Variational Calculus and its Applications in Control Theory and Nano mechanics Professor

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Lecture – 02

Introduction –Euler Lagrange Equations Part-2

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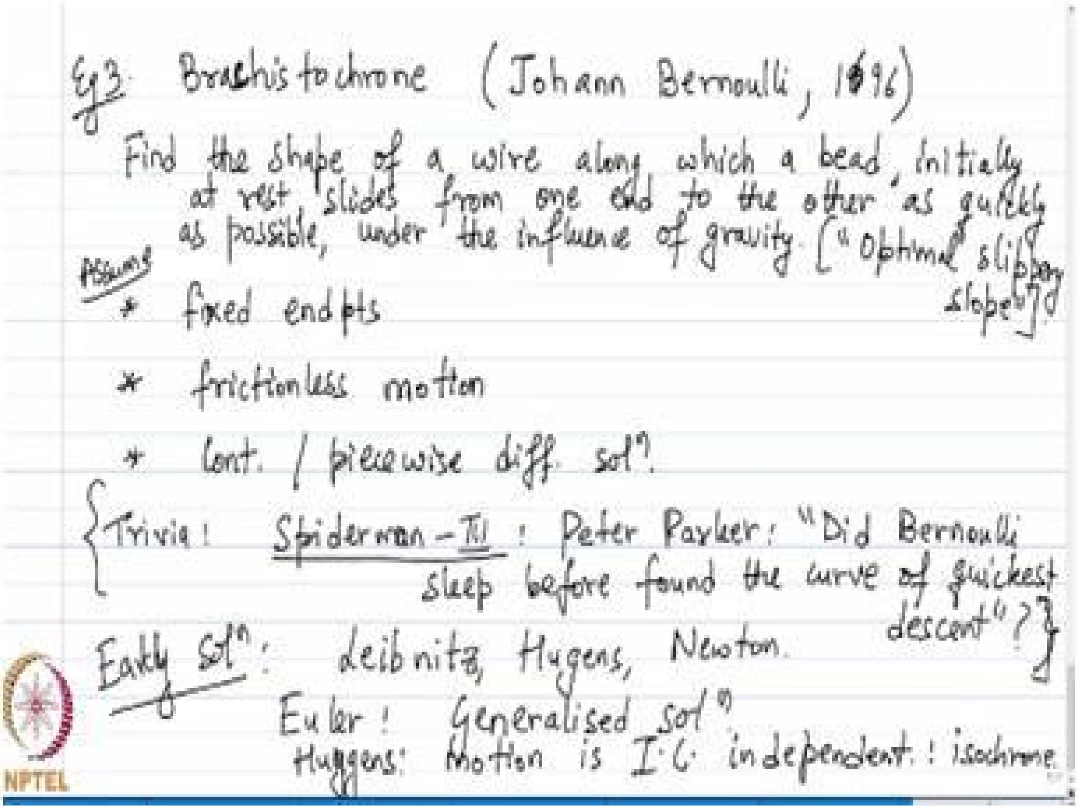

That is an another famous example of the Brachistochrone and the problem of Brachistochrone was introduced by the younger of the Bernoulli brothers Johann Bernoulli in 1696. In fact, there was a competition between the two brothers as to who will solve each other problems. So when Jacob introduced his problem of catenary, Johann came up with the problem of Brachistochrone.

So there was a rivalry among these two brothers. So the problem says that we have to find the shape, of a wire along which is initially a bead at rest, slides from one end to the other bead initially at rest slides from one end to the other as quickly as possible under the influence of gravity. In short, this particular problem of Brachistochrone was also termed as the optimal slippery slope problem.

The optimal slippery slope problem or optimal slippery dip. Now, there are few assumptions in this problem, again we assume that the curve is such that we have fixed end points, so these are the assumptions in the problem fixed end points and further we assume that the motion is such that it is frictionless, that is, friction does not contribute to the motion of the particle.

Further we assume that whatever we are trying to optimize or whatever solution we are trying to find, the solution is continuous and piecewise differentiable.

Now, in fact, just a matter of trivia so popular was this problem that it was in fact introduced in a famous Hollywood problem that is in Spiderman III, where it was said by the hero Peter Parker that did Bernoulli. He said that did Bernoulli sleep before he found the curve of quickest descent. So, in this case Peter Parker was referring to the problem of the Brachistochrone which we are going to talk about in depth over the course of our lecture. So, just a little bit more trivia, the early solutions to this problem was given by Leibnitz, Huygens and Newton himself Isaac Newton.

And then it was Euler who provided the most general framework of the class of Brachistochrone problems. So, Euler provided the generalized solution to this class of problems and finally it was Huygens who discovered that the motion of the problem in Brachistochrone is initial condition independent. Lets denote initial condition by I.C so the motion is initial condition independent also known as the isochrone problems. So just a bit of set up in this problem, we see that in this case.

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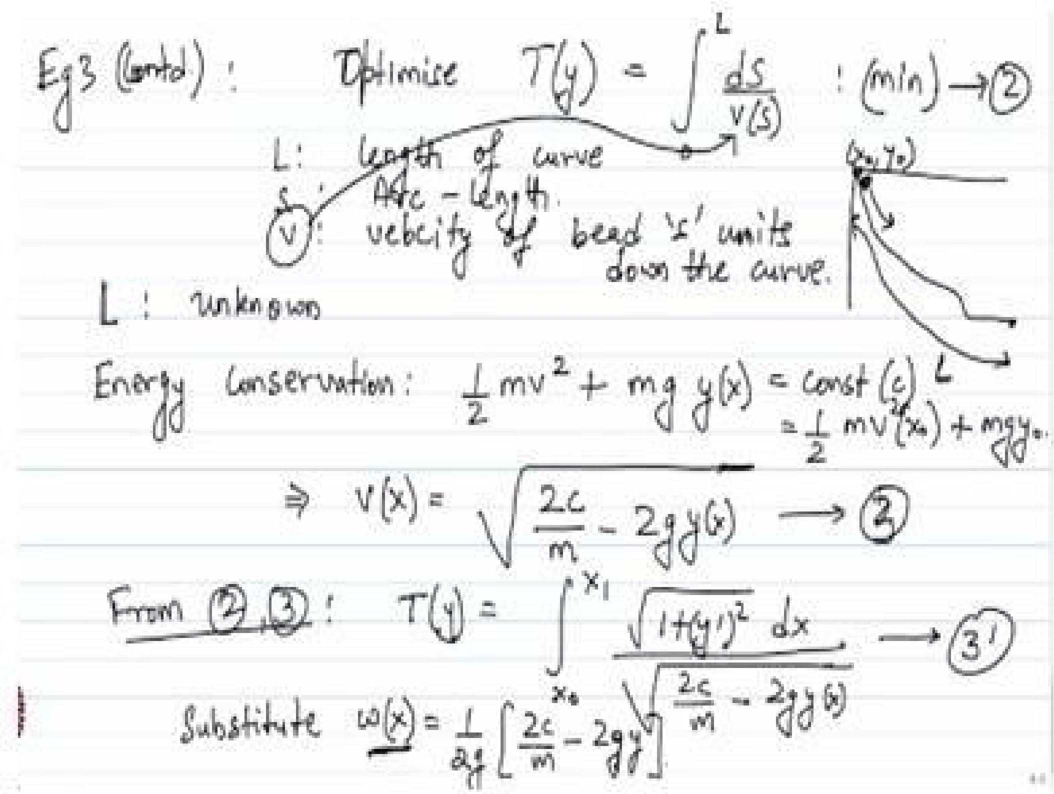

Again example 3 continued. We see that in this case that the quantity that we are trying to optimize or minimize is the total time in which the particles slides to the bottom of the curve. So we are trying to optimize the following integral which is the total length of the curve divided by the velocity of the curve which is a function of arc length itself and the total integral of the total length is from 0 to L and we want to minimize this particular functional.

$$Optimize \qquad T(y) = \int_0^L \frac{ds}{v(s)} \qquad (min) \qquad 2$$

So again, here L is the length, so let me draw a very basic figure to this problem. So let us say we have a curve and the starting point of the curve is x_0, y_0 and we have a bead which slides along this curve to come to rest at the ground level. So let us say the total length of this curve is L and s is the arc length of this curve and v is the velocity of the bead and bead at s units down the curve velocity of the bead at s units down the curve.

So we are not going to provide the solution at this stage, but we will look at in more depth we will look at the solution at a later stage. Further, let me just give a bit of slightly a bit more setup of this problem. Now, we know that again in this problem we know that L is an unknown of the problem. We do not know that the total length of the curve so to find this velocity to be substituted in this integral we utilize the conservation of energy.

So the energy conservation arguments states that the total energy of the system in this case the sum of the kinetic energy plus the potential energy is constant. Let me denote it by c and let us say this is equal to the value of this energy at the initial point

energy conservation = $\frac{1}{2}mv^2 + mgy(x)$ = Constant= c (say) = $\frac{1}{2}mv^2(x_o) + mgy_o$

So from this argument, I can directly find what is the value of this velocity v(x) turns out to be this is equal to taking the right hand side equal to c.

You see that this particular expression comes out to be

$$\Rightarrow \qquad v(x) = \sqrt{\frac{2c}{m} - 2gy(x)} \tag{3}$$

So from **2** and **3** we see that time functional is as follows.

$$T(y) = \int_{x_o}^{x_1} \frac{\sqrt{1+(y')^2}dx}{\sqrt{\frac{2c}{m} - 2gy(x)}} \tag{3`}$$

Now, this particular integrant is quite complicated so we do a little bit of simplification. We substitute the following variable.

$$w(x) = \frac{1}{2g}\left[\frac{2c}{m} - 2gy\right]$$

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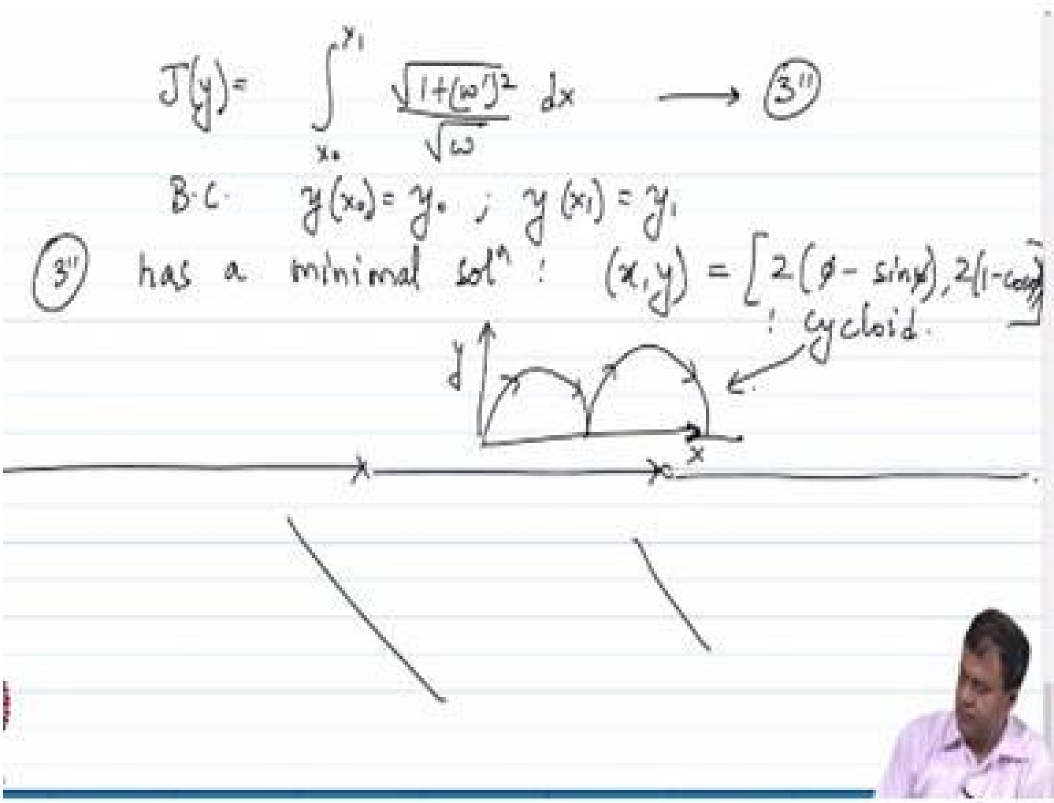

And when we do that, we get the following functional

$$J(y) = \int_{x_o}^{x_1} \frac{\sqrt{1+(w')^2}}{\sqrt{w}} dx \qquad \mathbf{3}"$$

So this is the quantity that we are trying to minimize, of course we have the boundary conditions which are given by $y(x_o) = y_o$ and $y(x_1) = y_1$. We see that let me call this as **3**" . Well, I am going to right away reveal the solution to this problem.

Although, we are going to give a detailed step by step solution later on. So, **3** has a minimal solution which is a cycloid. So, I am going to give a parametric representation of this cycloid which is as follows

$$(x,y) = [2(\varphi - \sin\varphi), 2(1 - \cos\varphi)]$$

Why the name is cycloid if we try to track the motion the locus of these points on the x, y plane, we see that the points are such that the locus are located the rim of a bicycle wheel and hence the name cycloid. So, let us move on to another example in our case study.

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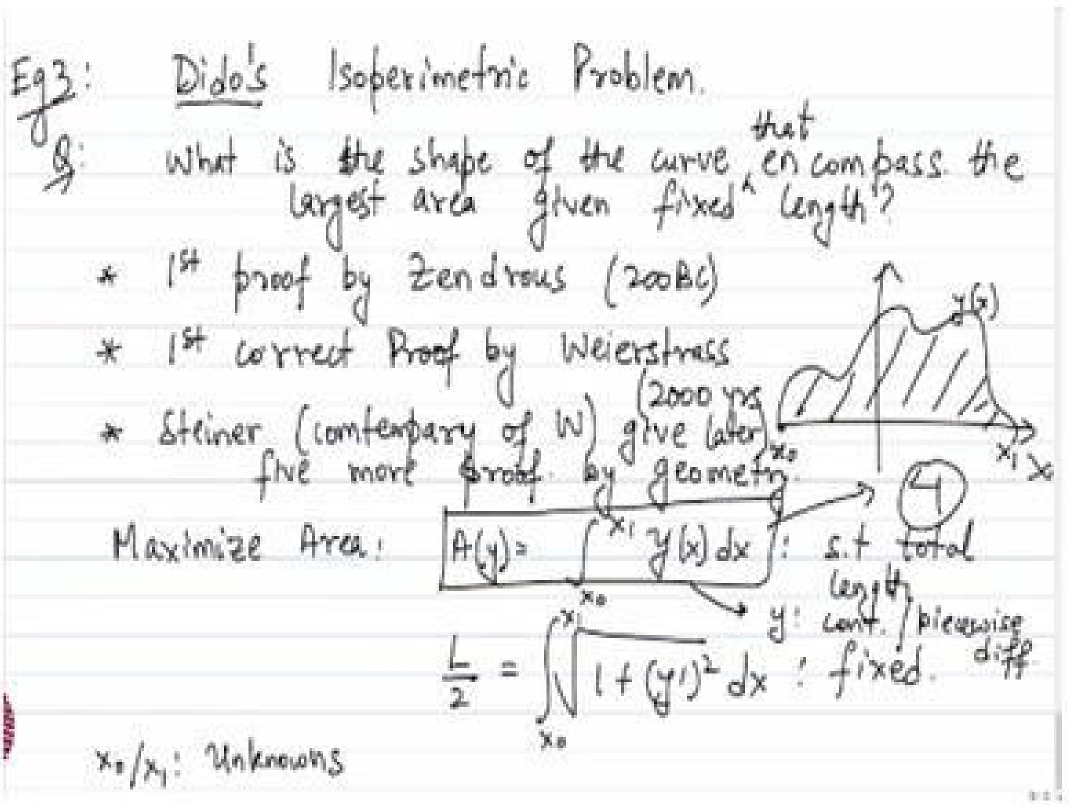

The third example that I have in mind is the Dido's Isoperimetric problem. So a little bit of trivia here. So it turns out that Dido was an ancient Egyptian queen in the early 300 BC and she was thrown away from her empire by her wicked brother and it turns out that she came to another land known as carthage and in that land the landlord gave his queen a piece of rope known as the bulls hide and ask the queen to figure out an area using this rope which will be her empire initial area of land which will be her empire.

Now, it turned out that this queen was very intelligent she figured out that the rope has to be organized in such a way that it swipes the area of the circle and hence known as the Dido's isoperimetric problem. So it turns out so the problem is as follows the problem is what is the shape of the curve which encompasses that encompasses the largest area given fixed length of the curve.

So in terms of the diagram, let us say that we are given a curve and we have to figure out the shape of this curve such that it encompasses the area swiped which is the largest among all possible area. So it turned out that the earliest proof was given at the time of Dido herself by a Greek scientist by the name of Zendrous in 200 BC. However the proof was very sketchy the first version of the correct proof was given by Weierstrass himself about 2000 years later.

It turned out that Steiner a German Mathematician who was a contemporary of Weierstrass also gave more proof using contemporary of Weierstrass also gave more proofs using the standard geometry arguments. In fact, he gave 5 more proofs by geometry consideration. So just to give a quick overview in this particular case we want to maximize the area

$$A(y) = \int_{x_0}^{x_1} y(x)dx \qquad \textbf{4}$$

such that the total length in this case it is in the diagram it is $\frac{L}{2}$.

$$\frac{L}{2} = \int_{x_0}^{x_1} \sqrt{1+(y')^2}dx$$

So this is certainly a case of isoperimetric problem and further we assume that the solution y is continuous and piecewise differentiable. Here varaibles x_o and x_1 are unknowns. So let me call this particular functional and denote it by **4**.

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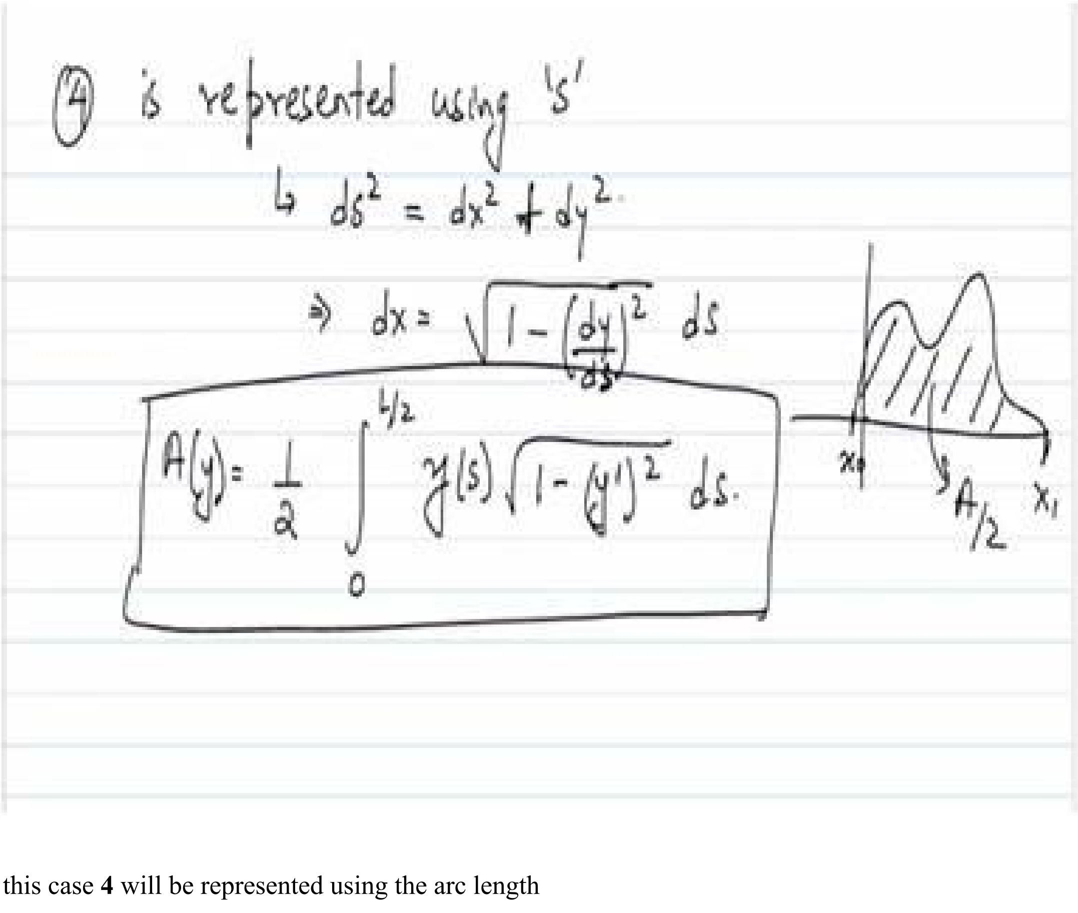

So in this case **4** will be represented using the arc length

$$ds^2 = dx^2 + dy^2$$

$$dx = \sqrt{1 - \left(\frac{dy}{ds}\right)^2} ds$$

$\Rightarrow$

Here A(y) is area functional.Now, Again let me redraw the figure that I had in my previous slide. So this is my area which is half of the total area that we are trying to optimize.

So here area is half the total integral of over the total length, in this case the total length is again $\frac{L}{2}$ of y(s), so I am representing this functional in terms of the arc length

$$A(y) = \frac{1}{2} \int_0^{\frac{L}{2}} y(s) \sqrt{1 - (y')^2} ds$$

So this is the functional that we need to optimize, of course Dido gave the solution to this optimal solution to this functional and the optimal solution intuitively comes out to be a semi circle.