Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 21
Isoperimetric Problems 3

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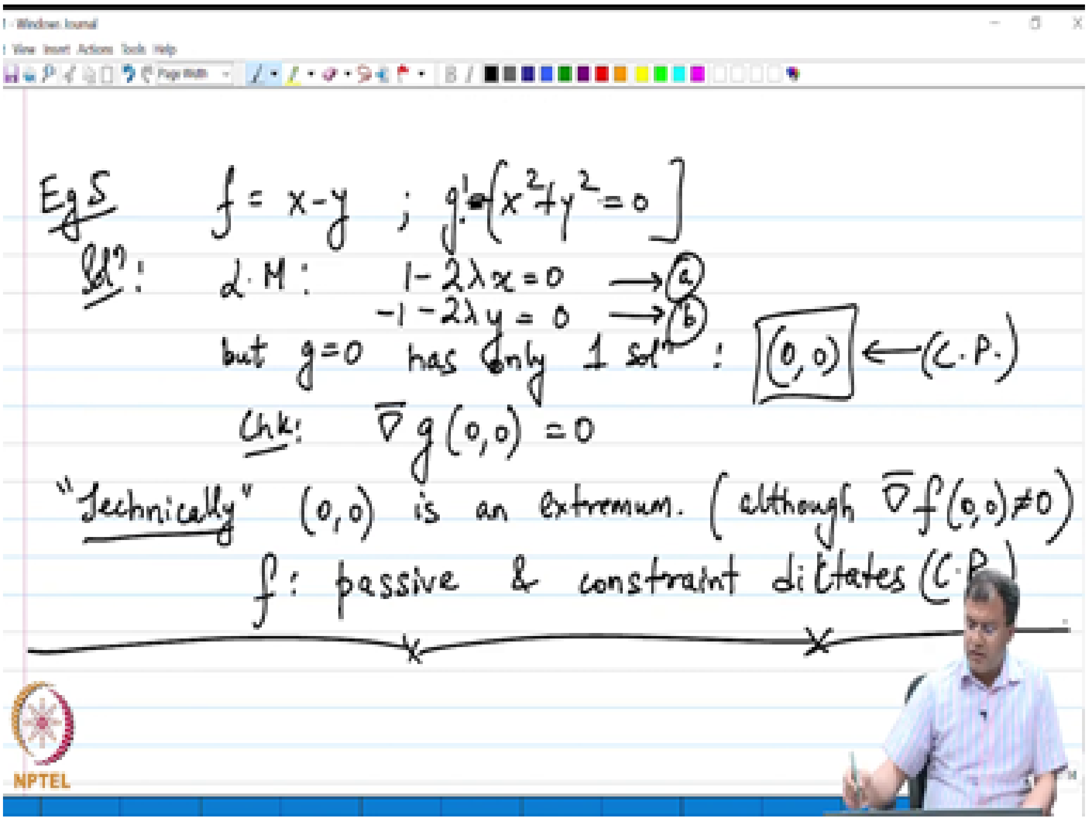

Let us now look at another Example: f = x - y; g = $x^2 + y^2$ = 0 So, again my Lagrange Multiplier method for this objective function and this constraint g = 0 . The Lagrange Multiplier method gives me two equations $1 - 2\lambda x = 0$ **a**
and $-1 - 2\lambda y = 0$ **b**
but g = 0 has only one solution, 0 because it is a real valued function and it will achieve only one solution when both x and y vanishes. So, now, it turns out that 0 is a solution to g, but 0 is not a solution to the Lagrange multiplier method. So, further check that the gradient at this point also vanishes. Even in the previous example, the gradient vanishes. We could go and check that. But in this problem, the gradient vanishes.

So this is an abnormal problem, where the only point we are getting does not satisfy the Lagrange Multiplier, 0 an extremum? Well, technically, yes, but it is not satisfying the Lagrange Multiplier. So, what I am saying is technically, so that is why I am using the word technically because I cannot show it via the Lagrange method that this is the minimum, because this is the only point that we have under consideration minimum or not.

So, technically 0 is an extremum because it is a minimum of the constraint. So, it is a extremum although I can show that $\bar{\nabla}(0, 0) \neq 0$, we see that the gradient is (1, -1), so it is never zero, but the only choice that we have is this point which means that the role of f is not much, it is the constraint which is governing the final answer here.

So, this is a case where f the objective function is passive, and it is a constraint which dictates my critical point. In this case, this is my critical point under consideration. So, we have all sorts of problems for the abnormal case. So, let me, just summarize our entire discussion for the abnormal case.

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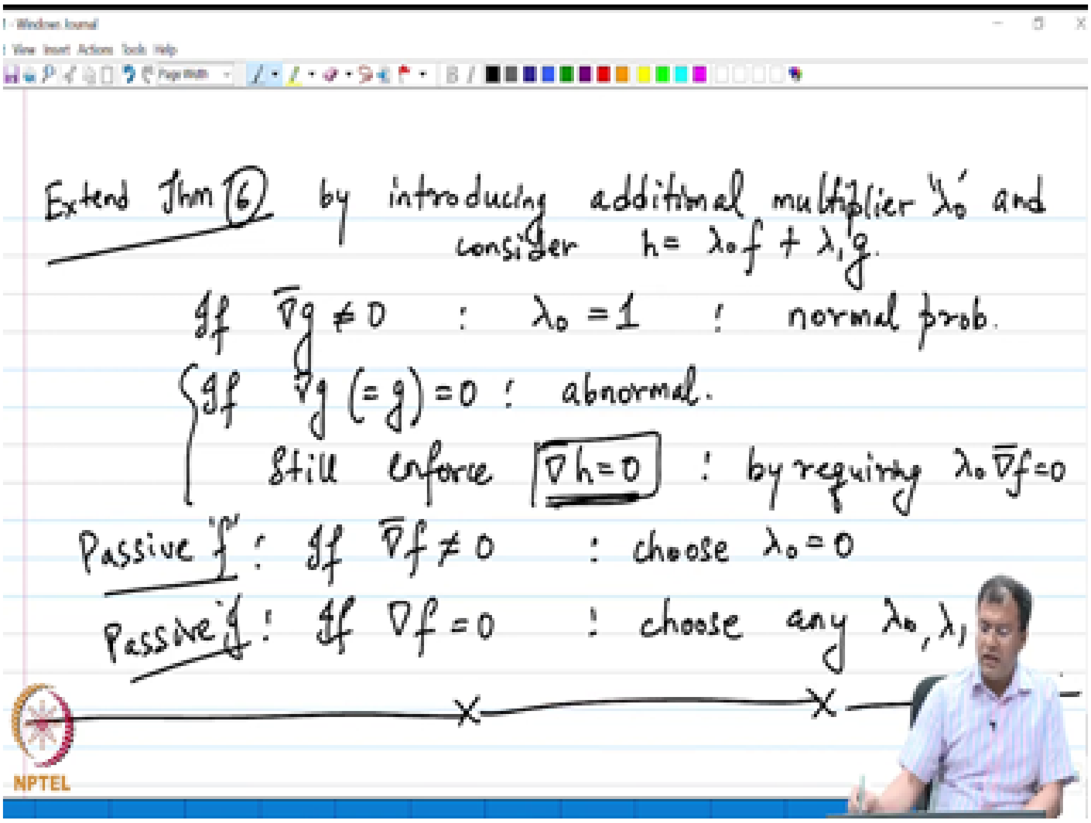

So, I am going to extend Theorem 6, my result for the normal case to the abnormal case by introducing, I need to develop a similar to the Lagrange Multiplier method for normal case, I need to develop an equivalent Lagrange Multiplier for the abnormal case so that it holds for these cases as well.

we extend Theorem 6 by introducing an additional multiplier λ_o and we consider that $h = \lambda_o f + \lambda_1 g$ and if $\bar{\bigtriangledown} g \neq 0$ that is a normal problem, we can very happily choose $\lambda_o = 1$ that is the standard Lagrange Multiplier method, this is the case of normal problems.

Suppose if I have $\bar{\bigtriangledown} g (= g) = 0$, then we are in the abnormal problem case and we have various scenarios, we can still enforce $\bar{\bigtriangledown} h = 0$ for finding the critical points by requiring $\lambda_o \bar{\bigtriangledown} f = 0$.

We have seen that in the abnormal case there were two subcases, one the objective function passive, the other constraint being passive. So, if I have the case where f is passive, so if I have passive objective function, then in that case, that $\bar{\bigtriangledown} f \neq 0$ the objective function is passive, I am going to choose $\lambda_o = 0$

So, that this particular condition still is enforced, on the other hand, if I have passive constraint or passive g, then and $\bar{\bigtriangledown} f = 0$, I can choose any λ_o or λ_1, this condition is going to be enforced. So let me now club all this summarized result in the form of a theorem.

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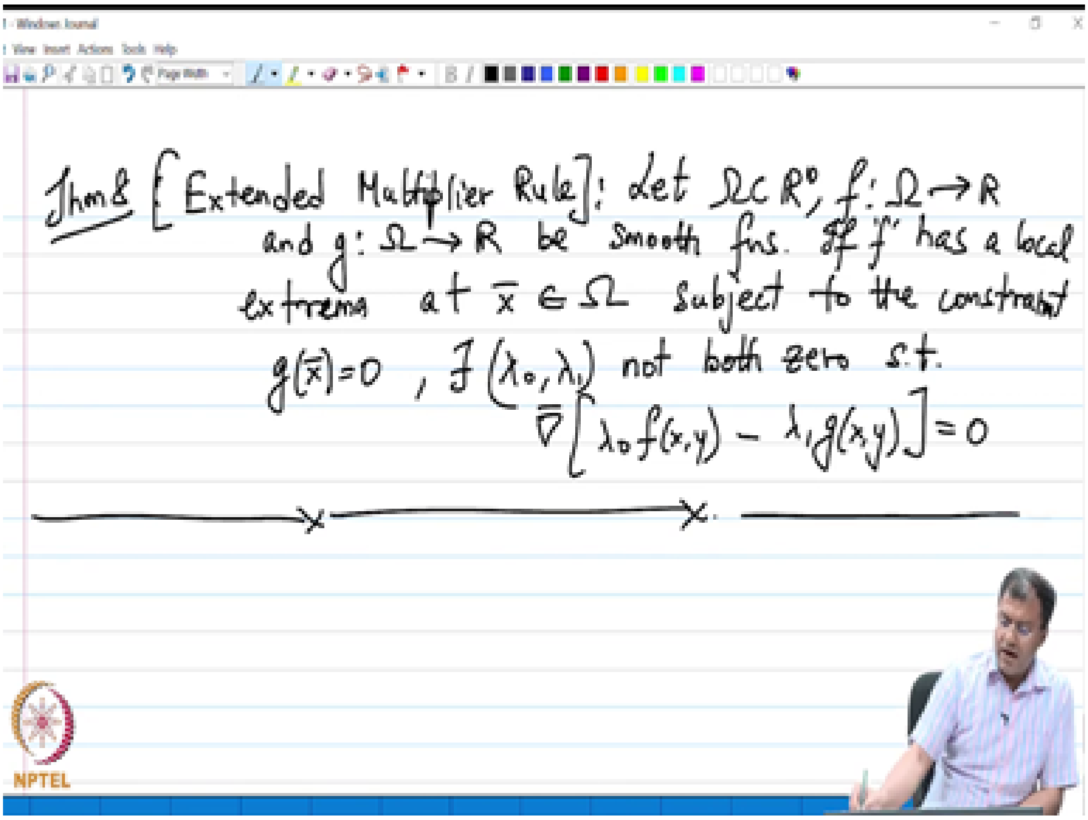

Theorem 8: [Extended multiplier rule] let $\Omega \subset \mathbb{R}^n$ and f : $\Omega \to \mathbb{R}$ and g : $\Omega \to \mathbb{R}$ be smooth functions, If f has a local extrema at this point $\bar{x} \in \Omega$ subject to the constraint $g(\bar{x}) = 0$, $\exists\ (\lambda_o, \lambda_1)$ not both zero such that $\bar{\bigtriangledown} [\lambda_o f(x, y) - \lambda_1 g(x, y)] = 0$
Suppose, we are in this situation then we have extended our standard Lagrange Multiplier using another multiplier λ_o in addition to the existing multiplier λ_1.

Now I think this background of finite dimensional calculus using Lagrange Multiplier is sufficient for us to look at problems involving constrained functional optimization. So, we are going to start our discussion on Isoperimetric problems consists of finding extremals of J satisfying Boundary condition
let $J : C^2[x_o, x_1] \to \mathbb{R}$ be a functional of the form $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'})dx$ $\quad \gamma_1$

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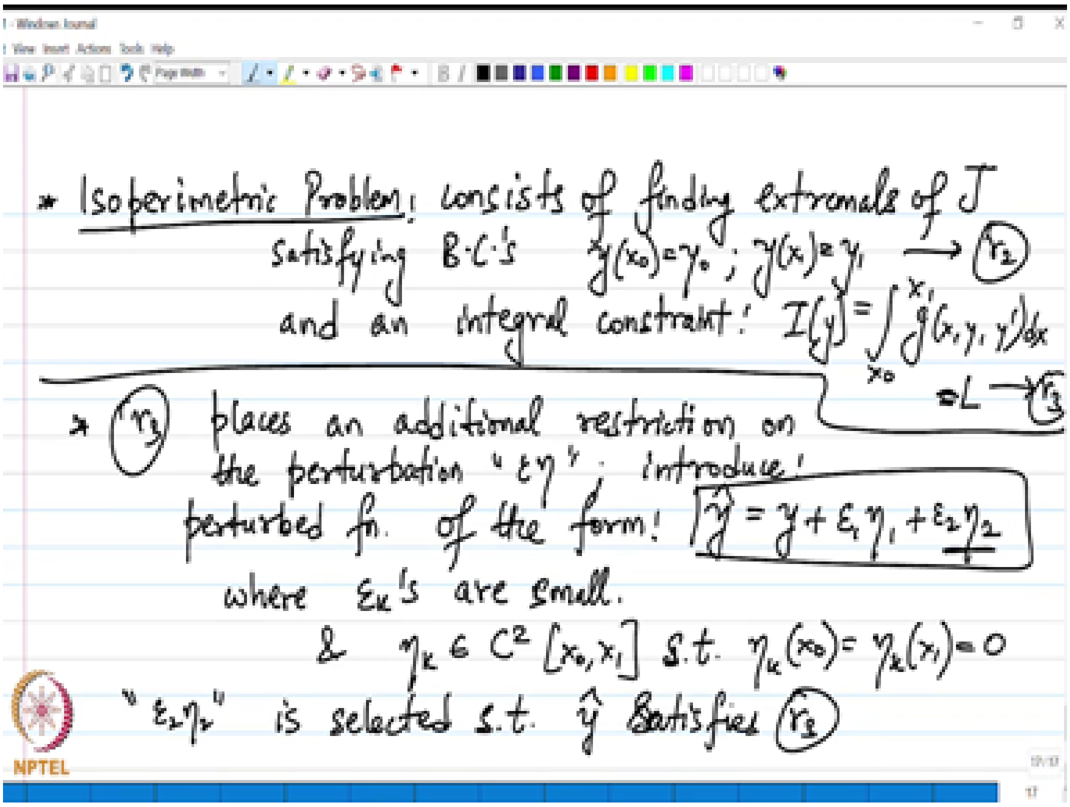

For Isoperimetric problem, I need to describe a constraint, it consists of finding extremals of J satisfying the boundary conditions, That is a fixed end point conditions $y(x_o) = y_o : y(x_1) = y_1$ $\quad \gamma_2$

and we have an additional integral constraints of the form I(y) $= \int_{x_o}^{x_1} g(x, y, y^{'})dx = L$ $\quad \gamma_3$
This particular functional is equal to a fixed value L is the constraint

So the problem now is how to find the extremum in this constraint optimization. Well, let us recall how did we find the extremum in the unconstrained case, we use to perturb our function by introducing a perturbation of the form $\epsilon\eta$ and then we use to do the Taylor series expansion of the perturbed function and then integrate and then cancel and so on so forth.

Now, the fact that we have a constraint, we cannot just perturb freely, we have to introduce an additional perturbation, so, this particular constraint is always satisfied. So, what I just said is γ_3 places an additional restriction, so let me just separate this out, restriction on the perturbation, it places an additional restriction on the perturbation $\epsilon\eta$.

We introduced the perturbed function of the form, the perturbed function of the form $\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2$, What have we done here is we have now introduced an additional perturbation $\epsilon_2\eta_2$ which is able to satisfy $\hat{y}$ also satisfies the constraint γ_3.

Now we have two switches ϵ_1 and ϵ_2 such that the constraint is also satisfied, where my ϵ case are small, I take them relatively small so that my first variation involves only the first term or the second term of the Taylor Series, which is the order ϵ term and $\eta \in C^2[x_o, x_1]$ such that $\eta_k(x_o) = \eta_k(x_1) = 0$.

So, what I have said is the following $\epsilon_2\eta_2$ is selected such that $\hat{y}$ satisfies γ_3, Now we are ready to find the first variation of the functional and hence the extremum of the functional in this case the Isoperimetric case.

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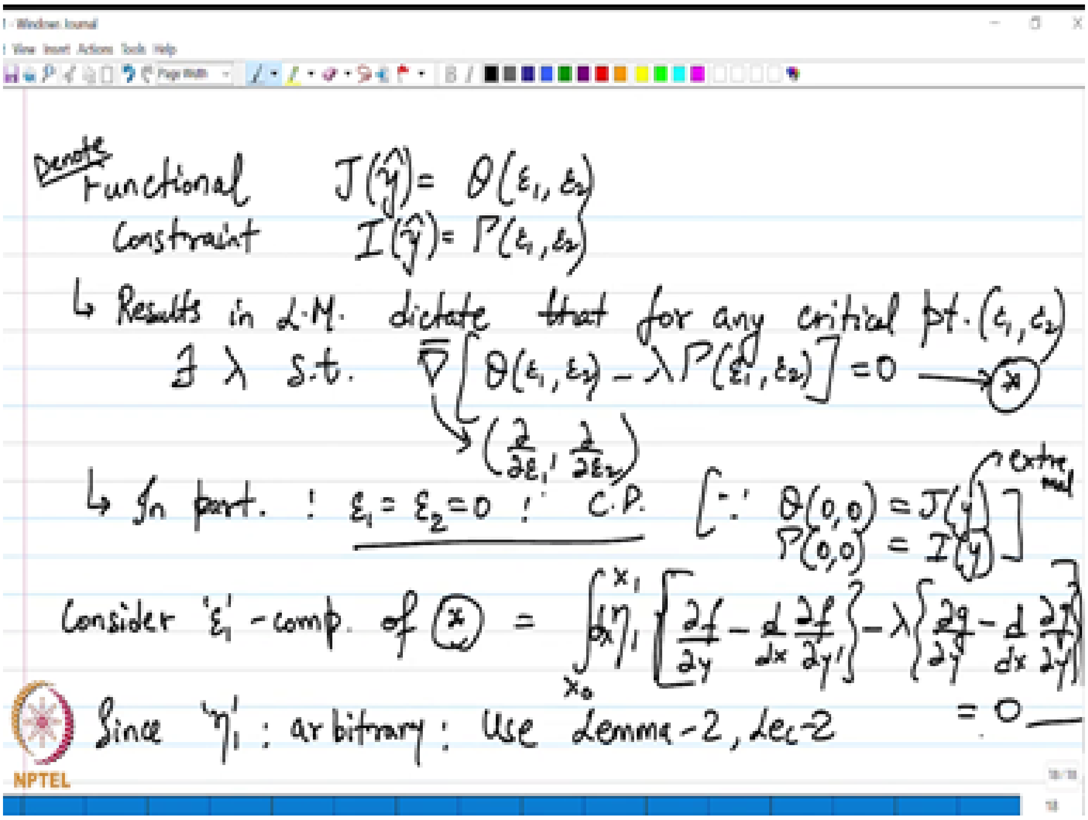

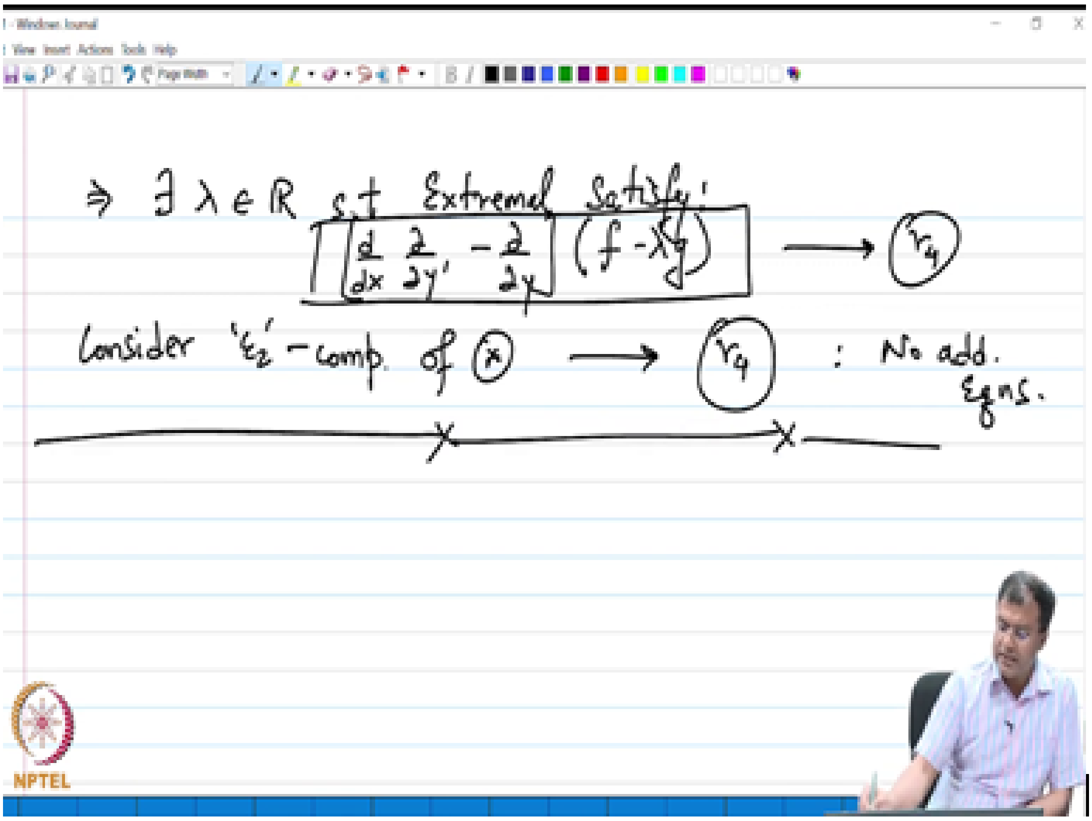

, Now, my functional will involve two unknown constants ϵ_1, ϵ_2, functional $J(\hat{y}) = \theta(\epsilon_1, \epsilon_2)$

5

and constraint $I(\hat{y}) = \Gamma(\epsilon_1, \epsilon_2)$

We see that the results in the Lagrange Multiplier dictate that for any critical point (ϵ_1, ϵ_2) $\exists\ \lambda$ such that $\bar{\bigtriangledown}\left[\theta(\epsilon_1, \epsilon_2) - \Gamma(\epsilon_1, \epsilon_2)\lambda\right] = 0$ $\quad *$

In particular check out that $\epsilon_1 = \epsilon_2 = 0$ satisfies both the functional and the constraint and this is indeed a critical point because both the functional as well as the functional attains an extremal that vanishes, but the functional attains an extremal as well as the constraint is also satisfied. So, this becomes I(y), where I(y) is an extremal.

So, plugging in $\epsilon_1 = \epsilon_2 = 0$, we are going to get the extremal of the functional, so certainly this is a critical point. So, let us look at the x component of this gradient. So, when we take the gradient, so let us consider the ϵ_1 component of *, this is actually a set of two equations, the derivative with respect to ϵ_1, the other with respect to ϵ_2.

Consider the ϵ_1 component of * equals to $\int_{x_o}^{x_1} \eta_1 \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} - \lambda\left\{\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right\}\right] dx = 0$ since my perturbation η_1 is arbitrary, I am going to use lemma 2 of lecture 2 to invoke the fact that integral constraint, well, this is set to 0 actually.

because of the * here, the integral constraint can be reduced to the differential constraint as follows that $\exists\ \lambda \in \mathbb{R}$ such that the extremal satisfies the following differential equation which is

$$\left[\frac{d}{dx}\frac{\partial}{\partial y'} - \frac{\partial}{\partial y}(f - \lambda g)\right] = 0 \qquad \gamma_{\mathbf{4}}.$$

So, this operator on the function $f - \lambda g$ where f is objective function and g is the constraint.

Now, consider ϵ_2 component of * we are going to again get the same relation $\gamma_{\mathbf{4}}$. So, we get no additional equations. So, what have we found is that for Isoperimetric problem, the necessary condition is $\gamma_{\mathbf{4}}$ or the Euler Lagrange equation. So, let me wrap up this lecture by giving two major results and try to summarize the case of Isoperimetric problem.

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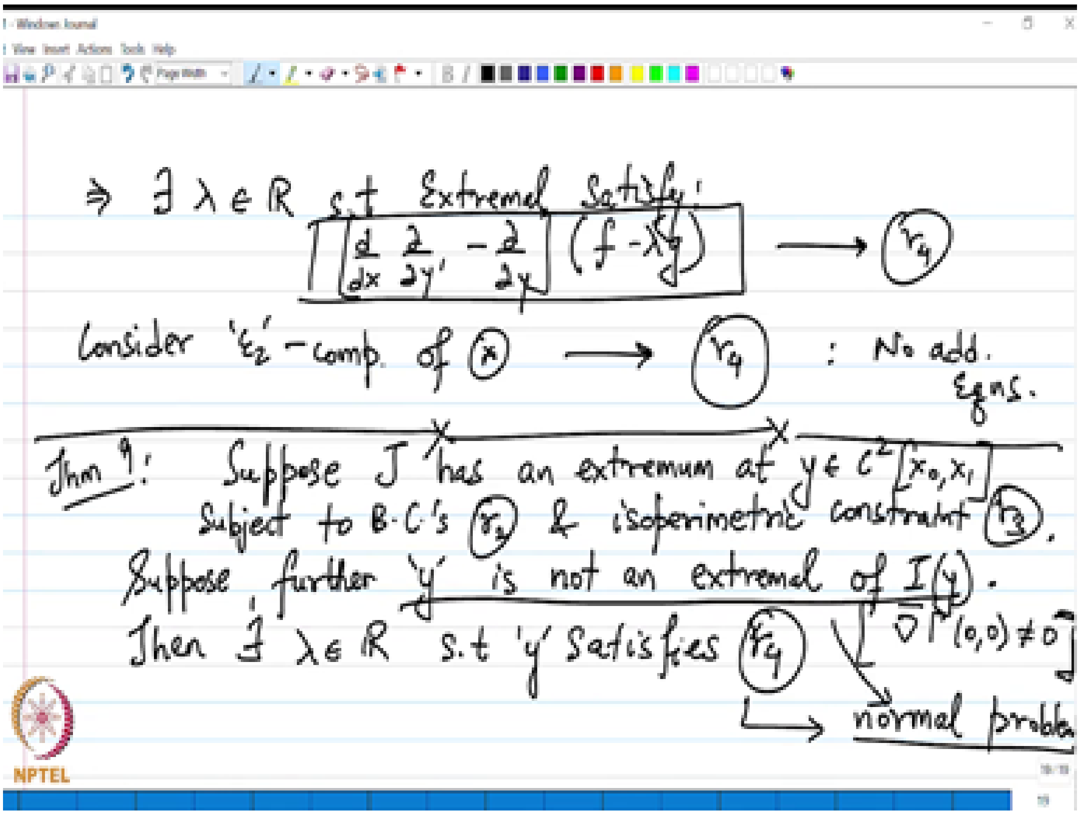

Theorem 9 Suppose J has an extremum, suppose J has an extremum at $y \in C^2[x_o, x_1]$ which is a second order differentiable function subject to the boundary condition γ_2 and the Isopermetric constraint γ_3

Then suppose further y is not an extremal of I, we are making sure that this particular condition is making sure that the gradient of the Isoperimetric constraints do not vanish at $\epsilon_1 = \epsilon_2 = 0$ which is equivalent to the finite dimensional constraint that $\bar{\bigtriangledown} g \neq 0$.

So, we are making sure that you are not dealing with the so called rigid extremal. So, y is not an extremal of the constraint itself, then there exists a $\lambda \in \mathbb{R}$ such that the extremal y satisfies the equation γ_4 or the Euler Lagrange equation γ_4 So, this is for the normal problem because in the normal problem we have assumed that y is not an extremal of I. How about the abnormal problem?

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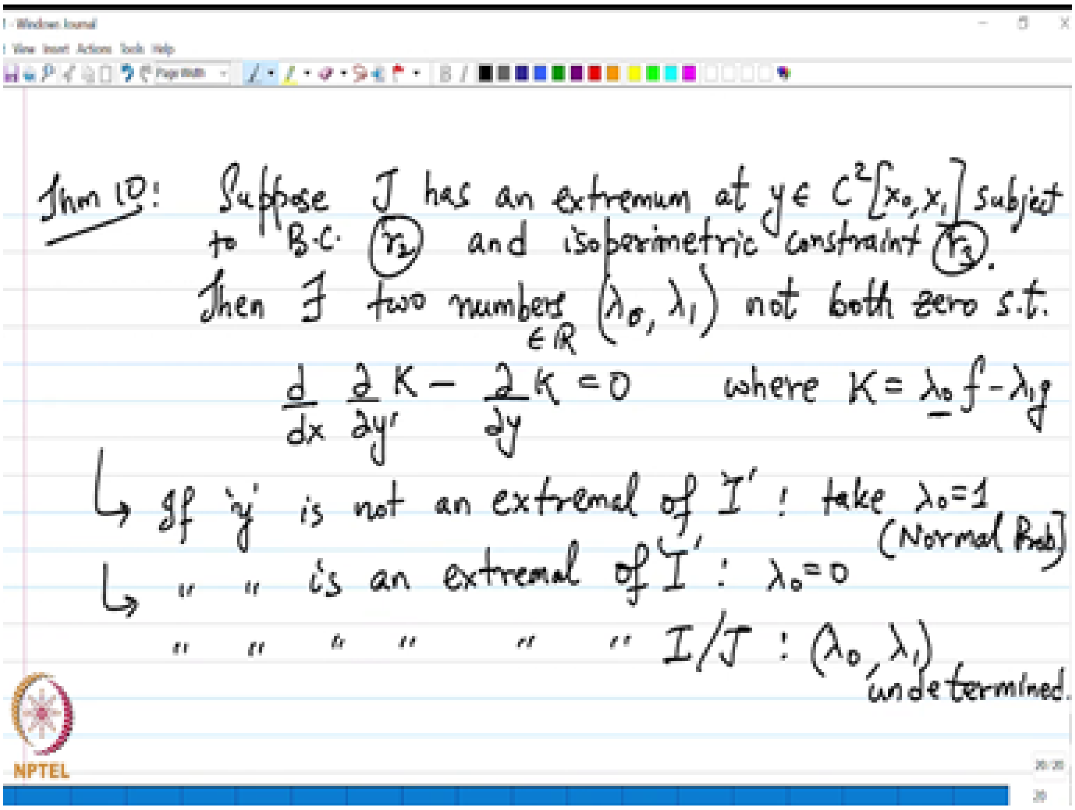

Theorem 10: Suppose J has an extremum at $y \in C^2[x_o, x_1]$ subject to the boundary condition γ_2 and the Isoperimetric constraint γ_3, then $\exists$ two numbers (λ_o, λ_1) not both 0, such that $\frac{d}{dx}\frac{\partial}{\partial y'}K - \frac{\partial}{\partial y}K$ equation holds, where function $K = \lambda_o f - \lambda_1 g$. So we have extended our previous result in Theorem 9, by introducing a new function with this new set of two constants λ_o and λ_1, this covers all sets of problems including normal problems.

If y is not an extremal of I the constraint we can take $\lambda_o = 1$ (normal problem) and the result reduces to Theorem 9. On the other hand, if y is an extremal of the constraint I, then we can take $\lambda_o = 0$ and if I have that y is an extremal of both I as well as J, then it turns out that both λ_o and λ_1 are undetermined.

We do not worry about what is undetermined, we do not care about what is the value of λ_o and λ_1. It just does not matter. So, it will not play any role in our optimization. In the next lecture I am going to look at a few other examples of this isoperimetric problem, including the normal as well as the abnormal problems.