is the isoperimetric constraint that we have.

To find the extremal, let us assume although this assumption is not necessary in the general case but we are trying to simplify the problem, let us assume without loss of generality that $x_o = 0$ and $x_1 = 1$, my total domain length along the x axis is 1 and further we assume that the height of the poles are also fixed, so poles that is the same height is also fixed and this is positive.

It means that my fixed boundary conditions are y(0) = y(1) = h. Further we also assume that the length of the cable here is greater than 1 because if L is equal to 1 then the only solution that is possible is a straight line which is parallel to the x axis and if L is less than 1 then there would not be any solution, it is quite intuitive to see that. So, then we are going to directly use this result above, so we are going to use this result * to find the extremal.

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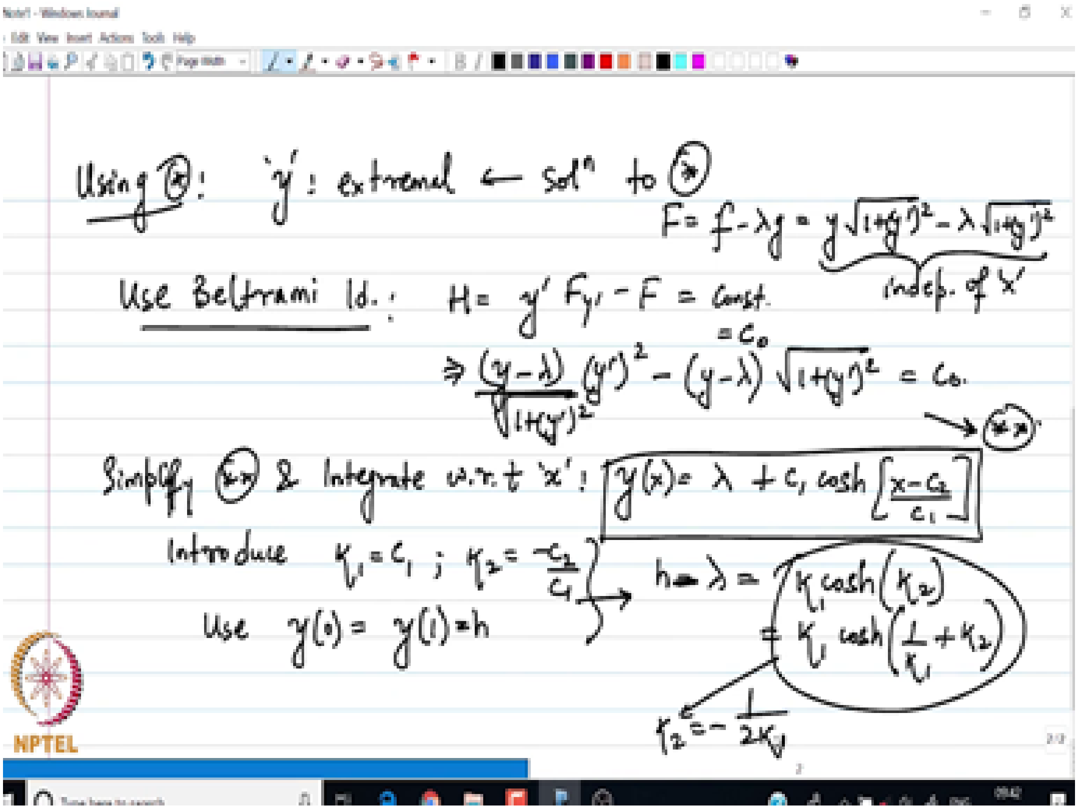

Using * I see that y which is the extremal is a solution to * and in that case I I get that it must be the solution to * and notice with our

$$F = f - \lambda g = y\sqrt{1+(y')^2} - \lambda\sqrt{1+(y')^2}$$

Notice that this quantity is independent of the variable x , which means we can use the Beltrami Identity which says that $H = y^{'}F_{y'} - F=$ constant, where the subscript $y^{'}$ denotes the derivative of capital F with respect to $y^{'}$

$$\Rightarrow \frac{(y-\lambda)}{\sqrt{1+(y')^2}}(y^{'})^2 - (y-\lambda)\sqrt{1+(y')^2} = C_o \qquad **$$

From here I can simplify this result a little bit further in order to finally integrate y with respect to x.

Simplify $**$ and integrate with respect to x, we get

$$y(x) = \lambda + C_1 \cosh\left(\frac{x - C_2}{C_1}\right)$$

that is my extremal, this time the extremal with the Lagrange constraint λ.

Now the next set of steps is to look at this solution and see what are the plausible values of C_1 and C_2, so to do that I am going to change my set of constants, I am going to introduce new set of constants $\kappa_1 = C_1$ and $\kappa_2 = -\frac{C_2}{C_1}$, the new set of constants and with this set of constants I use my boundary condition y(0) = y(1) = h

From here I get two sets of equation

$$h - \lambda = \kappa_1 \cosh\kappa_2 = \kappa_1 \cosh\left(\frac{1}{\kappa_1} + \kappa_2\right) \Rightarrow \kappa_2 = -\frac{1}{2\kappa_1}$$

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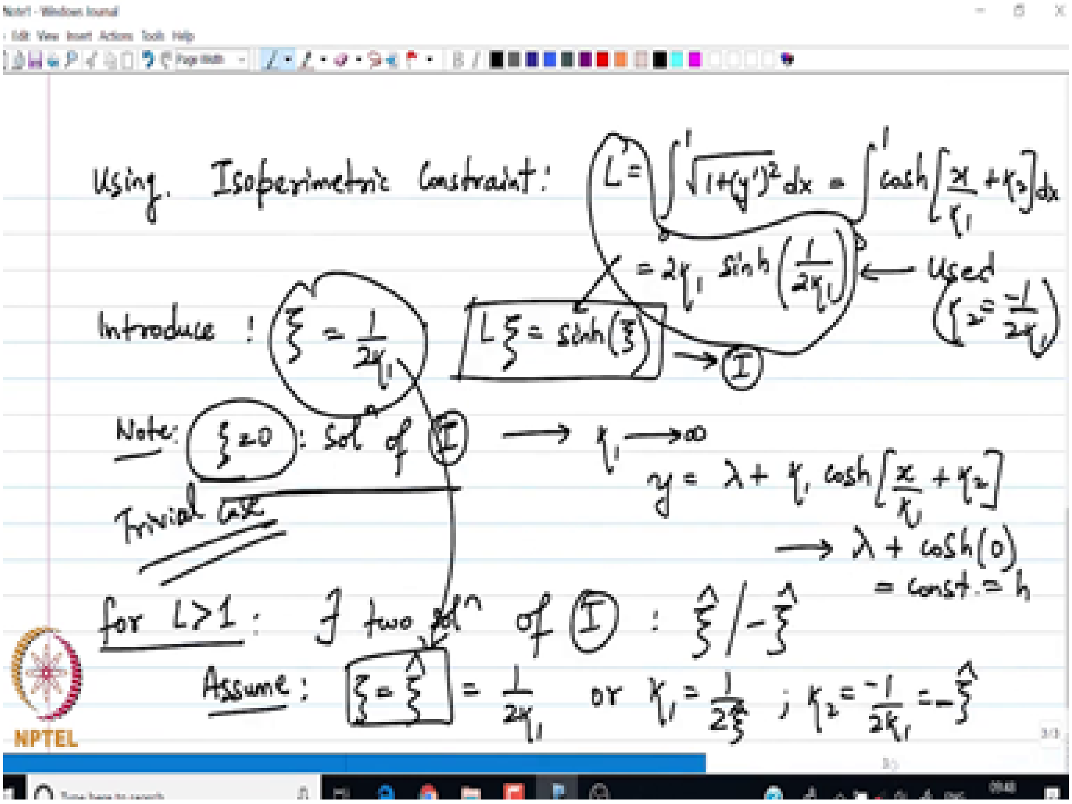

Using the isoperimetric constraint I see that

$$L=\int_0^1\sqrt{1+(y')^2}dx=\int_0^1\cosh\frac{x}{\kappa_1}+\kappa_2 dx=2\kappa_1\sinh\frac{1}{2\kappa_1},\quad \text{Used}\left(\kappa_2=-\frac{1}{2\kappa_2}\right)$$

Introduce another set of variable $\xi=-\frac{1}{2\kappa_1}$ from this Isoperimetric constraint I have this new equation

$$L\xi=\sinh(\xi) \qquad \mathbf{I}$$

Now we need to solve this equation for ξ, notice that this is a transcendental equation with the left-hand side a polynomial and the right-hand side a exponential function and in few lectures back we have shown that in general we will have two solutions if L is beyond a critical value, so notice that, note $\xi=0$ is a solution.

Since $\xi=\frac{1}{2\kappa_1}$, this means that $\kappa_1\to\infty$ then $y=\lambda+\kappa_1\cosh\frac{x}{\kappa_1}+\kappa_2\Rightarrow\lambda+\cosh(0)=\text{constant}=\text{h}$

If I have $\xi=0$ the trivial solution then y is a constant and that constant cannot be anything else other than h because at the boundary y = h, which means that $\xi=0$ gives me the trivial solution which is a line parallel to the x axis with y component h, and this is not the solution that we are looking after, so this is the trivial case and we are going to discard that case.

Now we know that for $L>1\exists$ two solutions of equation **I** and that has been shown few lectures back that L beyond a critical value in general there exists two solutions, we see that if we have a solution let us say $\hat{\xi}$ then even the $-\hat{\xi}$ is the solution because this equation **I** is even with respect to ξ.

Let us assume ξ is one of the solutions, in that case

$$\xi=\hat{\xi}=\frac{1}{2\kappa_1}\quad\text{or}\quad\kappa_1=\frac{1}{2\xi};\kappa_2=-\frac{1}{2\kappa_1}=-\hat{\xi} \qquad \mathbf{A}$$

all these set of conditions as my condition number A.

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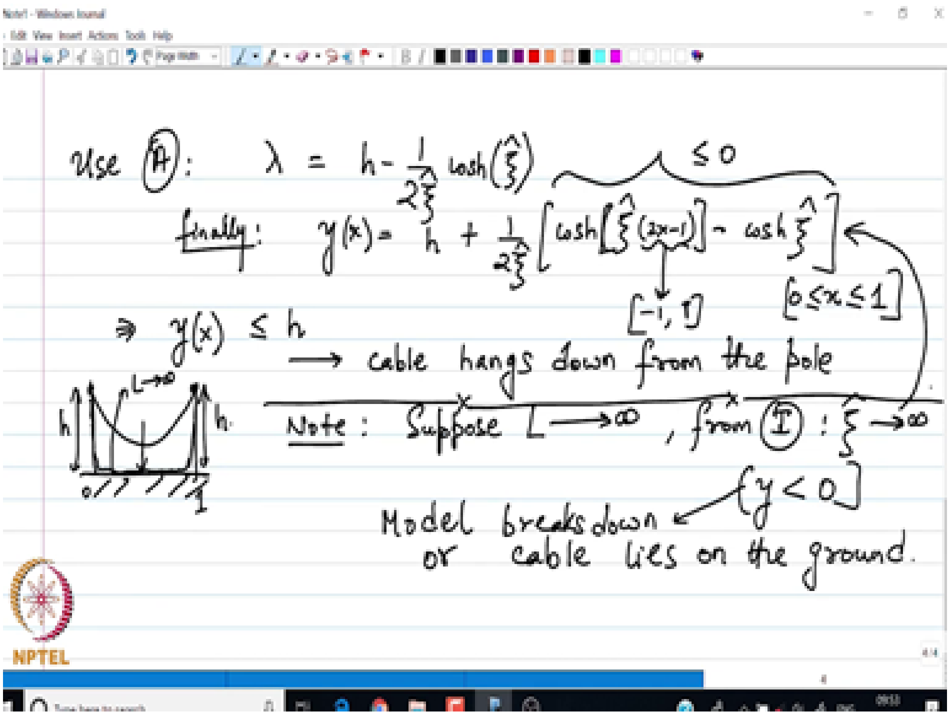

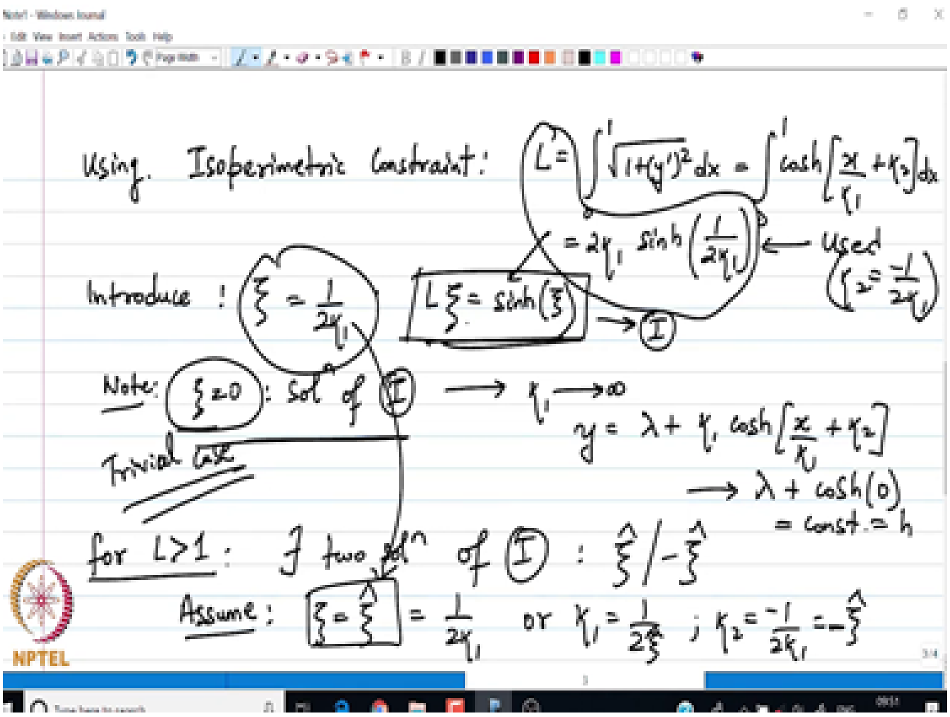

$$\text{Use}\mathbf{A}: \qquad \lambda = h - \frac{1}{2\xi}\cosh\hat{\xi}$$

$$\text{finally the solution is} \qquad y(x) = h + \frac{1}{2\xi}\left[\cosh\left[\hat{\xi}(2x-1)\right] - \cosh\hat{\xi}\right]$$

Since $0 \le x \le 1 \Rightarrow -1 \le 2x - 1 \le 1$ which means that

$$\cosh\left[\hat{\xi}(2x-1)\right] - \cosh\hat{\xi} \le 0 \rightarrow y(x) \le h$$

which means that the shape of the cable that we are trying to find hangs down from the pole, which means that if we were to draw the cable where the cable is hanging from two poles of the same height, where the x coordinates are 0, 1, we expect that the cable is hanging down.

Further suppose L is very-very long, so for a cable extremely large we see that the solution to this particular equation **I** will give some erroneous results

From equation **I** $\hat{\xi} \rightarrow \infty$, then plugging it into this solution I see that that y ¡ 0, which means that this is an abnormal result or what it is saying is that the model that we have, the model breaks down or in other words the cable lies on the ground, so there is nothing in terms of potential energy which is governing the shape of the cable, so in that case the cable will be completely lying on the ground.

So this is the case where $L \rightarrow \infty$, the cable is lying on the ground. So that completes the analysis of this example and this is the first example with the Isoperimetric constraint set up.

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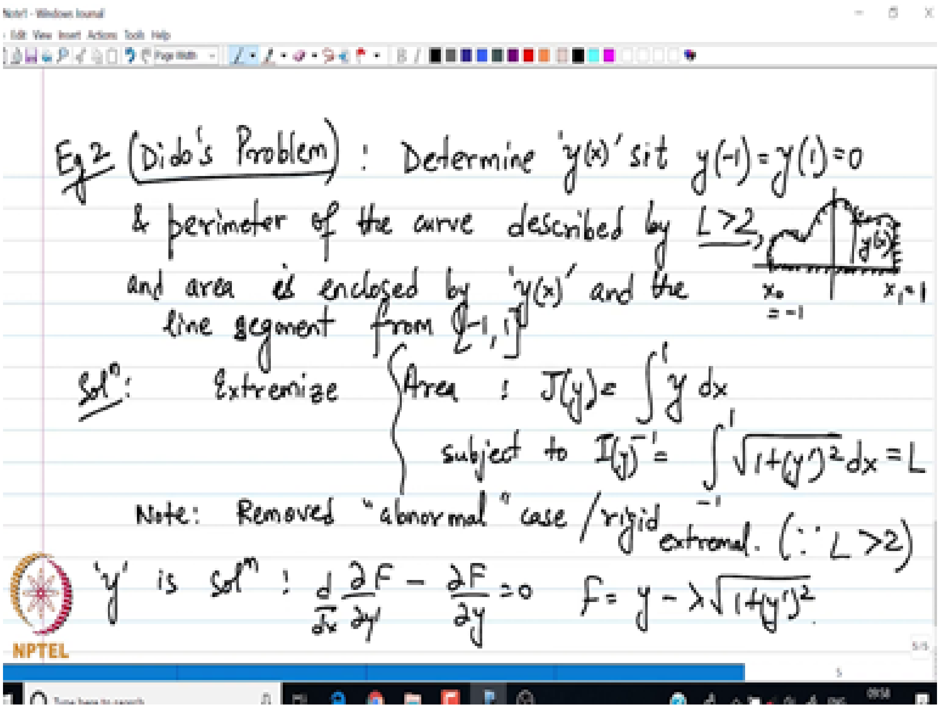

Let us look at more examples of this similar category, the second example that I have is on Dido's problem, so we have looked at Dido's problem on several occasion, so now we are going to look at the same problem with a length constraint set up on the curve. So, the problem again says we have to determine the height of the curve

let us draw a curve, these are end points x_o, x_1 and this is the height of the curve y(x), we need to determine y(x), let us also say that $x_o = -1$ and $x_1 = 1$ such that y(-1) = y(1) = 0 and the perimeter of the curve described by L > 2 .

And the reason we have taken L > 2 is because if L is equal to 2 then the only length will be lying parallel to the x-axis because the length along the x-axis is 2 and L less than 2 does not make sense we would not get a curve. So, L has to be necessarily greater than 2 and the area that we described is enclosed by this function y(x) and the line segment from - 1 to 1, so the length of the curve that we are trying to describe is this one and this one.

To find the extremal, we have to extremize the area

$$J(y) = \int_{-1}^{1} y dx$$

subject to the length constraint which is

$$I(y) = \int_{-1}^{1} \sqrt{1 + (y')^2} dx = L$$

Since we have already taken care of the fact that L > 2, we have removed certain abnormal situations where the solutions or the extremal is unique or the extremal is does not exist or have already removed.

note that we have already removed abnormal case or the case with rigid extremal because my total length is greater than 2, so some of the length will be above the x-axis, so we will get a shape with a nonzero area, which what we have, the extremal y will be the solution to the Euler Lagrange

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \qquad F = y - \lambda\sqrt{1 + (y')^2}$$

And again we can go ahead by observing that this quantity F is independent of the variable x explicitly, which means we can again use the Beltrami Identity.

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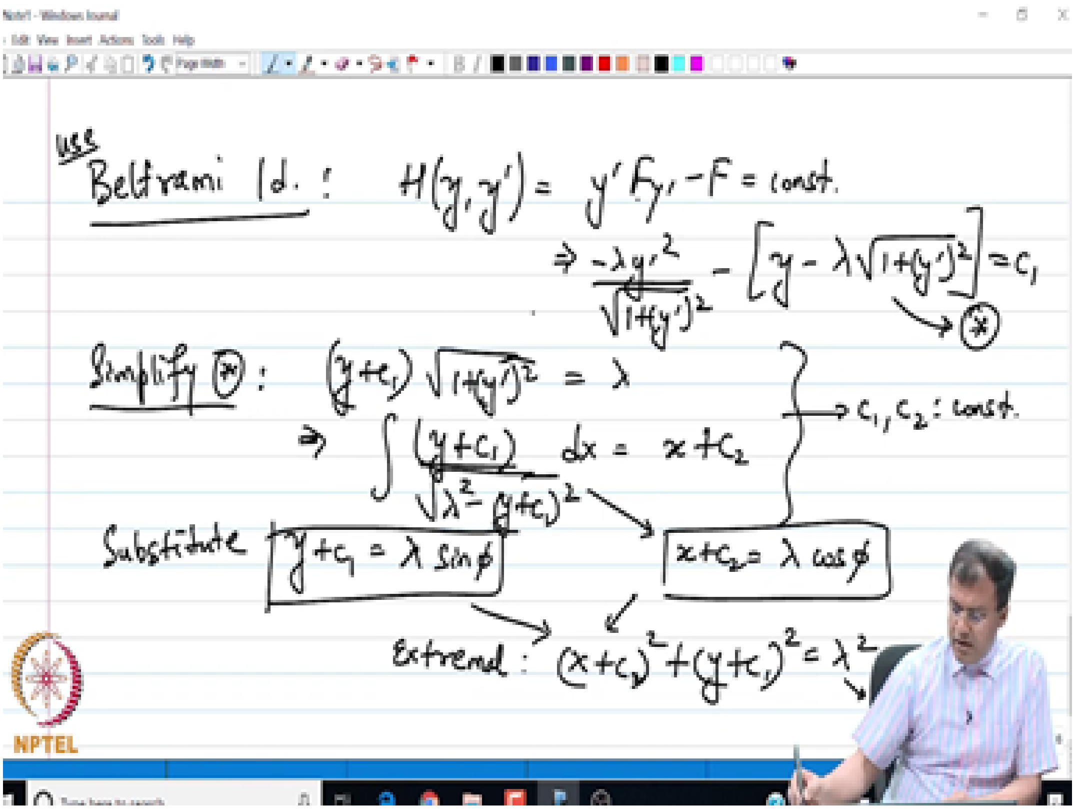

$$\text{Beltrami identity} \qquad H(y,y') = y'F_{y'} - F = \text{ constant}$$

$$\Rightarrow \frac{-\lambda y'^2}{\sqrt{1+(y')^2}} - \left[y - \lambda\sqrt{1+(y')^2}\right] = C_1 \qquad *$$

we simplify this expression and then solve for y, when we solve for y and we get the following expression $(y + C_1)\sqrt{1+(y')^2} = \lambda$

$$\Rightarrow \int \frac{(y+C_1)}{\sqrt{\lambda^2-(y+C_1)^2}}dx = x + C_2$$

C_1 and C_2 are constants of integration. So, the next set of steps in integrating will involve a certain substitution $y + C_1 = \lambda \sin\phi$ and the moment we do that this integral becomes very straightforward and we see that this particular integration leads to $x + C_2 = \lambda\cos\phi$

what we get is that the extremal is such that we have $(x + C_2)^2 + (y + C_1)^2 = \lambda^2$, this implies that the extremal lies on the circle with radius λ. all it remains now is to eliminate these constant C_1 and C_2 by using our boundary conditions that we have.

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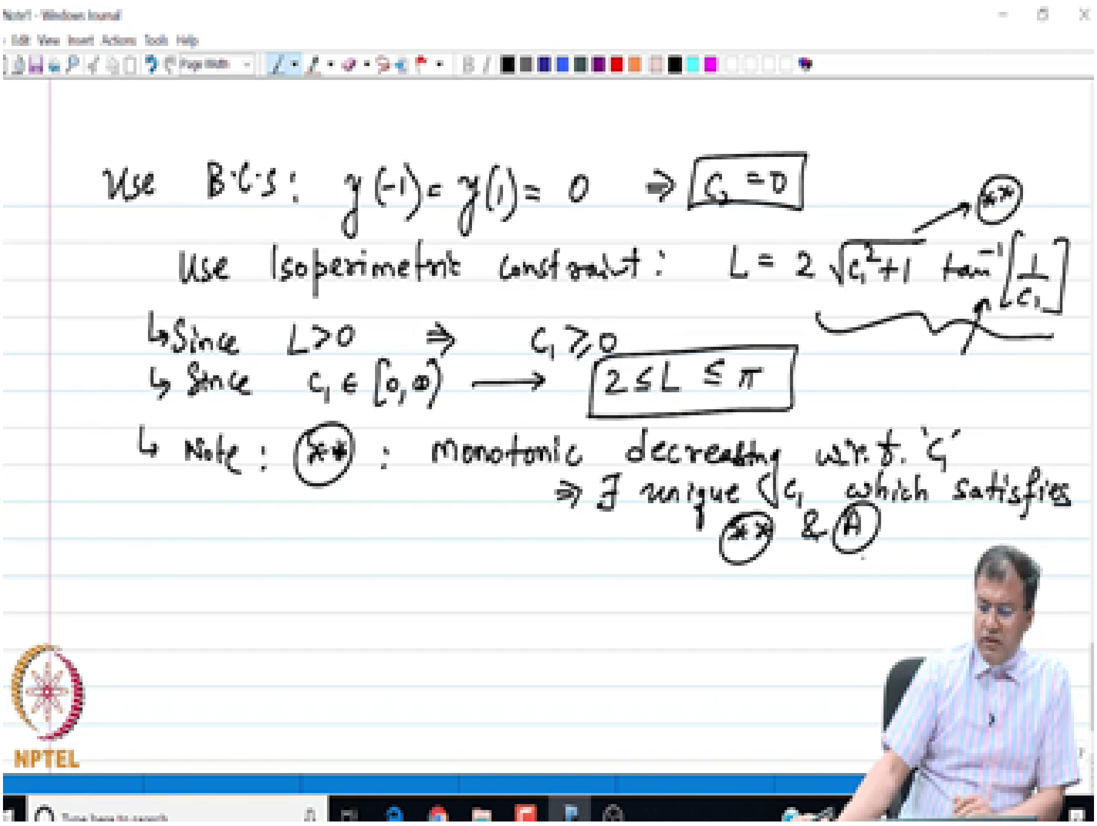

Boundary conditions y(-1) = y(1) = 0 $\Rightarrow C_2 = 0$
we have not used our isoperimetric constraint

$$L = 2\sqrt{C_1^2+1}\tan^{-1}\left[\frac{1}{C_1}\right] \qquad **$$

Since length L > 0 $\Rightarrow C_1 \geq 0$ otherwise since the sign of this expression on the right is completely governed by the $\tan^{-1}$ and that is positive only when C_1 is positive.

since $C_1 \in [0, \infty) \to 2 \leq L \leq \pi$ further note that this relation ** is monotonically decreasing with respect to C_1 which means that there exists a unique C_1 which satisfies ** and also the set of conditions, so we do not at this stage explicitly find the constant C_1 except by showing that C_1 is unique which lies on the circle as well as which satisfies the Isoperimetric constraint and saying that C_1 can readily be found. So we end the discussion on this problem.